QUANTUM THEORY OF MODULATION OF AN ELECTRON BEAM AT OPTICAL FREQUENCIES

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A quantum theory is presented for the modulation of an electron beam passing through a plate along which an electromagnetic wave propagates. It is shown that at optical frequencies the decisive role is played by quantum effects. The main result of the theory is that the degree of modulation has an entirely different spatial dependence than in the classical theory. The regions of strong modulation repeat periodically in space over large distances behind the plate, where, according to the classical mechanics, there is no longer any modulation. Radiation produced when a modulated electron beam interacts with the metallic screen is considered, and it is shown that it has a strongly anisotropic character.

1. INTRODUCTION

MODULATION of a beam of electrons at frequencies in in the radio band has been investigated from all points of view, both theoretically and experimentally, and is widely used in radio engineering. As to the optical band, modulation was obtained here only quite recently in the experiments of Schwarz and Hora.[1] A beam of fast electrons passed through a thin film placed in the field of the light wave from a laser. The resultant modulation led to the appearance of glow with the same frequency as the laser radiation when the beam was incident on a non-luminescent screen.

The classical theory used to describe modulation in the radio band is not suitable at optical frequencies, for the decisive role is played in this region by quantum effects. In the present paper we give a quantum theory of modulation of an electron beam and calculate the characteristics of the radiation excited by this beam. Some of the results of this theory and an explanation of the experiment of [1] were published by us earlier.[2]

The intersection of an electron beam with a laser beam in the absence of an additional body cannot in itself lead to beam modulation, since the absorption of one or several photons from the laser radiation by the electron is forbidden by the conservation laws. The presence of a film makes this process possible, since the electron can receive the required momentum from the film.

In the field of a laser wave, the material of the film becomes polarized and surface charges are produced. Interacting with these charges, the electron acquires (or loses) energy and momentum. In a sense, the surface of the film can be likened to a pair of grids to which a potential difference that alternates at the optical frequency ω is applied.

Let us compare the classical and quantum descriptions of the optical modulation of an electron beam passing through a film. In the classical description, the electron passing through the film is accelerated or decelerated, depending on the phase of the field at the instant it crosses the surface of the film. The electrons emitted from the films at different instants of time have different velocities. The fast electrons overtake the previously emitted slow electrons. This leads to spatial bunching, so that the current density is modulated at the frequency ω and its harmonics.

According to quantum mechanics, an electron passing through a film does not have definite energy or momentum. Its wave function represents a superposition of states resulting from the stimulated emission or absorption of n quanta $\hbar\omega$ ($n = 0, \pm 1, \pm 2, \ldots$). The modulation of the density and of the electron current is due to interference of these states. The main result of the quantum-mechanical analysis is a spatial dependence of the degree of modulation essentially different from that in the classical theory (formula (19)). The discrete character of the absorption or emission of the photons causes regions where the modulation is large to repeat periodically in space at large distances behind the film, where, according to classical mechanics, there is no longer any modulation (see Fig. 2). Only at small distances do the results of the quantum and classical analyses coincide.

The characteristic distance l over which quantum effects are significant depends on the frequency like ω^{-2} (formula (16)). In the radio band, the distance l is quite large, and the spatial bunching of the electrons is usually missing because the initial beam is not monochromatic and the quantum effects do not come into play. In the case of modulation at optical frequencies, l is small.

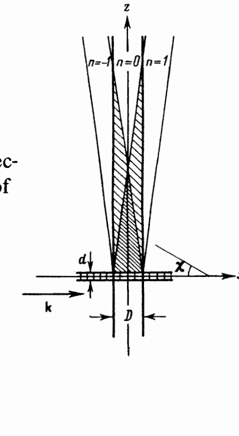

FIG. 1. Splitting of an electron beam passing through a plate along which an electromagnetic wave propagates, as a result of absorption and emission of photons with momentum ħk.

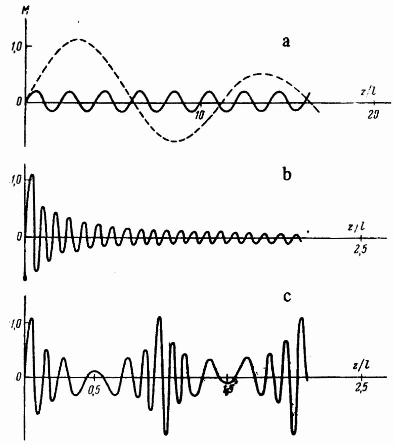

FIG. 2. Dependence of the amplitude of the modulation M of the current at the frequency ω on the distance z: a—N = 0.1, dashed—classical theory, b—N = 10, classical theory, c—N = 10, quantum theory (formula (19)).

