BEHAVIOR OF LIGHT BEAMS IN NONLINEAR MEDIA

V. E. ZAKHAROV, V. V. SOBOLEV, and V. C. SYNAKH

Computation Center, Siberian Branch, USSR Academy of Sciences

Submitted July 28, 1970

Zh. Eksp. Teor. Fiz. 60, 136–145 (January, 1971)

An analysis and numerical modeling of the propagation of homocentric light beams in a cubic medium and in a medium with nonlinearity saturation is carried out in the parabolic equation approximation, involving two integrals of motion I_1 and I_2. It is shown that the asymptotic behavior of the beam is determined by the sign of I_2, which in turn depends on the degree of initial focusing of the beam. For $I_2 > 0$, one or more foci are formed, after which the beam spreads out. For $I_2 < 0$, part of the beam becomes self-trapped. In the case of nonlinearity saturation, the shape and intensity of the channel near the axis oscillate. Collapse is observed in the case of a cubic medium. The amplitudes and powers of the first fifteen stationary solutions of the parabolic equation are found. The stability of the first and second stationary states in a cubic medium is investigated. It is shown that small additions to the initial distribution lead to instability.

INTRODUCTION

In 1962, Askar'yan[1] made the proposal that a regime of waveguide propagation of light could exist in nonlinear media—self focusing of the light. Subsequently, Talanov[2] and also Garmire, Townes, and Chiao[3] calculated the structure of a self-focused beam along its axis. However, there remained untouched the problem as to the manner in which the trapping of an arbitrary beam in a self-focusing regime takes place, and whether it occurs generally. Dyshko, Lugovoĭ and Prokhorov[4], on the basis of numerous experiments, drew the conclusion that in the self-focusing of the light, only one or a few foci are formed and self-trapping of light beams (filaments) does not generally take place.

The present research is devoted to the investigation of the problem of the behavior of stationary light beams in a self-focusing nonlinear medium. In Sec. 1 of the paper, the propagation of the beam in the medium is investigated theoretically, in Sec. 2, the results of numerical modeling are given. We shall show that, depending on certain characteristics of the beam entering the nonlinear medium, formation of one or a finite number of foci, or the capture of the beam in an entrapment regime is possible. In this latter case, the beam becomes homogeneous along the axis at large distances; here a structure is formed that is close to that calculated in[2,3].

For realization of waveguide propagation, small departures of the nonlinearity of the medium from cubic are necessary (in the direction of weaker nonlinearity); for a purely cubic nonlinearity, "collapse" of the beam takes place.

In media that are close to cubic, a spontaneous contraction of the beam takes place down to very small dimensions, which can be interpreted as the effect of formation of a "superfine" filament.

1. THEORY OF STATIONARY SELF-FOCUSING

The propagation of stationary light beams in a nonlinear medium is studied within the framework of the parabolic equation

$$2i\partial u/\partial z = \Delta_\perp u + f(|u|^2)u. \tag{1}$$

Here $f(\xi)$ is a function characterizing the nonlinearity of the medium. We shall assume that for small light intensities, the nonlinearity is cubic, so that $f(\xi) \to \xi$ as $\xi \to 0$. Furthermore, we shall assume that $f(\xi)$ is a function that is convex upwards, so that $f''(\xi) < 0$.

It is easy to prove by direct calculation that Eq. (1) has the integrals of motion

$$I_1 = \int |u|^2 dr, \tag{2}$$

$$I_2 = \int \{|\nabla u|^2 - F(|u|^2)\} dr, \quad F(\xi) = \int_0^\xi f(\xi)\,d\xi, \tag{3}$$

i.e., $\partial I_{1,2}/\partial z = 0$.

