CONTRIBUTION TO THE THEORY OF FORMATION OF TRANSITION X-RADIATION IN A STACK OF PLATES

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An analysis is presented of the formation of transition radiation in the transoptical region of frequencies when an extremely relativistic particle passes through a stack of plates. It is shown that the radiation is described by different formulas, depending on the concrete physical conditions. The frequencies at which radiation maxima take place are determined, and the influence of the stack on the formation of radiation in the plate is investigated. Results are also presented of some numerical calculations of the spectral distribution of the number of the transition-radiation quanta, and the question of the possible influence of multiple scattering on these distributions is briefly discussed.

THE formation of transition[1] and Cerenkov radiation in a stack of plates was considered by a number of authors. In [2,3] this problem was solved without approximations; in [2] the solution was obtained for a medium with an infinite number of plates, while in [3] the number of plates was arbitrary.[1)] In these papers, the general formulas were not given in sufficiently simple and lucid form, since it was assumed that the transition radiation should be concentrated in the optical frequency region, where the formulas are quite complicated. After it was established that for extremely relativistic particles the bulk of the transition radiation emitted forward is concentrated in the transoptical region,[5,6] where the dielectric constant is $\epsilon(\omega) = 1 - \sigma/\omega^2$ ($\sigma = 4\pi n e^2/m$), investigations were made of transition radiation in a layered medium in this frequency region. Ter-Mikaelyan and Gazazyan,[7,8] using an approximation where $\epsilon(\omega)$ is close to unity, obtained a formula for the transition in a layered medium (called in these papers "resonant" radiation), and carried out a quantitative analysis of this formula. Amatuni and Korkhmazyan[9] considered a medium with a density that varies periodically in space, in the approximation wherein the density of the medium changes little, and obtained in the particular case of transoptical frequencies the first two terms of the expansion of the radiation field; these terms were derived by another method in [8]. It was shown in [10] that by making the approximation $\epsilon(\omega) \sim 1$ in the formulas of [3], it is possible to arrive at the result obtained in [7,8].

Using the same formulas of [3], but considerably altered and reduced to a more convenient form in [11], and also those obtained by another method in [12], we carry out in the present paper a detailed analysis of these expressions in the frequency region where $\epsilon(\omega) = 1 - \sigma/\omega^2$. An important role is played here by the circumstance that in [11,12] they were able to express the sought fields in terms of combinations of Chebyshev polynomials. This has made it possible to use in the analysis of the present problem those representations of these polynomials[12] which are most convenient for our purposes. As a result we arrive at the conclusion that the formula obtained in [8,10] has narrower limits of applicability than those heretofore assumed in the cited papers and repeated in a recently published book,[13] and we show which formulas should be used and when. In addition, we find the frequencies at which the radiation emitted by one plate located in a stack turns out to be larger than the radiation emitted likewise by one plate, but isolated, and we demonstrate that the physical reason for such an enhancement of the transition radiation is the coincidence of the maxima of the corresponding angular distributions.

The present investigation was also stimulated by the fact that recently experimental research has been carried out on transition radiation in the x-ray regions.[14-16] On the other hand, the formula for the transition radiation in a layered medium can be simplified under certain conditions and reduced to the formulas for the transition radiation from one plate or even from one boundary.[17] Failure to take this circumstance into account in [14,15] has caused the authors of these references to complicate greatly the calculation of the theoretical curves, and in addition, they deprive themselves of the possibility of simply verifying the calculated curves, some of which, as will be shown in the present article, call for revision. In the last section of the article we consider briefly the influence of multiple scattering on the formation of the transition or resonant radiation in a layered medium.

1. Obviously, when a charge passes through a stack consisting of N plates, the radiation fields are produced both in the space ahead of the stack and behind it. Expressions for these fields were obtained in [3,12]. We assume in these formulas certain approximations, connected with the fact that $\beta = v/c \to 1$, where v is the particle velocity, as well as with the fact that we are considering a frequency region where the dielectric constant of the medium is close to unity and is given by the formula $\epsilon(\omega) = 1 - \sigma/\omega^2$. The backward radiation can be neglected, and for the radiation emitted forward it is convenient to use formula (41) of [12], provided the stack does not consist of very thin plates. We also introduce the quantity $r = (\sqrt{\epsilon} - 1)/(\sqrt{\epsilon} + 1)$, it being clear clear that $|r| = \sigma/4\omega^2 \ll 1$.

[1)]This problem was solved in [4] for layered medic consisting of ferrodielectric and crystalline plates.