Thus, in the experiments of Schwarz and Hora,[1] the value of l, according to our estimates, was ~ 0.9 cm, and the distance from the film to the screen was 25 cm. Thus, the effect observed in these experiments had a quantum nature.

In Sec. 2 of the present paper we find the wave function of an electron passing through a plate along which a modulating wave propagates. In Sec. 3 we determine the density of the electron current and calculate the depth of modulation. In Sec. 4 we consider the radiation produced when a modulated electron beam interacts with a metallic screen.

2. FORMULATION OF PROBLEM AND SOLUTION OF THE SCHRÖDINGER EQUATION

An electron beam can be modulated by different methods. To obtain the modulation it is necessary that the electrons pass through a region with a periodic field with frequency ω changes strongly over a length shorter than or of the order of the distance traversed by the electron during the period of the oscillations. Such a sharp change in the field is possible only near the surface of a dielectric or a conductor. The concrete characteristics of a modulated beam depend on the geometry of the experiment, namely the orientation of the initial beam of electrons relative to the dielectric or conducting body and the structure of the electromagnetic field. The real structure of the field is determined by the form and quality of the surface, and also by the conditions under which the radiation enters and leaves.

For concreteness, we consider the passage of an electron beam directed along the z axis normally to an infinite plane-parallel dielectric plate. Let the modulating electromagnetic wave propagate along the x axis (see Fig. 1) and let it be described by a vector potential (a gauge with zero scalar potential):

$$\mathbf{A}_M(\mathbf{r}, t) = \mathbf{A}(z) \sin(\omega t - kx). \quad (1)$$

The explicit form of the function $\mathbf{A}(z)$ and the value of the wave number $k = \omega/c_M$ depend on the mode excited in the plate, and c_M is the phase velocity of the wave.

On the surface of the plate, the normal component of the vector $\mathbf{A}$ experiences a jump

$$A_{z1}(-d/2) = \varepsilon A_{z2}(-d/2), \quad \varepsilon A_{z2}(d/2) = A_{z3}(d/2), \quad (2)$$

where ϵ is the dielectric constant of the plate, and the indices 1, 2, and 3 pertain respectively to the regions $z \le -d/2$, $-d/2 \le d/2$ and $z \ge d/2$.

We denote by e, m, $\mathbf{p}_0$, and E_0, respectively, the charge, the mass, the initial momentum, and the energy of the electron, and we introduce the amplitude of the electromagnetic field of the wave $\mathscr{E}(z) = -\omega A(z)/c$.

We shall assume the following conditions to be satisfied:

1) The electron is not relativistic, $E_0/mc^2 \ll 1$, $p_0 = mv$.

2) The amplitude of the variation of the electron velocity in the electromagnetic wave is small compared with the initial velocity, $|e\mathscr{E}/mv\omega| \ll 1$.

3) The quantum energy is small compared with the electron energy, $\hbar\omega/E_0 \ll 1$.

4) The electron wavelength $\hbar/p_0$ is small compared with the characteristic distance (on the order of the atomic distance) at which the actual field jump described by formula (2) takes place at the surface of the plate.

The modulation phenomenon is complicated by side effects—diffraction of the electrons and their slowing down in the plate. We shall not consider these effects here, but only the non-diffracted beam, assuming the plate to be sufficiently thin to be able to neglect the electron energy losses.

We find the wave function of an electron passing through a field (1) from the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{ie\hbar}{mc}\sin(\omega t - kx)\mathbf{A}(z)\nabla\psi. \quad (3)$$

The terms proportional to A^2 and div $\mathbf{A}$ have been discarded by virtue of conditions 2 and 4, respectively. When $z \to -\infty$, the wave function ψ, which describes the initial beam, is of the form

$$\psi = \exp\left(-\frac{iE_0t}{\hbar} + \frac{ip_0z}{\hbar}\right), \quad z \to -\infty. \quad (4)$$

We seek a solution of (3) in the form

$$\psi(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} f_n(\mathbf{r}) \exp\left(-\frac{iE_nt}{\hbar} + \frac{i\mathbf{p}_n\mathbf{r}}{\hbar}\right), \quad (5)$$

where

$$E_n = E_0 + n\hbar\omega, \quad \mathbf{p}_n = (n\hbar k, 0, \sqrt{p_0^2 + 2nm\hbar\omega - n^2\hbar^2k^2}); \quad E_n = p_n^2/2m. \quad (6)$$

Actually, by virtue of the condition 2, a decisive role will be played in the sum (5) by terms with $|n| \ll E_0/\hbar\omega$.