Equation (1) admits of stationary solutions of the form $u = \varphi(r)e^{-i\lambda z/2}$, where $\varphi(r)$ obeys the equation

$$\Delta_\perp\varphi + f(|\varphi|^2)\varphi - \lambda\varphi = 0. \tag{4}$$

Among these solutions there are those that are real and axially symmetric. For them, Eq. (4) reduces to the form

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\varphi}{\partial r} - \lambda\varphi + f(\varphi^2)\varphi = 0. \tag{5}$$

Of greatest interest is the monotonically decreasing (as $r \to \infty$) solution of Eq. (5), which we shall denote by $\varphi_0(r, \lambda)$. The quantity λ has the meaning of the reciprocal of the square of the characteristic dimension of the beam. As $\lambda \to 0$, the maximum of the function φ_0 also tends to zero. Here $\varphi_0(r, \lambda) \to \psi_0(r, \lambda) = \sqrt{\lambda}q(\sqrt{\lambda}r)$. In this case, $\psi_0(r, \lambda)$ is a monotonically decreasing solution of the equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi_0}{\partial r} - \lambda\psi_0 + \psi_0^3 = 0.$$

The function $g(y)$ was computed by Townes and his co-workers.[3]

The value of the integral I_1 for $\psi_0(r, \lambda)$, which, as is easily seen, does not depend on λ, will be designated by I_0; numerically, $I_0 = 1.86$.

We shall assume that the distribution of the amplitude $u_0(r)$ is given for $z = 0$ and shall investigate the behavior of this amplitude as $z \to \infty$. Here we shall also assume that for the initial distribution $I_1 \sim I_0$, since the light beam is unstable for $I_1 \gg I_0$[3] and breaks up into separate beams, for which $I_1 \sim I_0$. The basic question of interest to us is under what conditions the light beam goes over into a self-trapping mode.

It has been assumed previously[5] that sufficiently intense beams, for which $I_1 > I_0$, enter into the self-trapping mode. (In this connection, we call the quantity I_0 the critical power of the beam.) However, it is clear that this condition can only be a necessary one. If, for $I_1 > I_0$, the beam is strongly focused or defocused, then it cannot enter into a self-trapping regime. We shall show that the sufficient condition of entrapment of the beam in a self-trapping regime is a negative value of I_2.

Let $I_2 < 0$. We note that by virtue of the convexity of the function $f(\xi)$, we have the inequality

$$\frac{1}{2}\int |u|^4 d\mathbf{r} > \int F(|u|^2)\,d\mathbf{r}.$$

Moreover,

$$\int F(|u|^2)\,d\mathbf{r} = \int |\nabla u|^2 d\mathbf{r} - I_2 > |I_2|,$$

whence

$$\max |u|^2 \int |u|^2 d\mathbf{r} > \int |u|^4 d\mathbf{r} > 2|I_2|.$$

Finally,

$$\max |u|^2 > 2|I_2|/I_1. \tag{6}$$

The maximum value of the intensity of the ray over the radius is thus bounded by some z-independent constant, which also indicates self-trapping of the beam.

We separate the amplitude and phase in u: $u = Ae^{i\Phi}$; then

$$I_2 = \int \{(\nabla A)^2 + A(\nabla\Phi)^2 - F(A^2)\}\,d\mathbf{r}. \tag{7}$$

For the same amplitude distributions, the smallest value of I_2 exists for that beam for which $\nabla\Phi = 0$, i.e., the plane parallel, unfocused beam. For a homocentric, axially symmetric beam $\Phi = \pm r^2/2R^2$, where R is the distance to the focus of the beam. From (7) we have the minimal admissible value of R for which the integral I_2 is still negative:

$$R^2 > \int_0^\infty r^2 A^2 r\,dr \Big/ \int \{F(A^2) - (\nabla A)^2\}\,d\mathbf{r}. \tag{8}$$

Sufficiently strong focused or defocused (diverging) beams have a positive integral I_2 and cannot go over into a self-trapping regime.