For the tangential component of the Fourier transform of the electric field of the transition radiation emitted forward we obtain

$$E_{N,t}(\mathbf{k};N)=\frac{ei\varkappa}{2\pi^2}\frac{\omega^2c^{-2}(1-\varepsilon)}{\Lambda\Lambda_0}\exp\{i\varphi_0'(a+b)(N-1)+i\varphi_0'a\}$$
$$\times\left(1-\exp\left\{-i\left(\frac{\omega}{v}-\lambda\right)a\right\}\right)\frac{1-\exp\{-i(\varphi-\chi)N\}}{1-\exp\{-i(\varphi-\chi)\}},\qquad(1)$$

where a is the thickness of the plates, b is the distance between them, and

$$\Lambda=k^2-\frac{\omega^2}{c^2}\varepsilon,\quad \Lambda_0=k^2-\frac{\omega^2}{c^2},\quad k^2=\varkappa^2+\frac{\omega^2}{v^2},\quad \varphi_0'=\left(\frac{\omega}{v}-\lambda_0\right),$$
$$\varphi=\frac{\omega}{v}(a+b),\quad \chi=\lambda a+\lambda_0 b,\quad \lambda_0^2=\frac{\omega^2}{c^2}-\varkappa^2,\quad \lambda^2=\frac{\omega^2}{c^2}\varepsilon-\varkappa^2.$$

The foregoing simplifications limit the region of applicability of formula (1) by means of the following inequalities:

$$\sin^2\vartheta+(1-\beta^2)\ll 1,\qquad(2)$$
$$(rN)^2\ll 1,\qquad(3)$$

where ϑ is the radiation angle. Condition (2) imposes a limitation on the emission angles, whereas (3) relates the quantity $(\sqrt{\epsilon}-1)$ with the number of plates in the stack. This condition arises formally when the exact formula for the field $E'_{N;t}(\mathbf{k};N)$ is expanded in powers of the small quantity r^2 (these operations were carried out in detail in [18]).

We note immediately that in the case of extremely relativistic particles, the condition (2) is in fact not a limitation. The reason is that in this case the main contribution to the radiation is made by small angles. Therefore we are actually left only with condition (3). It is also clear from (1) that the field is maximal when

$$\varphi-\chi=2\pi n,\qquad(4)$$

where n are integers.

2. Let us now calculate the Poynting-vector flux through the plane z = const in the space behind the stack of particles, during the time of flight of the particle. Just as in [7, 8, 10, 12], assuming the material of the plates to be non-absorbing, we obtain in accordance with [11]

$$S_N'=\frac{8e^2}{\pi c}\int_0^\infty\int_0^{\pi/2}\frac{(1-\varepsilon)^2\sin^3\vartheta\,d\vartheta\,d\omega}{(1-\beta^2\cos^2\vartheta)^2[1-\beta^2(\varepsilon-\sin^2\vartheta)]^2}$$
$$\times\sin^2\left(\frac{a}{2}\left(\frac{\omega}{v}-\lambda\right)\right)\left[\frac{\sin NX}{\sin X}\right]^2,\qquad(5)$$
$$X=\left(\frac{\omega}{v}-\lambda\right)\frac{a}{2}+\left(\frac{\omega}{v}-\lambda_0\right)\frac{b}{2}.$$

This formula was derived in [8, 10], but without the condition (3). When n = 1, formula (5) gives the radiation by one plate.[19]

The expression in the square brackets of (5) is usually replaced by a sum of δ functions, using the well-known formula

$$\lim_{N\to\infty}\frac{1}{N}\frac{\sin^2 Nx}{\sin^2 x}=\sum_n\delta\left(\frac{x}{\pi}-n\right),\qquad(6)$$

and δ functions are then used to integrate with respect to the angle ϑ in formula (5). Such a calculation procedure calls, however, for the exercise of a certain caution.

First, according to (3), the value of N for a given frequency is bounded from above in principle, and therefore the transition to δ functions with the aid of (6) is not rigorous. Much more important, however, is the fact that N is a finite quantity under the experimental conditions ($N\sim 10^2-10^3$). These two circumstances cause the width of the function $\sin^2 Nx/\sin^2 x$ to be finite in the vicinity of those discrete values of the angle ϑ where $x=\pi n$ and the function has a maximum. Replacement of functions having finite widths at the maxima by a sum of δ functions is valid if the remainder of the integrand in (5) is a smooth function of the angle ϑ. Second, it is necessary that this remaining part of the integrand, which henceforth will be denoted by f, remain essentially unchanged as a function of the angle ϑ within a finite width about the maxima of the function $\sin^2 Nx/\sin^2 x$. We note that the authors of all the preceding papers on the theory of transition radiation in a layered medium (or resonant radiation) overlooked this circumstance.

We note also that the condition (3) is replaced in [13] by the requirement $r\ll 1$. This condition is brought about by the fact that in the quasiclassical approximation, which is used in [13], it is necessary to be able to neglect effects connected with reflected waves. The condition $r\ll 1$ is precisely the smallness of the reflection, but at one boundary, i.e., in [13] they lost sight of the fact that reflection takes place at each plate and it is necessary to stipulate that the entire reflected field be small compared with the transmitted field, which is equivalent to condition (3).