Substituting the series (5) in (3), we obtain a system of differential equations for the functions $f_n(\mathbf{r})$

$$\frac{i\hbar}{m}(\mathbf{p}_n\nabla f_n) + \frac{\hbar^2}{2m}\nabla^2 f_n = \frac{ie}{2mc}\mathbf{A}(\mathbf{p}_{n+1} - i\hbar\nabla)f_{n+1}\exp[i(\mathbf{p}_{n+1} - \mathbf{p}_n)_z z/\hbar]$$
$$-\frac{ie}{2mc}\mathbf{A}(\mathbf{p}_{n-1} - i\hbar\nabla)f_{n-1}\exp[i(\mathbf{p}_{n-1} - \mathbf{p}_n)_z z/\hbar] \quad (7)$$

with the boundary condition

$$f_n(\mathbf{r}) = \delta_{n0} \text{ as } z \to -\infty. \quad (8)$$

As seen from (7) and (8), the functions f_n depend only on the coordinate z. According to condition 4, the functions f_n change noticeably over distances much larger than the electron wavelength; therefore the terms containing $\hbar\nabla f_n$ can be neglected in (7) compared with $p_n f_n$. In addition, the quantities $p_{n\pm 1}$ and p_n can be replaced by p_0 everywhere with the exception of the arguments of the exponentials. This results in an error on the order of $|n|\hbar\omega/E_0 \ll 1$. in the determination of the amplitudes f_n. We retain in the arguments of the exponentials the first term of the expansion of the difference $(p_{n\pm 1} - p_n)_z$ in powers of $n\hbar\omega/E_0$. The next terms of the expansion will be discarded, assuming that the region where the field $A(z)$ differs from zero is bounded by the inequality $|z| \ll l/|n| = (4\pi/|n|)(v/\omega) \times (E_0/\hbar\omega)$ (for values of n that play an important role in different cases, see below). Following the indicated simplifications, the system of equations (7) reduces to

$$\frac{df_n}{dz} = \frac{e}{2\hbar c} A_z(z)\left[f_{n+1}\exp\left(i\frac{\omega}{v}z\right) - f_{n-1}\exp\left(-i\frac{\omega}{v}z\right)\right]. \quad (9)$$

As can be verified by direct substitution, the system (9) subject to the boundary condition (8) has the following solution:

$$f_n(z) = J_n(\rho(z))\exp[-in\varphi(z) + in\pi], \quad (10)$$

where J_n is a Bessel function of n-th order, and the real functions $\rho(z)$ and $\varphi(z)$ represent the modulus and the phase of the complex expression

$$\rho(z)\exp[i\varphi(z)] = \frac{e}{\hbar c}\int_{-\infty}^{z} dz_1 A_z(z_1)\exp\left(i\frac{\omega}{v}z_1\right). \quad (11)$$

We are interested in the wave function ψ of an electron passing through a plate in a region where there is no modulating field $A(z)$. In this region, the functions $\rho(z)$ and $\varphi(z)$, and consequently also the amplitudes $f_n(z)$, become constant. We put $\rho(\infty) = N$, $\varphi(\infty) = \Phi$, so that

$$N\exp(i\Phi) = \frac{e}{\hbar c}\int_{-\infty}^{\infty} dz_1 A_z(z_1)\exp\left(i\frac{\omega}{v}z_1\right). \quad (12)$$

Finally, the wave function of the electron takes in the indicated region the form

$$\psi(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} J_n(N)\exp\left[-\frac{iE_n t}{\hbar} + \frac{i\mathbf{p}_n\mathbf{r}}{\hbar} - in\Phi + in\pi\right]. \quad (13)$$

The quantity $J_n^2(N)$ is the probability of absorption or stimulated emission of n photons, in accordance with the fact that $\sum_{n=-\infty}^{\infty} J_n^2(N) = 1$. The physical meaning of the parameter N is determined by the equation

$$\sum_{n=1}^{\infty} n^2 J_n^2(N) = N^2, \quad (14)$$

from which we see that N is the rms number of modulating-radiation photons absorbed by the electron. The rms number of emitted phonons is also N, since $J_n^2(N) = J_{-n}^2(N)$. It is seen from the last equation that the average energy acquired by the electron during the modulation process is equal to zero, so that no field energy is consumed.