The requirement of a negative value of the integral I_2 is, as we have shown, simultaneously the condition put on the intensity I_1, which in this case is seen to be automatically greater than I_0. For this purpose, we set the variation equal to zero:

$$\delta(I_2 - \lambda I_1) = 0. \tag{9}$$

As is easy to see, Eq. (9) is identical with Eq. (5), so that all the stationary points of the functional $I_2 - \lambda I_1$ are contained among the solutions of Eq. (5). Absolute minimum of the functional guarantees a monotonic solution $\varphi_0(r, \lambda)$. By computing the integrals $I_1(\lambda)$ and $I_2(\lambda)$ for the solution $\varphi_0(r, \lambda)$, and eliminating λ, we find the boundary of the region on the (I_1, I_2) plane in which the admissible values of these integrals lie.

Multiplying Eq. (5) by $r\varphi$ and integrating from zero to infinity, we get

$$\lambda I_1 = \int_0^\infty r\varphi^2 f(\varphi^2)\,dr - \int_0^\infty r\varphi_r^2\,dr. \tag{10}$$

Differentiating (5) with respect to r, multiplying by $r^2\varphi$ and integrating, we obtain

$$\lambda I_1 = 2\int_0^\infty r\varphi^2 f(\varphi^2)\,dr - \int_0^\infty rF(\varphi^2)\,dr - 2\int_0^\infty r\varphi_r^2\,dr. \tag{11}$$

Subtracting (11) from (10), we obtain the relation

$$I_2 = \int_0^\infty r\{\varphi^2 f(\varphi^2) - 2F(\varphi^2)\}\,dr = \int_0^\infty r\left\{\varphi^2\frac{\partial F(\varphi^2)}{\partial\varphi^2} - 2F(\varphi^2)\right\}dr. \tag{12}$$

For the convex function $f(\xi)$, the value of the integral I_2, calculated from Eq. (12), is negative. In the case of a cubic medium, when $f(\xi) = \xi$, $I_2 = 0$. Thus, at all stationary points of the integral I_2 in a cubic medium, its value is equal to zero.

In the cubic medium, the integral I_1, computed for the function $\varphi_0(r, \lambda)$ does not depend on λ and is equal to I_0. The problem of the minimization of the integral I_2 for a given I_1 is degenerate and has meaning only for $I_1 = I_0$. Then the minimum value of I_2 is zero, while the minimum is achieved on any of the functions $\varphi_0(r, \lambda) = \psi_0 = \sqrt{\lambda}\,g(\sqrt{\lambda}r)$. For $I_1 < I_0$, the integral I_2 can take on any positive values, for $I_1 > I_0$, any values either positive or negative.

Let us consider a medium that is close to cubic:

$$f(\xi) = \xi - 3\alpha\xi^2, \quad F(\xi) = \xi^2/2 - \alpha\xi^3, \quad \alpha\xi \ll 1.$$

We then have, from Eq. (8),

$$I_2 = -\alpha\int_0^\infty \varphi^6 r\,dr = -\alpha\lambda^2 C_1, \qquad C_1 = \int_0^\infty g^6(y)\,y\,dy. \tag{13}$$

In Eq. (13), we have made the approximate substitution $\varphi_0(r, \lambda) = \psi_0 = \sqrt{\lambda}\,g(\sqrt{\lambda}r)$. More accurately, we should write $\varphi_0(n, \lambda) = \sqrt{\lambda}\,g(\sqrt{\lambda}r) + \delta\varphi$, where $\delta\varphi$ satisfies the equation

$$\frac{1}{2}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\delta\varphi - \lambda\delta\varphi + 3\lambda g(\sqrt{\lambda}r)\,\delta\varphi = 3\alpha\psi_0(r, \lambda). \tag{14}$$