Before we proceed to further calculations, it is advantageous to change over in (5) to the small-angle approximation, i.e., to make the substitution $\sin\vartheta\to\vartheta$, and to introduce a new variable $y=\vartheta^2$. Then formula (5) is rewritten in the form

$$S_N'=\frac{4e^2\sigma^2}{\pi c}\int_0\frac{d\omega}{\omega^4}\int_0^\infty\frac{y\sin^2 Y}{(\xi+y)^2(\eta+y)^2}\left[\frac{\sin NX}{\sin X}\right]^2dy,\qquad(7)$$

where

$$\omega'_{(a,p)}=\frac{4\pi v}{(a,p)(1-\beta^2)},\qquad \omega''_{(a,p)}=\frac{(a,p)\sigma}{4\pi v},\qquad(8)$$
$$p=a+b,\quad \xi=1-\beta^2,\quad \eta=1-\beta^2+\frac{\sigma}{\omega^2},$$
$$X=\pi\left(\frac{\omega_a''}{\omega}+\frac{\omega}{\omega_p'}+\frac{p\omega}{4\pi v}y\right),\quad Y=\pi\left(\frac{\omega_a''}{\omega}+\frac{\omega}{\omega_a'}+\frac{a\omega}{4\pi v}y\right).$$

Let $f=(4e^2/\pi c)f_1f_2$, with

$$f_1=\frac{\sigma^2}{\omega^4}\frac{y}{(\xi+y)^2(\eta+y)^2},\qquad f_2=\sin^2 Y.\qquad(9)$$

An analysis of f_1 shows that the maximum of this function is reached at $y\sim(1-\beta^2)$, and the width of this maximum is also of the order of $1-\beta^2$.

On the other hand, it is easily seen that the function f_2 has maxima at the following values of y:

$$\bar{y}_s=\frac{\omega_a'}{\omega}(1-\beta^2)\left[s-\left(\frac{\omega_a''}{\omega}+\frac{\omega}{\omega_a'}-\frac{1}{2}\right)\right],\qquad(10)$$

where the integers s assume values starting with the integer larger than $(\omega_a''/\omega+\omega/\omega_a'-1/2)$. Therefore, if we introduce a frequency region such that

$$\omega\gg\omega_a',\qquad(11)$$

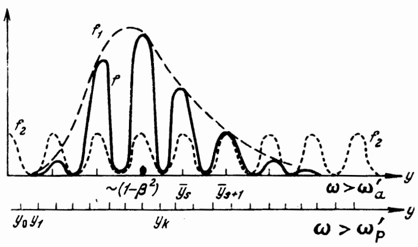

FIG. 1

then the zeroes due to f_2 are superimposed on the values of the function f_1 (see Fig. 1 with the upper abscissa axis, where the dashed lines represent the functions f_1 and f_2, and the solid line is proportional to f). These zeroes are located at distances

$$\Delta y = \frac{\omega_a'}{\omega}(1-\beta^2) \tag{12}$$

from one another. On the other hand, in the region of frequencies satisfying a condition inverse to (11), namely

$$\omega \lessapprox \omega_a', \tag{13}$$

the function f_2 will pass approximately once through zero where the function f_1 assumes its essential values (see Fig. 2a and b with the upper abscissa axis). It is clear that in this case the interval where the function f experiences significant changes is

$$\Delta y \sim (1-\beta^2). \tag{14}$$

Since it is clear from formula (7) that $S_1' = \int f\, dy\, d\omega$ is the intensity of the forward-emitted radiation per plate, it is meaningful to consider the question of the behavior of this quantity as a function of the frequency. It is seen from Figs. 2a and 2b that in the frequency region (13) this quantity has clearly pronounced maxima as functions of the position of $\bar{y}_s$ relative to $(1-\beta^2)$. On the other hand, it is seen from Fig. 1 that in this case the change of the frequency, i.e., the shift of the maxima of the f_2 curve, does not result in considerable changes in the value of $\int f\, dy$. Therefore, the dependence of this quantity on the frequency will be smooth or else will have small maxima.

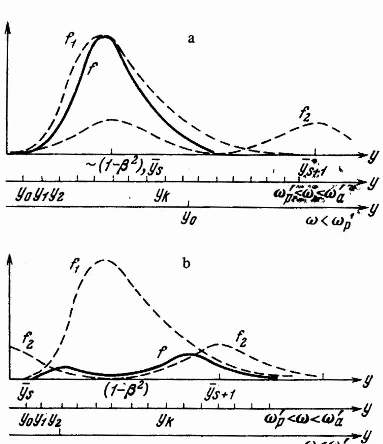

FIG. 2

The positions of the maxima, as seen from Fig. 2a, should be determined from the condition $\bar{y}_s = 1-\beta^2$. In view of the fact that there are practically no maxima when $\omega > \omega_a'$, we take that solution of the last equation which occurs in the frequency region $\omega \ll \omega_a'$, namely