Expression (13) is a general solution of our problem for the class of fields that can be represented in the form (1). The values of N and Φ determined by formula (12) depend on the concrete form of the field. We present by way of an example these values for the case of a TM wave propagating along a thin ($kd \ll 1$) dielectric plate. In this case $k \approx \omega/c$, $c_M \approx c$, and

$$A_z(z) = -\frac{c}{\omega}\mathscr{E}_z\begin{cases}\exp[q(z + d/2)], & z < -d/2,\\ \varepsilon^{-1}, & -d/2 < z < d/2,\\ \exp[-q(z - d/2)], & z > d/2,\end{cases}$$

where $q = (\epsilon - 1)(2\epsilon)^{-1}(kd)k \ll k$, $\mathscr{E}_z$ is the normal component of the electric field on the outer side of the plate surface. We then get from (12)

$$N\exp(i\Phi) = \frac{\varepsilon - 1}{\varepsilon}\frac{2e\mathscr{E}_z v}{\hbar\omega^2}\sin\frac{\omega d}{2v}.$$

We see therefore that by virtue of condition 2 the inequality $N \ll E_0/\hbar\omega$ holds. We note that when $\omega d/2v \ll 1$ the quantity N does not depend on the electron velocity and constitutes the ratio of the additional potential difference $e\mathscr{E}_z d(\epsilon - 1)/\epsilon$, resulting from the polarization of the plate to the quantum energy $\hbar\omega$.

3. CALCULATION OF THE ELECTRON CURRENT

In the preceding section we calculated the probabilities of stimulated emission or absorption of n photons. In this section we consider the interference between states resulting from absorption or emission of different numbers of photons. This interference leads to modulation of the density and of the current of the electrons at a frequency ω and its harmonics.

Upon absorbing (emitting) a photon, an electron acquires a momentum component equal to $\hbar k$ (or minus $\hbar k$) along the x axis (see formula (6)). This causes the partial currents corresponding to different n in (13) to have different components along the x axis. These components, however, are quite small compared with the current components along the z axis, since $|n|\hbar k/p_0 \lesssim N\hbar k/p_0 \ll 1$. Therefore the current density per electron and per unit volume can be written in the form $\mathbf{j}(\mathbf{r}, t) \approx e\mathbf{v}|\psi(\mathbf{r}, t)|^2$. Effects connected with nonparallelism of the partial currents will be considered below.

We start with an important particular case, when the parameter $N \ll 1$. Only such a situation has been realized so far experimentally.[1] We note that the classical description of modulation with $N < 1$ is certainly not valid. In this case the modulation at the frequency ω is determined only by single-photon transitions. We retain in formula (13) only terms with n = 0 and ± 1, and confine ourselves to the first term of the expansion of the Bessel function $J_n(N)$ in terms of N. We then obtain

$$j(\mathbf{r}, t) = j_0\left[1 + 2N\sin\frac{\pi z}{l}\sin\omega\left(t - \frac{x}{c_M} - \frac{z}{v} + \frac{\Phi}{\omega}\right)\right], \quad (15)$$

where j_0 is the current density in the initial beam. Expression (15) can also be obtained by solving the Schrödinger equation (3) in first order of perturbation theory.

An important feature of formula (15) is the periodic dependence of the depth of modulation on the distance z, with a period l. Accurate to small terms of order $(v/c)^2$ and $(\hbar\omega/E_0)^2$, we have

$$l = 4\pi\frac{v}{\omega}\frac{E_0}{\hbar\omega}. \quad (16)$$

This essentially quantum periodicity, first noted in [2], has not yet been observed experimentally.[1)] It has the following origin: interference of states with $n = 0$, $n = 1$, and $n = 0$, $n = -1$ leads to the formation of two traveling modulation waves with frequency ω. These waves have identical phase-velocity components along the x axis, equal to c_M. However, the phase-velocity components of these waves along the z axis, equal to $\hbar\omega/(p_1 - p_0)_z$ and $\hbar\omega/(p_0 - p_{-1})_z$, differ by a small quantity $v\hbar\omega/2E_0$ (see formula (6)). The beats between these modulation waves in the region of their overlap are the cause of the periodic dependence of the depth of modulation on the distance z.