For each λ the integral I_1 becomes $I_1 = I_0 + \delta I_1$. For

$$\delta I_1 = 2\int_0^\infty r\delta\varphi\psi_0(r, \lambda)\,dr$$

we have, from Eqs. (6), (7),

$$\lambda\sigma I_1 = -2\int r\psi_{0r}\delta\varphi_r\,dr + 4\int r\psi_0^3\delta\varphi\,dr - 3\alpha\int\psi_0^6 r\,dr,$$

$$\lambda\delta I_1 = -4\int r\psi_{0r}\delta\varphi_r\,dr + 6\int r\psi_0^3\delta\varphi\,dr - 6\alpha\int\psi_0^6 r\,dr. \tag{15}$$

Multiplying Eq. (10) by $\psi_0(r, \lambda)$ and integrating, we get

$$\lambda\delta I_1 = -2\int r\psi_{0r}\delta\varphi_r\,dr + 6\int r\psi_0^3\delta\varphi\,dr - 6\alpha\int\psi_0^6 r\,dr. \tag{16}$$

From (15) and (16), we have

$$\delta I_1 = 2\alpha\lambda C_1, \tag{17}$$

whence we find the connection between I_1 and the minimum value of I_2:

$$I_1 = I_0 + 2(-\alpha C_1 I_2)^{1/2}. \tag{18}$$

Equation (18) solves the problem of the possible values of the integrals I_1 and I_2. For $I_2 < 0$, the integral I_1 can take on values that are bounded below by the quantity (18); for $I_2 > 0$, it is bounded below by any positive value. The region 1 on Fig. 1 is the region of admissible values of I_1 and I_2.

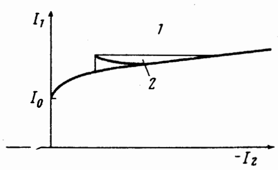

FIG. 1

We now investigate the problem of the stability of the stationary solution $\varphi_0(r, \lambda)$. As is seen from Fig. 1, the solution $\varphi_0(r, \lambda)$ is the absolute minimum of the integral I_2 for given I_1, and also the absolute minimum of the integral I_1 for given (negative) value of the integral I_2. (The latter is invalid, however, in the case of a cubic medium, when one and the same value of I_1, equal to I_0, is permissible for all $\varphi_0(r, \lambda)$.) Then, in accord with the known Lyapunov criterion (see, for example,[6]), it follows that the solution $\varphi_0(r, \lambda)$ is stable in the medium in which $f''(\xi) < 0$. So far as a medium with a cubic nonlinearity is concerned then, the problem of the stability of the stationary solution $\varphi_0(r, \lambda)$ for it cannot be solved on the basis of the analysis of the integrals I_1, I_2.

It is natural to suppose that as $z \to \infty$, the light beam approaches one of the stationary states $\varphi_0(r, \lambda)$. However, we note that Eq. (1) is conservative, so that the problem arises of the mechanism and character of the approach to the stationary state.

Such a mechanism can be the radiation of energy at infinity. Part of the light leaves the paraxial region and enters the region of large values of the radius r, thus carrying off a certain amount of the integral I_1. The amplitude of the radiating light is small, and its nonlinearity can be neglected. Therefore, the radiating light carries with it a positive value of the integral I_2. Hence, the values of the integrals I_1 and I_2 in the paraxial region will decrease (here the integral I_2 will increase in absolute value) until it reaches the boundary of the region of admissible values of the integrals.

Figure 1 shows the region in which the integrals in the paraxial part of the beam can vary; in the same drawing, the number 2 indicates a hypothetical trajectory of the beam in the (I_1, I_2) plane.

We note that, as is seen in Fig. 1, the integral I_2, for small departure of the nonlinearity of the medium from cubic, can take on values whose modulus is greater than its initial value. The maximum value of the integral I_2 is

$$|I_2|_{max} = (I_1 - I_0)^2 / \alpha C_1.$$

As follows from Eq. (13), the integral I_2 is uniquely connected with the quantity λ and the reciprocal of the square of the dimension of the beam. For minimum dimension of the beam, we have

$$r_{min} = \frac{1}{\sqrt{\lambda}} = \left(\frac{\alpha C_1}{I_1 - I_0}\right)^{1/2}. \tag{19}$$

It is very probable that, after radiation of the excess energy from the paraxial part, the beam exists down to dimensions of the order of r_{min}. In media that are close to cubic, r_{min} can be a very small quantity and the considerations set down above can explain the effect observed experimentally of the formation of superfine filaments in the self-focusing.