$$\bar{\omega}_s = \omega_a'' / (s + 1/2). \tag{15}$$

Since $\bar{\omega}_s \ll \omega_a$, it is necessary that the integers s satisfy the inequality

$$(s + 1/2)/(1-\beta^2) \gg a^2\sigma/(4\pi v)^2. \tag{16}$$

On the other hand, in view of the fact that the function f_1 has a sharp boundary at the frequency $\omega_b = \sqrt{\sigma}/\sqrt{1-\beta^2}$ (see [5,6]), the radiation is sufficiently intense at the maxima if

$$(s + 1/2)/\sqrt{1-\beta^2} > a\sqrt{\sigma}/4\pi v. \tag{16'}$$

For all the values of s except s = 1, the condition (16) is more stringent than (16′).

3. Let us now analyze the function $\sin^2 NX/\sin^2 X$. The maxima of these functions will occur at those values $y = y_n$, for which $X = \pi n$. From (8) we obtain

$$y_n = \frac{4\pi v}{p\omega}\left(n - \frac{\omega_a''}{\omega} - \frac{\omega}{\omega_p'}\right).$$

If we put $n_{min} = \{\omega_a''/\omega + \omega/\omega_p'\}$, where the curly bracket denotes an integer larger than the number inside the bracket, then it follows from the last formula that n should assume the following values: $n = n_{min} + k$, where $k = 0, 1, 2, \ldots$. On the other hand, we can write

$$n_{min} = \left(\frac{\omega_a''}{\omega} + \frac{\omega}{\omega_p'}\right) + d(\omega), \tag{17}$$

where the function $d(\omega)$ $(0 \le d(\omega) \le 1)$ complements the quantity $\omega_a''/\omega + \omega/\omega_p'$ to form an integer:

$$d(\omega) = \left\{\frac{\omega_a''}{\omega} + \frac{\omega}{\omega_p'}\right\} - \left(\frac{\omega_a''}{\omega} + \frac{\omega}{\omega_p'}\right),$$

and has the form shown in the lower part of Figs. 3 and 4. Then

$$n = \left(\frac{\omega_a''}{\omega} + \frac{\omega}{\omega_p'}\right) + k + d(\omega), \tag{18}$$

from which we see that the discrete values of the square of the emission angle (16) are best marked by the index k:

$$y_k = y_n = \frac{\omega_p'}{\omega}(1-\beta^2)(k+d). \tag{19}$$

Let us find also the distance between these discrete values:

$$\Delta = y_{k+1} - y_k = \frac{\omega_p'}{\omega}(1-\beta^2). \tag{20}$$

To determine the width Δ_N of the maxima of the investigated function, we write $y = y_k + \Delta_N$, and substitute such an expression in (8) for X and then determine Δ_N from the requirement $(p\omega/4\pi v)\Delta_N \sim 1/N$, whence

$$\Delta_N \approx \frac{1}{N}\frac{4\pi v}{p\omega} = \frac{1}{N}\frac{\omega_p'}{\omega}(1-\beta^2). \tag{21}$$

To be able to change over from $\sin^2 NX/\sin^2 X$ to a sum of δ functions, we should stipulate satisfaction of the condition

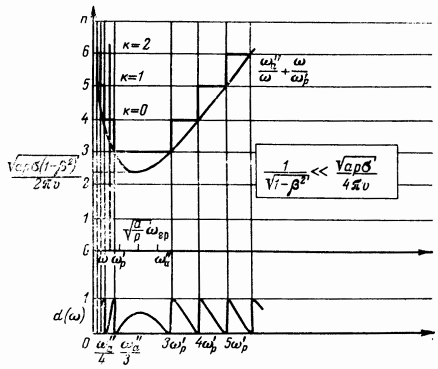

FIG. 3

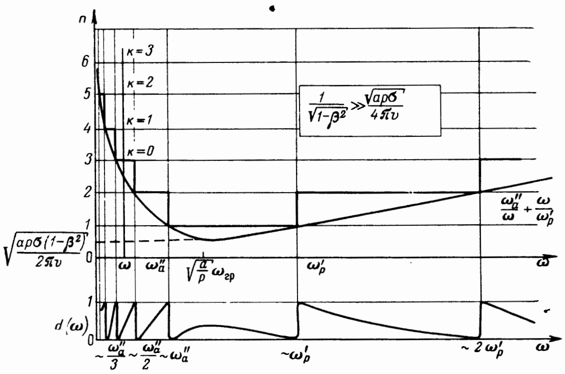

FIG. 4

$$\Delta_N \ll \Delta y, \tag{22}$$

where Δy is determined by formulas (12) and (14), depending on the frequency region we have in mind. It is easily seen that at frequencies determined by the condition (11), practically no condition is imposed on N by (22). On the other hand, if condition (13) is satisfied, then it follows from (22) that

$$N \gg \omega_p'/\omega. \tag{23}$$

In both cases, N must also satisfy the condition (3). From the requirement that the conditions (3) and (23) be compatible in the frequency region (13), we obtain the following inequalities:

$$\left(\frac{a}{p}\right)^{3/2}\frac{\sigma^{1/2}}{2} \ll \omega_p' \ll 4\frac{\omega^3}{\sigma}. \tag{24}$$