Let us now consider the general case of arbitrary N. The principal role is then played in the sum (13) by the terms with $|n| \sim N$. The electron density is determined by the expression

$$|\psi(\mathbf{r}, t)|^2 = \sum_{nm} J_n(N) J_m(N) \exp[-i(n-m)(\omega t - kx + \Phi - \pi) + i(\mathbf{p}_n - \mathbf{p}_m)_z z/\hbar]. \quad (17)$$

At not too large values of z, expression (17) can be written in a more compact form, by expanding the difference $(p_n - p_m)_z$ in powers of the small parameter $\hbar\omega/E_0$ up to the quadratic term inclusive ($v^2/c^2 \ll 1$):

$$(\mathbf{p}_n - \mathbf{p}_m)_z \approx \frac{(n-m)\hbar\omega}{v} - (n^2 - m^2)\frac{\hbar\omega}{v}\frac{\hbar\omega}{4E_0}. \quad (18)$$

The current density can then be represented in the form of the following series of harmonics with frequencies $s\omega$:[2)]

$$j(\mathbf{r}, t) = j_0\left\{1 + 2\sum_{s=1}^{\infty} J_s\left(2N\sin\frac{\pi sz}{l}\right)\cos\left[s\left(\omega t - kx - \frac{\omega z}{v} + \Phi - \frac{\pi}{2}\right)\right]\right. \quad (19)$$

where the parameters N and Φ are determined by formula (12), and the length l by formula (16). In the derivation of (19), after making the substitution $m = n + s$, the summation over n was carried out with the aid of the formula[3]

$$\sum_{r=-\infty} J_r(N) J_{r+\nu}(N) \exp(ir\Phi) = J_\nu\left(2N\sin\frac{\varphi}{2}\right)\exp[i\nu(\pi - \varphi)/2].$$

Expression (19) contains formula (15) as a particular case corresponding to $N \ll 1$.

An estimate of the terms discarded from the argument of the exponential (17) when using the approximate expression (18), with allowance for the fact that $|n|$, $|m| \sim N$, gives the condition for the applicability of formula (19). Namely, the expression for the amplitude of the s-th harmonic in (19) is valid when $sz \ll (l/N^2) \times (E_0/\hbar\omega)$.

The constant-modulation-phase surfaces are parallel to the y axis and propagate at an angle $\chi = \tan^{-1} v/c_M$ to the x axis.

When $sz \ll l$, the sine function in the argument of the Bessel function can be replaced by $\pi sz/l$. Then the expression for the amplitude of the s-th harmonic of the current ceases to depend on the Planck constant $\hbar$, and goes over to its classical limit $J_s(sz/l_0)$. Here

$$l_0 = \frac{l}{2\pi N} = 2\frac{v}{\omega}E_0 \times \left|e\int_{-\infty}^{\infty}\mathscr{E}_z(z_1)\exp\left(i\frac{\omega}{v}z_1\right)dz_1\right|^{-1} \quad (20)$$

is the characteristic length over which the depth of modulation changes significantly in accordance with the classical theory.[4] When $z \gg l_0$, the depth of modulation becomes small. The remarkable difference between the quantum formula (19) and the classical one lies in the fact that the regions of large modulation at the frequency $s\omega$ repeat periodically in space with the period l/s. The most surprising thing is that when $N \gg 1$ a high degree of modulation arises periodically at distances ($z \sim l, 2l, 3l \ldots$ for $z = 1$) at which there is practically no modulation in accordance with classical mechanics ($l \gg l_0$, see Fig. 2).

The origin of the periodicity has already been discussed earlier for the case $N \ll 1$. When $N \gg 1$ there are many modulation waves at the frequency $s\omega$, due to the interference between the states with indices n and $n \pm s$ in the sum (13). For these waves, the components of the phase velocities $s\hbar\omega/(p_{n+s} - p_n)_z$ along the z axis differ in the approximation (18) for different n by an amount that is a multiple of $(v/2)(\hbar\omega/E_0)$. The beats between all these modulation waves give rise to the spatial periodicity with period l/s. When $sz > (l/N^2) \times (E_0/\hbar\omega)$, the terms discarded in the expansion (18) come into play, and the periodicity is violated.

So far we have described the initial beam of electrons by means of a plane monochromatic wave. Let us now consider effects connected with nonmonochromaticity of the beam and its boundedness in the xy plane. The scatter of the values of the momenta of the electrons in the initial beam (Δp_z and $\Delta p_\perp$) leads to a smearing of the modulation at large distances. It is seen from (19) that in order to observe modulation with frequency $s\omega$ at a distance z it is necessary to have

$$\Delta p_z / p_0 < v / s\omega z. \quad (21)$$

It can be shown that the condition imposed on the scatter of the momentum components in the xy plane is somewhat weaker:

$$\Delta p_\perp / p_0 < c / s\omega z. \quad (22)$$

Obviously, the scatter $\Delta p_\perp$ and the beam diameter are connected by the uncertainty relation $\Delta p_\perp D \gtrsim \hbar$.