In a cubic medium, $r_{min} = 0$ and there are no reasons to prevent the beam from being compressed to any arbitrarily small dimension. Actually, nothing prevents the complete "collapse" of the beam in a cubic medium.

2. NUMERICAL MODELING OF STATIONARY SELF-FOCUSING

The process of propagation of homocentric light beams has been modeled by means of the nonlinear parabolic equation (1) with the boundary conditions

$$\frac{\partial u}{\partial r}(0, z) = 0, \quad u(\infty, z) = 0. \tag{20}$$

As an initial distribution for (1), we used the Gaussian distribution

$$u(r, 0) = a_0 \exp(i\gamma - r^2 / l^2), \quad \gamma = \beta r^2. \tag{21}$$

A cubic medium was considered:

$$f(|u|^2) = \sigma |u|^2, \quad \sigma > 0 \tag{22}$$

and also a medium with saturation of the nonlinearity:

$$f(|u|^2) = \sigma \varkappa^{-1}[1 - \exp(-\varkappa |u|^2)], \quad \sigma, \varkappa > 0. \tag{23}$$

As $\kappa \to 0$, the medium (23) becomes cubic. The numerical experiment was carried out for $\kappa = 0.1$ and $\kappa = 0.01$.

Equation (1) was approximated by a finite-difference scheme of second order in both coordinates. For the radial variable r, the grid was taken to be nonuniform, with a specified law of change of the integration step. Use of a variable step made it possible, on the one hand, to take into account the singularities of behavior of the field close to the axis and, on the other, to take the integral of the numerical integration over sufficiently large r (of the order of 15--20 half-widths of the original Gaussian distribution). The latter circumstance, together with the correct translation of the boundary condition from infinity[7] made it possible to model a medium that was infinite in r.

Control of the accuracy of the calculations was provided by preservation of the invariants of I_1 and I_2 with account of their flux through the boundary of the integration region. The importance of the conservation of the integral of motion I_2 must be especially emphasized, in view of the fact that, first, the asymptotic behavior of the beam is determined by it, and, second, it is very sensitive to errors of calculation, inasmuch as its integrand contains derivatives of r. In the calculations performed, the invariant I_1 is conserved with relative accuracy $\sim 2 \times 10^{-4}$ for the medium (23), with saturation of the nonlinearity, up to the maximum $z = 30$, and for the cubic medium (22) up to the moment of collapse. Similarly, the invariant I_2 is preserved with accuracy 2×10^{-3}. We note that in collapse (the amplitude on the axis increases 500–600 times in comparison with its original value) the first invariant is preserved to within 2%, and the second is violated. This indicates the uselessness of further calculation.

For the parabolic equation (1), in the case of a cubic nonlinearity, the amplitudes and powers of the first fifteen stationary states of $\varphi(r)$ were determined. These are given in the table. The invariants I_1 of the first five states differ from those found earlier[8] by less than 0.4%. The power of the first stationary state φ_0 is I_0. It is seen from the table that the invariant I_0 is the smallest. For all $\varphi(r)$, the second integral of motion I_2 was found to be equal to zero. The initial beams (21) were taken with power I_1 more than twice the value of I_0 ($a_0 = 2$, $l = 2$, $I_1 = 4$).

Stationary state	φ (0)	I_1	Stationary state	φ (0)	I_1
1	2,21	1,86	9	9,13	239,5
2	3,33	12,28	10	10,39	279,1
3	4,15	31,16	11	11,70	322,5
4	4,83	58,46	12	13,08	370,1
5	5,42	94,16	13	14,53	421,7
6	6,01	135,8	14	16,04	477,3
7	6,90	170,2	15	17,60	537,2
8	7,37	203,4			

The originally focused light beam ($\beta = 0.6$), which has a positive second invariant ($I_2 = 10$) in the medium with saturation of the nonlinearity ($\kappa = 0.1$), is focused on the axis, as is seen from Fig. 2, and then diffuses. This is in complete agreement with what was developed in Sec. 1.