4. We now change over to a sum of δ functions in formula (7). According to (6), we have

$$\frac{\sin^2 NX}{\sin^2 X} = N\sum_{k=0}\Delta\cdot\delta(y-y_k), \tag{25}$$

where y_k and Δ are determined by (19) and (20). We integrate in (7) with respect to y, using (25):

$$S_N' = N\int Q\,d\omega, \tag{26}$$

where

$$Q = \Delta\sum_{k=0}^{\infty} f(y_k). \tag{27}$$

Formula (26) can also be represented in the form

$$S_N' = N\frac{4e^2}{\pi c}\omega_p''\int\frac{d\omega}{\omega^2}\cdot \tag{28}$$

$$\cdot\sum_{k=0}^{\infty}\frac{(k+d)\sin^2[(\pi a/p)(k+d+\omega_p''/\omega+\omega/\omega_p')]}{(k+d+\omega_p''/\omega+\omega/\omega_p')^2(k+d+\omega/\omega_p')^2}.$$

The last formula was first obtained in a somewhat different form in [8, 10] (see also [13]), but without the aforementioned limitations (3), (23), and (24) on the region of its applicability.

If the condition inverse to (23) holds in the frequency region (13), then we have in this case $\Delta_N \gg \Delta y$. This means that now the function f can be regarded as a sharp function, and $\sin^2 NX/\sin^2 X$ as a smooth function. Taking this into consideration, we can easily estimate the integral (7). The result is that the intensity at the stack is in this case small and equal in order of magnitude to the intensity of radiation per plate.

Formula (28) can be simplified if certain conditions are satisfied. Namely, if we are interested in those cases when the summation over k in (28) can be replaced by integration, then we should stipulate for this purpose that the quantity under the summation sign in (28) change little when k changes by unity. To this end it is necessary to have

$$p/a \gg 1, \quad \omega \gg \omega_p', \tag{29}$$

and as a result, as shown in [17], we obtain after integrating with respect to k from 0 to ∞:

$$S_N' = NS_1', \tag{30}$$

where S_1' is equal to the radiation intensity per plate.[19]

We now proceed to obtain the same result in a different way, which not only has the advantage that it enables us to avoid direct integration, but also has the virtue that it enables us to examine the entire problem as a whole.

5. To this end we turn again to the geometrical interpretation of the operation of integration with respect to the variable y (see Figs. 1 and 2). In the case of one plate, the spectral distribution of the intensity of the transition radiation is given by the formula

$$dS_1'/d\omega = \int f(y)\,dy, \tag{31}$$

i.e., it is necessary to calculate the area under the curve $f \approx f_1 f_2$. In the case considered at present, the average radiation intensity from one plate, but now located in a stack of plates, is

$$\frac{1}{N}\frac{dS_N'}{d\omega} = Q, \tag{31'}$$

i.e., it is now necessary to calculate the sum Q defined by formula (27).

Using these illustrative concepts, we shall show in this section that the maxima in the spectral distribution $N^{-1}dS_N'/d\omega$ will occur at approximately the same frequencies as in $dS_1'/d\omega$, and we shall also consider the values of these maxima.

To this end, we recall that at frequencies $\bar{\omega}_S$ at which the maxima of $dS_1'/d\omega$ take place, the values of the angle variables corresponding to the maxima of the functions $f_1(y)$ and $f_2(y)$, coincide, namely $1-\beta^2=\bar{y}_S$. In the case $N^{-1}dS_N'/d\omega$, the angular dependence is contained not only in the already noted functions, but also in the arguments of the δ functions. Therefore the maxima now occur at frequencies such that at least one of the y_k coincides with $1-\beta^2=\bar{y}_S$.

If one can speak of resonance in radiation at all, then it can be stated that it occurs precisely at these frequencies in transition radiation in a layered medium. The condition (4) of the present paper, which was called in [7, 13] the "resonance" condition, determines the angles $\vartheta_k = \sqrt{y_k}$ at which radiation of arbitrary frequency ω can be emitted by a fast charged particle in a layered medium.[2)] Whether, however, the radiation intensity at this frequency will be maximal or minimal is now determined decisively by the values assumed at the angles ϑ_k by the quantity f, which describes the angular distribution of the intensity of the transition radiation produced in one plate.

The sum Q can be greatly influenced by two circumstances. The first is the interval Δ in the sum, and the second is the position of y_k, and particularly of y_0. Generally speaking, one can visualize a case in which one of the values of y_k is equal to $1 - \beta^2 = \bar{y}_s$, corresponding to the maximum of the function $f_1(y)f_2(y)$, and the interval Δ is in this case much larger than Δy. It is clear that in this case $Q \gg dS'_1/d\omega$.