On passing through the plate, the wave packet characterizing the real bounded beam splits into a series of wave packets propagating in slightly different directions at angles $n\hbar k/p_0$ to the z axis (see formula (6)). The modulation due to the interference of the wave packets with different n takes place only in the region of their overlap. The wave packets n and n + s overlap at $z < Dp_0/s\hbar k$ and, according to condition (22), the modulation in this entire region will not be smeared out if $\Delta p_\perp \sim \hbar/D$. The formula for the current density (19) is valid at distances z bounded by the conditions (21) and (22), in a region where all the wave packets with $|n| \lesssim N$ overlap. When $\Delta p_\perp \gtrsim N\hbar k$, such an overlap always takes place, and when $\Delta p_\perp \ll N\hbar k$ it takes place only at distances $z < Dp_0/2N\hbar k$. (In the case $N < 1$, N should be replaced by unity in the foregoing estimates.)

[1)] Note added in proof (24 November 1970). This quantum periodicity was recently observed experimentally (private communication from H. Schwarz).

[2)] The constant phase $-(s+1)\pi/2$ was omitted from the analogous formula of [2]. We are grateful to L. S. Cutler and B. M. Oliver for pointing out this circumstance.

In the general case, there exist regions where only certain wave packets overlap. In these regions, modulation exists if the conditions (21) and (22) are satisfied, but it is not described by formula (19). By way of an example, Fig. 1, which pertains to the case $N \ll 1$, $\Delta p_\perp < \hbar k$, shows the picture of the overlap of the wave packets $n = 0$ and ± 1. In the doubly-hatched region all three packets overlap, so that beats are produced between two traveling modulation waves at the frequency ω, and formula (15) is valid. In the singly-hatched regions there is only pairwise overlap of the packets $n = 0$, $n = 1$ with the packets $n = 0$, $n = -1$; here the degree of modulation does not depend on the distance. In the case $N \gtrsim 1$ the picture becomes more complicated, but it can be readily established in similar fashion.

In concluding this section, let us emphasize the very general character of the representation of the wave function of an electron interacting with an electromagnetic wave in the presence of an additional body, in the form of a superposition of states resulting from absorption or emission of n photons. Of course, the amplitudes of these states depend on the concrete form of the field and on the geometry of the experiment. But the character of the modulation of the electron current, and particularly the spatial dependence of the depth of modulation, is determined by purely kinematic relations. This circumstance greatly facilitates the analysis of different cases that might be of interest in connection with experiments on modulation. Thus, the results can readily be generalized to the case of relativistic beams. The use of the relativistic connection between the energy and momentum gives the following expression for the characteristic length l that enters in (15) and (19):

$$l = 2\pi \frac{v}{\omega} \frac{p_0 v}{\hbar\omega} \left[1 - \frac{v^2}{c^2}\left(1 - \frac{c^2}{c_M^2}\right)\right]^{-1}. \tag{23}$$

The velocity v in (15), (19), and (23) is now determined by the relation $v = p_0 c (p_0^2 + m^2c^2)^{-1/2}$. In the nonrelativistic limit, expression (23) goes over into (16).

4. RADIATION OF A MODULATED ELECTRON BEAM

When a modulated electron beam interacts with some body, a conductor or a dielectric, radiation is produced at the modulation frequency ω and its harmonics. This monochromatic radiation appears against the background of a broad continuous spectrum of the radiation existing in the absence of modulation (transition radiation, bremsstrahlung, and luminescence). Which of the radiation mechanisms is the most effective depends on the dielectric constant. The intensity of coherent radiation is proportional to the square of the current, whereas incoherent radiation is linear in the current. We are interested here only in coherent transition radiation due to modulation. In this section we calculate the intensity and the angular distribution of this radiation, and regard expression (19) as a classical current.

For concreteness, we consider a situation when the screen is an ideal conductor. The modulated electron beam directed along the z axis strikes a flat metallic screen placed perpendicular to the xz plane. (We recall that the x axis coincides with the direction of propagation of the modulating wave.) We denote by the letter α the angle between the normal to the surface of the screen and the z axis (see Fig. 3).

