RADIATIVE INSTABILITY OF A NONEQUILIBRIUM PLASMA IN MAGNETIC TRAPS

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We consider a plasma in a magnetic field, with a non-equilibrium plasma momentum distribution, namely, δ-like terms of all the components or the transverse components of the momentum (in the latter case $p_{\perp 0} \gg p_{\parallel 0}$). In such a plasma, oscillations with circular polarization and negative energy are possible and can propagate along the magnetic field. In systems bounded along the magnetic field, for example in magnetic traps, the negative-energy oscillations build up as a result of the outflow of energy to the vacuum. Thus, the magnetic trap can serve as a source of electromagnetic radiation. The considered mechanism for the buildup of the unstable oscillations turns out to predominate in sufficiently short systems and not too large a plasma density.

1. INTRODUCTION

A. As is well known, negative-energy oscillations are possible in systems that are not in thermodynamic equilibrium. Such oscillations grow if their energy is dissipated or is extracted in some way from the system. Thus, for example, it was shown in [1] that the boundary between supersonic flows is unstable, owing to radiation of oscillations into the interior of the liquid. In [2] there were considered electrostatic oscillations of a thermodynamically-nonequilibrium plasma in an inhomogeneous magnetic field. A stable solution of the wave equation was found; in the region of the magnetic-field minimum this solution describes cyclotron ion oscillations with negative energy,[3] and far from the minimum it describes electron Langmuir oscillations that carry energy away to the external part of the plasma.

We consider one more example of such an instability. In the present case, the unstable system can be used as a source of electromagnetic oscillations.

B. We consider a plasma in a magnetic field, with a non-equilibrium momentum distribution of the charged particles (electrons or ions)

$$f_0(\mathbf{p}) = \frac{1}{4\pi p_{\perp 0}}\delta(p_\perp - p_{\perp 0})\delta(|p_\parallel| - p_{\parallel 0}).$$

(The "parallel" and "perpendicular" symbols denote the direction with respect to the magnetic field.) We are interested in oscillations having a circular polarization, propagating along the magnetic field and having an electric vector that rotates in the electron or ion sides, depending on which of the plasma components is not in equilibrium. The energy of such oscillations is equal to

$$W = \left\{1 + \frac{\omega_p^2}{2(\omega-\omega_c)^2}\left[\frac{\omega_c}{\omega} - \frac{p_{\perp 0}^2}{m^2c^2}\frac{\omega_c}{\omega-\omega_c}\right]\right\}\frac{E^2}{4\pi}. \quad (1)$$

Here ω_p is the plasma frequency, ω_c the cyclotron frequency, the plasma and the magnetic field are assumed to be homogeneous, the electric field of the oscillations is chosen in the form $E_0 \exp(-i\omega t + ikz)$, the z axis is directed along the magnetic field, and the wavelength remains sufficiently large: $k^2 \ll \omega\omega_c/c^2$. In investigations of oscillations having a frequency close to the cyclotron frequency, one need take into account only one of the plasma components (electrons or ions). Both cases are analyzed in a similar manner, and we therefore omit the symbol j(e, i).

It follows from (1) that when $\omega > \omega_c$ the energy of the oscillations can become negative. Let us assume now that the plasma is bounded and directed along the magnetic field, say it is locked in a magnetic trap. In this case oscillations with negative energy will grow as a result of the radiation into vacuums. The energy of the oscillations becomes negative because of the relativistic dependence of the cyclotron frequency on the velocity. It is shown in [4] that this effect can lead to a buildup of oscillations in homogeneous systems, owing to the influence of phase focusing. In order to exclude this possibility and to observe the type of instability of interest to us in pure form, it is necessary to impose certain conditions, spelled out below, on the system parameters.

2. FUNDAMENTAL EQUATIONS

Let us obtain the equation which must be satisfied by the amplitude of the oscillations $E(z)$ in a bounded system (magnetic trap). We assume that the magnetic field is constant inside the trap, and that it increases jumpwise on its boundaries ($z = \pm L$).

From Maxwell's equations we have

$$E'' + \frac{\omega^2}{c^2}E + \frac{4\pi i\omega}{c^2}j = 0. \quad (2)$$

Here $E = E_x \mp iE_y$, $j_x \mp ij_y$, the "minus" sign must be taken for electrons and the "plus" sign for ions, we are using a left-hand coordinate system, and the primes denote differentiation with respect to z.

The perturbing current j is conveniently calculated by integrating along the trajectories

$$j = -\frac{e^2n_0}{2}\int d\mathbf{p}\frac{p_\perp}{m}\int_{-\infty}^{0} d\tau\left[\frac{\partial f_0}{\partial p_\perp} - \frac{i}{m\omega}\left(p_\perp\frac{\partial f_0}{\partial p_\parallel(\tau)} - p_\parallel(\tau)\frac{\partial f_0}{\partial p_\perp}\right)\frac{\partial}{\partial z(\tau)}\right]E(z(\tau))\exp\{i(\omega_c-\omega)\tau\}. \quad (3)$$

In the system under consideration, the particle velocity and the plasma density are constant for $|z| < L$. The longitudinal velocity reverses sign at the point $z = \pm L$ were the particles are reflected from the magnetic mirrors.