However, from formulas (12), (14), and (20) we can find that

$$\frac{\Delta}{\Delta y} = \frac{\omega_p'}{\omega} \gg 1 \text{ for } \omega \ll \omega_p', \tag{32a}$$

$$\frac{\Delta}{\Delta y} = \frac{\omega_p'}{\omega} \ll 1 \text{ for } \omega_p' \ll \omega < \omega_a', \tag{32b}$$

$$\frac{\Delta}{\Delta y} = \frac{a}{p} < 1 \text{ for } \omega_a' < \omega. \tag{32c}$$

From these formulas it is easily seen that when $\omega \gg \omega'_p$, the maxima of $N^{-1}\, dS'_N/d\omega$ occur at the frequencies $\bar{\omega}_s$. Indeed, in this case, according to (32b) and (32c), the values of y_k are quite close to one another (see Fig. 1, the strokes on the lower y axis, and Figs. 2a and 2b, the strokes on the middle y axes) and $Q = dS'_1/d\omega$. Here we have taken into account the fact that

$$y_0 = \frac{\omega_p'}{\bar{\omega}}(1-\beta^2)\,d(\bar{\omega}_s) \ll 1-\beta^2.$$

From the condition for the replacement of the sum Q by an integral, namely

$$\Delta \ll \Delta y, \tag{33}$$

we can obtain with the aid of (32b) and (32c) conditions that coincide with (29). For $p \sim a$ and $\omega \gg \omega'_p$, we have $Q \ll dS'_1/d\omega$ in view of the fact that $y_k \neq \bar{y}_s$.

We now turn to the frequencies $\omega \ll \omega'_p$. In this case, according to (32a), the values of y_k will be spaced far apart (see the lower abscissa axis of Figs. 2a and 2b), and since $\Delta \gg \Delta y$, the decisive role will be played by the value y_0. If it turns out that $y_0 \sim (1-\beta^2)$, then $Q \gg dS'_1/d\omega$, and if $y_0 \gg (1-\beta^2)$, then $Q \ll dS'_1/d\omega$. In fact, however, $y_0 = (\omega'_p/\omega)(1-\beta^2)/d(\omega)$, and therefore the values of the angle variables at which the maxima of the intensity occur will coincide at the frequencies $\bar{\omega}_s$ if

$$\frac{\omega_p'}{\bar{\omega}_s}\,d(\bar{\omega}_s) \sim 1. \tag{34}$$

However, it is easily seen that when $\bar{\omega}_s \ll \omega'_p$ we have

$$d(\bar{\omega}_s) = \left\{s + \frac{1}{2} + \frac{\bar{\omega}_s}{\omega'}\right\} - \left(s + \frac{1}{2} + \frac{\bar{\omega}_s}{\omega'}\right) = \frac{1}{2}. \tag{35}$$

Therefore the resonance condition (34) at $\bar{\omega}_s \ll \omega'_p$ will not be satisfied. On the other hand, if the values of $\bar{\omega}_s$ are in the region of ω'_p, and are both smaller or larger than this value, then the condition (34) can be satisfied for such $\bar{\omega}_s$ in order of magnitude. In this case, however, Δ is already of the order of Δy (see (32a)), and therefore Q is simply larger than $dS'_1/d\omega$. The inequality $\bar{\omega}_s \gtrsim \omega'_p$ can be satisfied provided $p \gg a$.

We now return again to the frequencies $\omega \ll \omega'_p$. Although the condition (34) is not satisfied when $\omega = \bar{\omega}_s$, if we take a frequency lower than $\bar{\omega}_s$ by an amount $\Delta\omega$, and such that

$$s + 1 > \frac{\omega_a''}{\bar{\omega}_s - \Delta\omega} > s + \frac{1}{2}, \tag{36}$$

then $d(\bar{\omega}_s - \Delta\omega)$ can decrease to such an extent, as a result of the fact that $\omega''_\alpha/\bar{\omega}_s - \Delta\omega)$ approaches an integer value, that we obtain as a result $y_0 \sim (1-\beta^2)$. We note that the left-hand side of the inequality (36) is due to the fact that at the frequency $\omega''_\alpha/(s+1)$ there occurs the case shown in Fig. 2b, that is, a minimum in the intensity; this case must be avoided. Therefore, if conditions (36) are satisfied, then the maxima of $f_1(y)$ and $\sin^2 NX/\sin^2 X$ will coincide at the frequency $\bar{\omega}_s - \Delta\omega$, but the maxima of $f_1(y)$ and $f_2(y)$ will be shifted. Thus, when $\omega \ll \omega'_p$ we have maxima at frequencies somewhat smaller than $\bar{\omega}_s$, and their values are $Q < dS'_1/d\omega$.