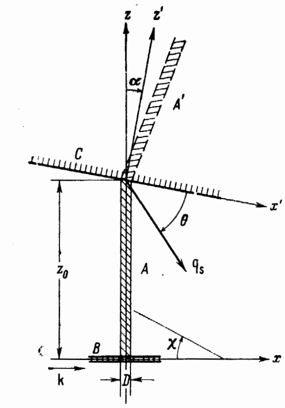

FIG. 3. Radiation with wave vector q_s is produced when the electron beam A, modulated in the plate B, strikes the metallic screen C. A′ is the mirror image of the beam.

We assume that the electrons are completely absorbed on the surface of the screen. Then, assuming the metal to be ideally conducting, we can calculate the radiation field by taking as the source the real modulated electron current and the current of the mirror image in the metal.

A similar approach was used in a paper by Rubin.[5]

It is convenient to introduce a coordinate system x′ y′ z′ connected with the screen, so that the origin coincides with the center of the "electron spot" on the screen, the z′ axis is in back of the screen normal to its surface, and the y′ and y axes coincide. The vector potential $\mathbf{A}_s(\mathbf{r}')$, which determines the radiation field of the s-th harmonic, is

$$A_s(\mathbf{r}') = \frac{\exp(iq_s r')}{cr'} \int \mathbf{j}_s(\mathbf{r}_1) \exp(-i\mathbf{q}_s\mathbf{r}_1)\, d^3\mathbf{r}_1, \tag{24}$$

where the wave vector $\mathbf{q}_s$ ($q_s = s\omega/c$) is directed from the origin of the system x′y′z′ to the observation point $\mathbf{r}'$. The current density entering in formula (24) is of the form

$$\begin{aligned} \mathbf{j}_s(\mathbf{r}') &= (-\sin\alpha,\ 0,\ \cos\alpha)\, j_s(x', y', z') \text{ for } z' < 0, \\ \mathbf{j}_s(\mathbf{r}') &= (\sin\alpha,\ 0,\ \cos\alpha)\, j_s(x', y', -z') \text{ for } z' > 0, \end{aligned} \tag{25}$$

where, in accordance with formula (19),

$$j_s(x'y'z') = j_0 J_s\left(2N \sin\frac{\pi s z_0}{l}\right) \exp[is\varkappa\mathbf{r}' - is\Phi - is\pi/2 + is\omega z_0/v];$$

$$\varkappa_{x'} = \frac{\omega}{v}(-\sin\alpha + \operatorname{tg}\chi\cos\alpha), \quad \varkappa_{y'} = 0, \quad \varkappa_{z'} = \frac{\omega}{v}(\cos\alpha + \operatorname{tg}\chi\sin\alpha);$$

$$\operatorname{tg}\chi = v / c_M. \tag{26}$$

The wave vector $\varkappa$ is directed normal to the surface of constant modulation phase of the electron beam. In the coordinate system xyz, the components of this vector are equal to ω/c_M, 0, and ω/v, respectively. The angle χ determines the inclination of these surfaces to the xy plane. z_0 is the distance between the origins of the systems xyz and x′y′z′. In formula (26), the variable z in the argument of the Bessel function has been replaced by z_0, since the main contribution to the integral (24) is made by the small region of the "electron spot" near $z' = 0$, with thickness on the order of $c/s\omega$. In a real situation, the linear dimensions of this region are small compared with the length l, so that the Bessel function remains practically unchanged here.

Two qualitatively different cases can occur, depending on whether the wavelength of the modulating radiation $2\pi/k$ is larger than or smaller than the diameter of the electron beam D. If $skD \gg 1$, then the phase of the modulation varies periodically within the limits of the spot in the direction of x', and travels over the screen. This leads to a sharp directivity of the excited radiation, with a divergence on the order of $(skD)^{-1}$. When $skD \ll 1$, the phase of the modulation of the electron current remains constant over the cross section of the beam. It is therefore obvious that the light will be emitted in a wide angle interval. The first case, $skD \gg 1$, is most interesting and will therefore be considered below.

Substituting the expression for the current density (25), (26), into (24), we obtain the vector potential $A_s(r')$ and we calculate the radiation intensity of the s-th harmonic per unit solid angle Ω

$$\frac{dW_s}{d\Omega}=\frac{8\pi}{c}\frac{I_s^2}{S\cos\alpha}\left(\frac{\sin(\chi-\alpha)}{\cos\chi}+\frac{v^2}{c^2}\sin\alpha\right)^2\delta(q_{sx'}-s\varkappa_{x'})\delta(q_{sy'}), \quad (27)$$

where

$$I_s(z_0)=j_0SJ_s\left(2N\sin\frac{\pi sz_0}{l}\right), \quad (28)$$

S is the cross section area of the electron beam.