An analysis shows that the effect of buildup of the oscillations as a result of the outflow of energy from the ends prevails provided the oscillation frequency is close to the cyclotron frequency, $|\omega - \omega_c| \ll p_{\|0}/mL$. In the opposite case the oscillations that build up are those considered in [4, 5], which are not significantly influenced by the end effects. If the condition $|\omega - \omega_c| \ll p_{\|0}/mL$ is satisfied, then the particle can traverse the trap many times during the time $(\omega - \omega_c)^{-1}$, so that the field $E(z)$ acting on it averages out. In this case we can use the approximate equation

$$\int_{-\infty}^{0} d\tau E(z(\tau)) \exp\{i(\omega_c - \omega)\} \approx \frac{i}{\omega - \omega_c} \langle E \rangle, \qquad (4)$$

where[1)]

$$\langle E \rangle = \frac{1}{2L} \int_{-L}^{L} dz E(z).$$

Taking into account the relation

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \frac{p_\|(\tau)}{m} \frac{\partial}{\partial z(\tau)},$$

and also the fact that the momentum distribution function depends on the modulus of $p_\|$, we get from (3)

$$j = \frac{ie^2 n_0}{2} \left\{ \frac{1}{\omega} \int d\mathbf{p} \frac{p_\perp}{m} \left(2p_\perp \frac{\partial f_0}{\partial p_\|^2} - \frac{\partial f_0}{\partial p_\perp} \right) E - \int d\mathbf{p} \frac{p_\perp}{m} \left[\frac{1}{\omega - \omega_c} \frac{\partial f_0}{\partial p_\perp} + \frac{1}{\omega} \left(2p_\perp \frac{\partial f_0}{\partial p_\|^2} - \frac{\partial f_0}{\partial p_\perp} \right) \right] \langle E \rangle \right\}. \qquad (5)$$

Substituting (5) in (2) and integrating over the velocities we obtain ultimately

$$E'' + \left[\frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \left(1 + \frac{p_{\perp 0}^2}{2p_{\|0}^2} \right) \right] E + \frac{\omega_p^2}{c^2} \left[\frac{\omega}{\omega - \omega_c} \left(-1 + \frac{p_{\perp 0}^2}{2m^2c^2} \frac{\omega}{\omega - \omega_c} \right) + \left(1 + \frac{p_{\perp 0}^2}{2p_{\|0}^2} \right) \right] \langle E \rangle = 0. \qquad (6)$$

3. STABILITY ANALYSIS

A. Let us consider a symmetrical solution of (6) $E(z) = E(-z)$, which goes over outside the plasma ($|z| > L$) into outgoing waves $E(z) = E_0 \exp(i\omega|z|/c)$. Some idea of the stability of such oscillations can be obtained without solving (6). Indeed, let us multiply (6) by $E^*(z)$ and average the result over z:

$$i\frac{\omega}{cL} |E(L)|^2 - \langle |E'|^2 \rangle + \frac{\omega^2}{c^2} \langle |E|^2 \rangle + \frac{\omega_p^2}{c^2} \frac{\omega}{\omega - \omega_c} \left(-1 + \frac{p_{\perp 0}^2}{2m^2c^2} \frac{\omega}{\omega - \omega_c} \right) |\langle E \rangle|^2 - \frac{\omega_p^2}{c^2} \left(1 + \frac{p_{\perp 0}^2}{2p_{\|0}^2} \right) (\langle |E|^2 \rangle - |\langle E \rangle|^2) = 0. \qquad (7)$$

The first term, which takes into account the outflow of energy from the system, was obtained here as a result of integration by parts.

If the plasma density is sufficiently high, so that the third term in (7) can be omitted, and the energy outflow is disregarded, then the oscillations are stable. To prove this statement, it suffices to take into account the relation $\langle |E|^2 \rangle - |\langle E \rangle|^2 > 0$. We then obtain from (7) two real values for the natural-oscillation frequencies. If we then regard the first term in (7) as a small correction, we find that the oscillations with

$$0 < \omega - \omega_c < \omega_c p_{\perp 0}^2 / m^2c^2$$

are unstable. Thus, in accordance with the statement made at the beginning of the paper, the energy outflow into vacuum can actually lead to a buildup of the oscillations.