Thus, whereas when the plate is placed in the stack, the maxima in the spectral distribution of its emission intensity become smoothed out in the frequency region $\omega \ll \omega'_p$ owing to the decrease of the maxima, the maxima of the angular distributions of the radiation produced in a plate are either equal to or at resonance with the maxima produced in a stack of plates at frequencies in the region of ω'_p, as a result of which the maxima at the frequencies $\bar{\omega}_s$ increase when $p \gg a$. As to $\omega \gg \omega'_p$, when $p \gg a$ there is independent addition of the intensities from all the plates. All these singularities in the behavior can be visualized by comparing the curves of either Fig. 5 of the present paper or of the corresponding figures in [17, 19].

6. The so-called "threshold" values of the energy, starting with which the n-th harmonic radiation, i.e., radiation at a discrete angle $\vartheta_n = \sqrt{y_n}$, appears, are introduced in [7, 13]. Energies having such a physical meaning are obtained firectly from Fig. 3, if one equates the value of the function $(\omega''_\alpha/\omega + \omega/\omega'_p)$ at the minimum, that is $\sqrt{ap\sigma(1-\beta^2)}/2\pi c$, to the integers n. In the present paper we used in place of n the integer values k, which are reckoned from the value n_{min} (the stepped line in Figs. 3 and 4). For greater clarity, Figs. 3 and 4 show for the arbitrary frequency ω both the values of k and the corresponding values of n.

From the point of view of the angular dependence of the emitted radiation, the results indicate that when $b \gg a$ and $\omega \gg \omega'_p$ we have a continuous distribution with a maximum at $(1-\beta^2)^{1/2}$, just as in the case of one plate. When these conditions are violated, the intensity at a fixed frequency ω can either increase or decrease, and in the angular distribution there will occur a concentration of the radiation in definite directions y_k with

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2) This condition was first obtained by Feĭnberg and Khizhnyak [2] and was called with full justification the dispersion equation for a layered dielectric.

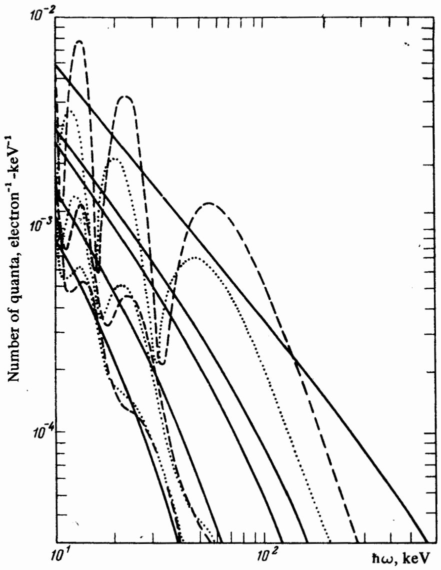

FIG. 5

widths Δ_N which depend on the number of plates. The intensity of radiation at the frequency ω, emitted at an angle whose square is equal to y_k, is determined by the k-th term of formula (28), and, as can be seen from this formula, the dependence of the intensity in these discrete directions on the particle energy can become very sharp. We note that the possibility of intensifying the dependence of the transition-radiation intensity on the particle energy by means of angular discrimination was first pointed out in [20] (see also [21]).

As to the connection between the transition radiation in a layered medium with transition radiation emitted by a plate or by one interface between media, it is stated in [7] that independent summation of the transition radiations from different plates corresponds to "resonance," i.e., in our terminology, to a discrete radiation angle, of high order. Yet it is seen from the foregoing that independent summation of the transition radiations occurs when all the discrete radiation angles merge into a continuous angular distribution, and by integrating over this distribution (and not by taking one discrete angle, as is stated in [7]), we obtain, if the conditions (29) are satisfied, the ordinary transition radiation for a plate multiplied by the number of plates.

A correct understanding of the relation between the ordinary transition radiation and the case when individual discrete angles appear in the radiation is not only of general interest, but is important from the practical point of view. Indeed, a numerical calculation of ordinary transition radiation in a plate or on one boundary is relatively simple, and if it is compared with the calculation by means of formula (28), where it is necessary to sum numerical series, these simple calculations are a measure of the correctness of the results obtained by such a summation. The curves given in [17, 19] satisfy these requirements.

On the other hand, curves are given in [14, 15] for the number of transition-radiation quanta in a layered medium, calculated only on the basis of formula (28). From the statements made above it is clear that such an approach is not the best from the point of view of either simplicity or reliability.