The radiation direction is determined by the two δ functions in (27). The radiation propagates in the x'z' plane at a certain angle to the x' axis, which we shall denote by θ, and is linearly polarized in the same plane ($A_{y'}=0$). The value of the angle θ is determined by the equations $\cos\theta = q_{sx'}/q_s$ and $q_{sx'} = s\kappa_{x'}$, from which we get with the aid of (26)

$$\cos\theta=\frac{c}{c_M}\frac{\sin(\chi-\alpha)}{\sin\chi}. \quad (29)$$

Thus, the direction of the radiation depends on the ratio of the phase velocity of the propagation of the modulating electromagnetic wave c_M to the velocity of light in vacuum c, and on the angles α and χ characterizing respectively the orientation of the screen and the propagation direction of the phase of the modulation wave. Radiation is not produced at any screen orientation, but only if α lies in a certain limited interval of angles. In the case when the difference between the values of c_M and c can be neglected, this interval is given by the inequality $0 \le \alpha \le 2\chi$. When the screen is rotated, corresponding to a variation of α from 0 to 2χ, the angle θ runs through values from zero to π. When $\alpha = \chi$, the plane of the screen coincides with the constant-phase plane of the modulation. If α is smaller (larger) than χ, then the modulation wave travels over the screen in the positive (negative) direction of the x' axis. Accordingly, when $\alpha < \chi$ we have $\theta < \pi/2$, and when $\alpha > \chi$ we have $\theta > \pi/2$.

In the nonrelativistic limit, $\chi = v/c_M \ll 1$, so that the permissible interval of angles α is very small. The radiation intensity varies in this case, in accordance with (27) and (29), like $\cos^2\theta$. We note, however, that the angular dependences given by formulas (27) and (29) also hold for relativistic beams, since these dependences are the consequence of relativistically invariant kinematic relations.

Strictly speaking, the δ functions in (27) are obtained in the limit $skD \to \infty$. At large but finite values of this parameter, the δ functions are replaced by sharp functions with width $\sim D^{-1}$. Thus, the radiation occurs in a narrow angle interval $\Delta\theta \sim (skD)^{-1}$ about the value of θ determined by formula (29). We note that the radiation direction is the same for all harmonics, but the angle divergence decreases with increasing number of the harmonic s.

The expression (28) for I_s is valid wherever formula (19) is valid, but formula (27) has a more general character. It describes radiation emerging from one region of intersection of the modulated beam with the screen. On the other hand, if the splitting of the electron beam which was discussed in Sec. 3 gives rise to several such regions, then each of them will serve as a source of radiation with an angular distribution given by the same formula (27). All that changes is the value of I_s, which is proportional to the depth of modulation in the given region.

In the experiment of Schwarz and Hora, the parameter $(kD)^{-1}$ was of the order of 0.1. They did not, however, observe the sharp directivity of radiation observed here. This can be attributed to the fact that the surface of the employed screen was not flat accurate to within the wavelength of light. Formulas (27) and (28) make it possible to estimate the intensity of the coherent transition radiation in this experiment, $W_1 \sim 10^{-14}$–10^{-13} W, corresponding to emission of 10^4–10^5 quanta per second. This quantity can readily be observed. However, if it is compared with the intensity of incoherent radiation, an essential difficulty arises in the interpretation of the Schwarz and Hora experiment.[1] 3) The ratio of the intensity W_1 to the incoherent transition radiation in the frequency interval $\Delta\omega$[6] is of the order of $(I_0/e\Delta\omega)(I_1/I_0)^2$. In the case of visual observation, $\Delta\omega \sim \omega$, and this ratio at a current $I_0 = 1\ \mu A$[1] is much smaller than unity, thus contradicting the data of Schwarz and Hora, according to which no glow of the screen was observed in the absence of the laser radiation. The line width of the excited beam of coherent radition $\Delta\omega_k$ is determined by the nonmonochromaticity of the modulating laser light. In the small frequency interval $\Delta\omega_k$, the intensity of the coherent radiation in [1] exceeded by many orders of magnitude the intensity of the incoherent radiation.

In conclusion we note that a large circle of problems connected with modulation of electron beams at optical frequencies has not yet been studied experimentally. It would be quite interesting to observe the features of this phenomenon which were considered theoretically in the present article, namely the periodic dependence of the depth of modulation on the distance, the splitting of the electron beam, the directivity of the resultant radiation, and others.

Further experimental research on this phenomenon is essential.

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3) B. Ya. Zel'dovich was the first to call attention to this circumstance.

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