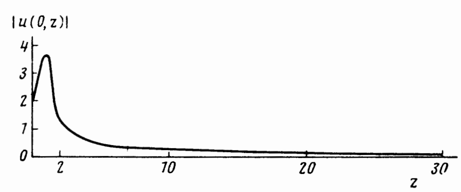

FIG. 2

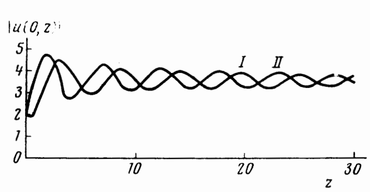

FIG. 3

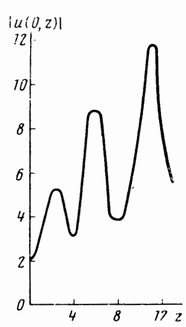

FIG. 4

If the second invariant I_2 is negative, then the light beam transforms into the self-trapped regime. (Fig. 3). This applies to the initially defocused ($\beta = -0.3$, curve I) and to the focused ($\beta = 0.3$, curve II) homocentric beams. It is seen from Fig. 3 that as $z \to \infty$, in the medium (23) with saturation of nonlinearity, the initial distributions with $I_2 < 0$ approach the stationary state, which is stable.

As κ decreases, the amplitudes of the oscillations on the axis increase (Fig. 4, $\kappa = 0.01$, $\beta = -0.1$), since the medium (23) approaches the cubic situation, in which, as is well known, collapse of the beam takes place with $I_1 > I_0$.

The characteristic profiles of the amplitude $u(r, z)$ in the case $\kappa = 0.01$, $\beta = -0.1$ are shown in Fig. 5 for different z. For maximum values of the amplitude on the axis (for z = 6 and 11), the beam contracts into the paraxial region which, in this case, makes the principal contribution to the power I_1 of the beam and forms a radiating filament, which is observed experimentally.[9]

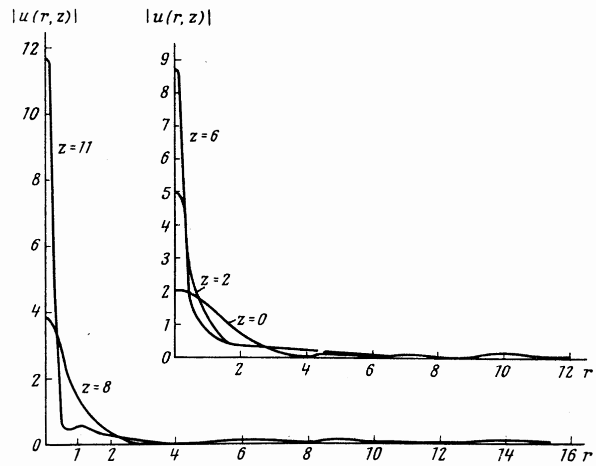

FIG. 5

In the course of the numerical experiment, the stability of the first and second stationary states of the parabolic equation (1) were studied in the case of a cubic medium. Here a perturbation of the form $u_1 = a_1 \exp\{-r^2/l_1^2\}$ is superposed on the stationary distribution, making a contribution δI_1 to the power of the beam of up to 10%. It was shown that even in the case of small positive contributions ($\delta I_1 \approx 0.01 I_1$) the first stationary state φ_0 is unstable (Fig. 6, curve 1). For large perturbations, the instability increases (curve 2, $\delta I_1 \approx 0.05\, I_1$). When the contributions are negative, then the beam diffuses (curve 3, $\delta I_1 \approx -0.01\, I_1$; curve 4, $\delta I_1 \approx -0.07 I_1$).