B. Let us consider the unstable oscillations in greater detail. The symmetrical solution of (6) at $|z| < L$ is of the form

$$E(z) = \operatorname{ch} \varkappa z + \frac{\alpha}{\varkappa^2 - \alpha} \frac{1}{\varkappa L} \operatorname{sh} \varkappa L. \qquad (8)$$

Here

$$\varkappa = \left[\frac{\omega_p^2}{c^2} \left(1 + \frac{p_{\perp 0}^2}{2p_{\|0}^2} \right) - \frac{\omega^2}{c^2} \right]^{1/2},$$

$$\alpha = \frac{\omega_p^2}{c^2} \left[\frac{\omega}{\omega - \omega_c} \left(-1 + \frac{p_{\perp 0}^2}{2m^2c^2} \frac{\omega}{\omega - \omega_c} \right) + \left(1 + \frac{p_{\perp 0}^2}{p_{\|0}^2} \right) \right].$$

Matching this solution at $z = \pm L$ with the outgoing waves $E_0 \exp(i\omega|z|/c)$, we obtain the dispersion equation for the frequency of the natural oscillations:

$$\frac{\alpha}{\varkappa^2} = \left[1 - \left(\varkappa L \operatorname{cth} \varkappa L + i\varkappa^2 L \frac{c}{\omega} \right)^{-1} \right]^{-1}. \qquad (9)$$

In the derivation of (9) we used the relation $p_{\|0}/mL \ll |\omega - \omega_c|$.

An analysis of (9) shows that the oscillations are unstable only if $|\omega_c/(\omega - \omega_c)| > p_{\perp 0}^2/2m^2c^2$, see also (1) and (7). In addition, it is necessary to exclude the possibility of the instability of the oscillations with $|\omega - \omega_c| \gg p_{\|0}/mL$, considered in [4, 5], the buildup of which is not connected with the boundary effect (outflow of energy into the vacuum), and therefore is possible also in homogeneous systems. All these conditions will be satisfied if the plasma density is not too large, $\omega_c/\omega_p \gg p_{\perp 0}/mc$, and the length of the system is sufficiently small

$$\frac{L\omega_c}{c} \ll \operatorname{Min}\left\{ 1;\ \left(\frac{\omega_c}{\omega_p} \frac{p_{\perp 0}}{mc} \right)^2;\ \frac{\omega_c}{\omega_p} \left(1 + \frac{p_{\perp 0}^2}{2p_{\|0}^2} \right)^{-1/2} \right\}.$$

For the frequency of the unstable oscillations, regardless of the sign of κ^2, we have the following simple expression:

$$\omega \approx \omega_c + \frac{1+i}{2} \omega_p \frac{p_{\perp 0}}{mc} \left(\frac{L\omega_c}{c} \right)^{1/2}. \qquad (10)$$

The amplitude of the considered oscillations inside the system is almost constant: $|\kappa L| \ll 1$.

Let us examine now the velocity distributions, other than those in the form of a δ-function, for which buildup of oscillations with negative energy is possible. To exclude the cyclotron absorption due to the relativistic dependence of the cyclotron frequency on the velocity, it is necessary to satisfy the condition

$$\frac{p_0 \Delta p_0}{m^2c^2} \ll \left| \frac{\omega - \omega_c}{\omega_c} \right|,$$

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1)If $E(z)$ is expanded in a Fourier series $E(z) = \sum_n E_n^{\exp(in\pi z/L)}$, as was done in the analysis of cyclotron oscillations in [6], and if it is recognized that $E_n \lesssim E_0$ (see below), then we really find that the additional terms in (4) are smaller by a factor $[(\omega - \omega_c)mL/p_{\|0}]^2$.

where Δp_0 is the thermal momentum spread. On the other hand, the oscillations with negative energy themselves exist only if

$$\frac{p_{\perp 0}^2}{m^2c^2} > \left|\frac{\omega - \omega_c}{\omega_c}\right|$$

Comparing these conditions, we obtain $p_0 \Delta p_0 \ll p_{\perp 0}^2$. Thus, the transverse-velocity distribution should be close to a δ-function. The same can be said also concerning the longitudinal-velocity distribution if $p_{\|0} \gtrsim p_{\perp 0}$. However, if $p_{\perp 0} \gg p_{\|0}$, then the scatter in $p_{\|}$ can be comparable with the average longitudinal velocity, as, for example, in the case of a Maxwellian distribution.

An analysis shows that for a Maxwellian longitudinal-velocity distribution

$$f_0(p_{\|}) = \frac{1}{\pi^{1/2} p_{\|0}} \exp\left\{-\frac{p_{\|}^2}{p_{\|0}^2}\right\},$$

the ratio $p_{\perp 0}^2/2p_{\|0}^2$ in (6) is replaced by $-p_{\perp 0}^2/p_{\|0}^2$. The corresponding changes must be made also in (8) and (9). It turns out, however, that neither the conditions for the existence of the instability, nor the expression for the frequency of the unstable oscillations (10) are altered thereby.

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[1] B. B. Kadomtsev, A. B. Mikhaĭlovskiĭ, and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **47**, 2266 (1964) [Sov. Phys.-JETP **20**, 1517 (1965)].

[2] H. L. Berk et al., Phys. Rev. Lett. **22**, 867 (1969).

[3] A. V. Timofeev and V. I. Pistunovich, in: Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 5, Atomizdat, 1967, p. 351.

[4] A. V. Gaponov, M. I. Petelin, and V. K. Yulpatov, Izv. Vuzov Radiofizika **10**, 1414 (1967).

[5] J. Sharer, Plasma Phys. **11**, 1 (1969).

[6] V. V. Arsenin, Zh. Tech. Fiz. **37**, 614 (1967) [Sov. Phys.-Tech. Phys. **12**, 442 (1967)].

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