Assuming, together with the authors of [14, 15], that the medium can be described in the considered region of frequencies by means of a dielectric constant in the form $\epsilon = 1 - \sigma/\omega^2$, let us turn to the curves of Fig. 1 of [14] (see also [13], Fig. 59). In this figure, the ordinates represent the numbers of quanta emitted by a charged particle per centimeter of layered medium. Since there is only one plate per centimeter in the layered medium ($b = 1$ cm, $a = 2 \times 10^{-2}$ cm), the curves of [14] give the number of quanta emitted by a single plate. On the other hand, from the statements made in section 5 it is easily seen that the curve for muons with energy 3.5×10^{11} eV, which generated transition radiation in the layered medium, should be close to the curve for a plate, since $\omega_p' = 2.55$ keV, and $b \gg a$. In Fig. 5, which shows the numbers of quanta per plate, the three lowest closely-lying curves (solid, dashed, and dotted) correspond to muons with this energy. The solid curve corresponds to a double boundary, the dashed one to a plate, and the dotted one is calculated from formula (28) with $N = 1$. The values of the numbers of quanta, in accordance with the corresponding curve of Fig. 1 of [14], are larger by approximately one order of magnitude up to energies 15–20 keV than those that follow from our curves, which were obtained by three methods. Figure 5 also shows four solid curves for a double boundary and for muons with energies 7×10^{11}, 2.1×10^{12}, 3.5×10^{12}, and 7×10^{13} eV. For the energies 7×10^{11} and 7×10^{13} eV, the figure shows, in addition, dashed curves for a plate and dotted curves calculated from formula (28). In this case, too, there is considerable disagreement, by approximately one order of magnitude, with the calculation of [14].

It also follows from Fig. 5 that the curves behave relative to one another in the manner that follows from the general analysis given above, and it is seen in particular that for the two lower energies the values of the maxima of the curves of (28) are higher than those of the curves for a plate, in view of the fact that in these cases we have respectively $\omega_p' - 2.55$ keV and $\omega_p' = 10.2$ keV, whereas for the maximum energy 7×10^{13} we have $\omega_p' = 1.02 \times 10^5$ keV, and therefore in this case the maxima of (28) are smaller than the maxima for the plate.

However, the presented results of a calculation by formula (28) for muons with energies 7×10^{13} eV, while correct in themselves, are not applicable to the experimental conditions of [14] for a layered medium consisting of 300 plates. Indeed, the region of applicability of (28) is bounded by the conditions (3), (23), and (24). It is easily seen that for this energy the condition (23) is not satisfied. In fact, since $\omega_p' = 1.02 \times 10^5$ keV, and from [14] it follows that $\omega = 20$–80 keV, we obtain $N \gg 0.5 \times 10^4$–10^3. For an average energy 3.5×10^{12} eV we have $\omega_p' = 255$ keV, and condition (23) is satisfied.

As to [15] (see also [13]), the physical conditions for observing transition radiation were such that it suffices to use the simplest formula for a double boundary in the calculation. Indeed, it is easy to verify that in this case we have both satisfaction of the conditions (29) and the fact that the plate thickness is much larger than the zone where the radiation is produced in the matter, the latter being equal to $c\omega^{-1}/(1 - \beta^2 + \sigma\omega^{-2})$. In spite of this, we have calculated, for greater reliability, the curves corresponding to Figs. 4 and 5 of [15], using all three formulas (double boundary, plate, and (28)), and, as expected, these gave results that agreed with good accuracy. In the comparison with Fig. 4 of [15], we obtained agreement with curve 5, whereas the remaining curves lie higher than our data, the maximum excess reaching 60%. The curves of Fig. 5 coincided with those obtained by us.

7. In conclusion, let us stop to discuss the question of allowance for the influence of multiple scattering on the formation of transition radiation in a layered medium. The appropriate theory was developed in [22] (see also [13]). It follows from this theory[23] that under those physical conditions which were realized in [15], the influence of multiple scattering is appreciable, and this leads to an increase in the theoretically calculated number of quanta produced in the layered medium, and as a result the appreciable discrepancy between theory and experiment is eliminated[15] (see Figs. 71 and 72 in [13]). However, as was shown in the preceding section, the physical conditions of the experiment[15] are such that we are dealing here with independent formation of radiation at each of the plate boundaries. Therefore, for the case of the experiments of [15], one can apply the theory of formation of transition radiation on an interface between a vacuum and a medium, where multiple scattering is taken into account. This question has been the subject of a number of investigations. According to the latest of them, namely [24], when $E \ll E_{crit}$, where

$$\frac{E_{crit}}{\mu c^2} = \frac{L\sqrt{\sigma}}{c}\left(\frac{\mu c^2}{E_s}\right)^2,$$

multiple scattering has no effect whatever (here L is the radiation unit of length, μ is the mass of the radiating particle, and $E_s = 21$ MeV). Assuming, in accordance with [15], L = 40 g/cm^2, a plate density 1.17 g/cm^3 and $\sqrt{\sigma} = 19$ eV, we can readily find that, if we are dealing with electrons, then $E_{crit} = 1.1 \times 10^4$ MeV. Since the maximum energy of the electrons in the experiments of [15] did not exceed 600 MeV, there could be no influence of multiple scattering on the formation of transition radiation in these experiments.

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