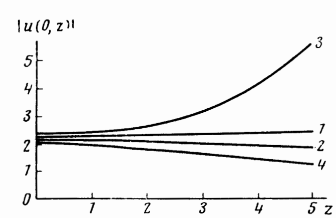

FIG. 6. Development of the instability of a homogeneous channel for values of $\delta I_1/I_0$ equal to +0.015 (curve 1), −0.013 (curve 2), +0.056 (curve 3), −0.070 (curve 4).

The second stationary state is unstable for any small perturbations, since the distributions obtained always have $I_1 > I_0$, which leads in the case of a cubic medium to a collapse of the beam. This confirms the numerical experiment.

All the calculations were carried out on the BÉSM-6 electronic computer of the Siberian-Division Numerical Center of the USSR Academy of Sciences.

CONCLUSION

Let us now discuss the conditions under which self-trapping of a monochromatic light beam can exist.

1. The intensity of the beam should be greater than critical, although it should not exceed it by a large factor, since in this case the picture of the light field becomes unstable and difficult to interpret.

The relation between the amplitude and dimension of the beam should be so chosen that the amplitude does not exceed threshold values for the appearance of stimulated Raman scattering and stimulated Mandel'shtam-Brillouin back scattering.

2. The value of the integral I_2 for the laser pulse should be controlled and negative. This places a limit on the degree of focusing of the beam and on the value of spatial coherence. As is seen from Eq. (8), the focal distance of the beam R should satisfy the inequality

$$R \gg d\delta n_{\text{nl}} / n_0.$$

Here d is the dimension of the beam, δn_{nl} the nonlinear correction to the index of refraction.

Similarly, for the angular spreading of the beam θ, we have

$$\theta \ll (\delta n_{\text{nl}} / n_0)^{1/2}.$$

3. It is necessary to be concerned about the stability of the self-focused beam relative to longitudinal modulation. As was shown in[10], in a medium in which the dispersion law for ω_k is convex below ($\omega_k'' < 0$), the stationary beam is unstable.

4. It is necessary that the nonlinearity of the medium differ from cubic (saturated at the lowest possible level). It is desirable that the medium also possess the longest possible time of relaxation of the nonlinearity, since this makes an additional reserve of longitudinal stability.[11]

[1] G. A. Askar'yan, Zh. Eksp. Teor. Fiz. 42, 1567 (1962) [Soviet Phys.-JETP 15, 1088 (1962)].

[2] V. I. Talanov, Izv. vuzov, Radiofizika 7, 564 (1964).

[3] R. Y. Chiao, E. Garmire, and C. H. Townes, Phys. Rev. Lett. 13, 479 (1964).

[4] A. L. Dyshko, V. N. Lugovoĭ and A. M. Prokhorov, ZhETF Pis. Red. 6, 655 (1967) [JETP Lett. 6, 146 (1967)].

[5] V. I. Bespalov and V. I. Talanov, ZhETF Pis. Red. 3, 44 (1966) [JETP Lett. 3, 26 (1966)].

[6] G. R. Gantmakher, Lektsiĭ po analiticheskoi mekhanike (Lectures on Analytical Mechanics), Nauka, 1966.

[7] V. V. Sobolev and V. S. Snyakh, PMTF, No. 6, 20 (1969).

[8] H. A. Haus, Appl. Phys. Lett. 8, 128 (1966).

[9] N. F. Pilipetskii and A. R. Rustamov, ZhETF Pis. Red. 2, 88 (1965) [JETP Lett. 2, 55 (1965)].

[10] V. E. Zakharov, Zh. Eksp. Teor. Fiz. 53, 1735 (1967), [Soviet Phys.-JETP 26, 994 (1968)].

[11] V. E. Zakharov ZhETF Pis. Red. 7, 290 (1968) [JETP Lett. 7, 226 (1968)].

Translated by R. T. Beyer
19