SELF-MODULATION OF WAVES IN NONLINEAR MEDIA

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Self-modulation of a finite-amplitude monochromatic wave in a nonlinear medium is studied. Solutions are obtained describing the self-modulation process in which the initial wave effectively excites only a single pair of waves. It is shown that such a process is periodic, the period being strongly dependent on the initial conditions. The conditions of applicability of the solutions are obtained. As an example, the process of the self-modulation of gravitational waves on the surface of a liquid is considered. The results of experiments are discussed.

INTRODUCTION

SELF-MODULATION is taken to mean the instability of finite-amplitude monochromatic waves to decay into a pair of waves with adjacent frequencies $\omega_0 \pm \Omega$ (ω_0 is the frequency of the monochromatic wave, $\Omega \ll \omega_0$). This instability takes place because of four-wave interactions (these correspond to the cubic terms in the equations of motion), if the dispersion law $\omega = \omega(k)$ is such that $V\omega''(k) < 0$, where V is the coefficient of four-wave interaction. Theoretically, this result is obtained both in general form[1] and in application to various types of waves: ion-acoustic oscillations of plasma,[1] gravitational waves on the surface of a liquid,[1-3] light,[4] and spin waves.[5] Self-modulation of monochromatic gravitational waves has been observed experimentally on the surface of water[3] and in the case of light.[6]

All the theoretical researches enumerated contain the results of the investigation of the initial stage of the process of self-modulation: the conditions of production of self-modulation are described, and the increments of growth of the wave pairs are found. The change in the amplitude of the wave at the fundamental frequency has been neglected in these investigations. Because of the law of conservation of energy, this is valid only when the amplitudes of the modulating waves are negligibly small, i.e., when $t \ll 1/\gamma$, where γ is the increment of growth of the amplitudes of the wave pairs.

In the present research, the investigation of the self-modulation process has been carried out over the time range $t \gtrsim 1/\gamma$ on the basis of the equations for slowly changing amplitudes (their derivation is contained in the first section). The case is considered in which only a single pair of waves is effectively excited in the self-modulation process. The equations for the amplitudes of this pair of waves and of the wave at the fundamental frequency are integrated. It is shown that the self-modulation process with the participation of only a single pair of waves is periodic. An estimate is made, showing that the role of waves with the combination frequencies $\omega_0 \pm n\Omega$, $n > 1$, is negligibly small in the self-modulation process.

Thus the problem of the self-modulation of a wave of finite amplitude with the effective excitation of only a single pair of modulating waves has been solved. Here the question of the applicability of the results is of importance. The point is that, in the presence of an intense wave, a large number (continuum) of wave pairs becomes unstable. This circumstance made it possible for Zakharov[1] to draw the conclusion that "the wave should become chaotic after a time of the order of $1/\gamma$." At the same time, Benjamin[3] assumed (and the experimental results which he puts forth confirm this) that only a single pair of waves is effectively excited in the self-modulation process. A discussion of this question is given in Sec. 3.

By way of example, the self-modulation of gravitational surface waves is considered in the last section, and the known experimental results are discussed.[3]

1. FUNDAMENTAL EQUATIONS

In a homogeneous, non-dissipative medium, it is convenient to describe the set of interacting waves in terms of the Hamiltonian formulation. With account only of four-wave interactions, the Hamiltonian can be written in the following form:

$$\mathscr{H} = {}^1/_2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} u_{\mathbf{k}}^* u_{\mathbf{k}} + {}^1/_2 \sum_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \times u_{\mathbf{k}}^* u_{\mathbf{k}_1}^* u_{\mathbf{k}_2} u_{\mathbf{k}_3} \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3). \quad (1.1)$$

Here the u_k are the spatial Fourier components of the wave field. The first component in the Hamiltonian corresponds to the linear approximation of the theory, and the second takes into account the interaction of the waves. The relative contribution of the second component in (1.1) will be assumed small. In the linear approximation, the frequency ω and the wave vector $\mathbf{k}$ are connected by the dispersion law

$$\omega = \omega(\mathbf{k}). \quad (1.2)$$

The interaction coefficients satisfy the symmetry relations

$$V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = V(\mathbf{k}_1,\mathbf{k},\mathbf{k}_2,\mathbf{k}_3) = V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2) = V(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k},\mathbf{k}_1). \quad (1.3)$$

We have assumed these to be real here without any loss of generality. The following equations of motion follow from (1.1) for the u_k:

$$i\dot{u}_{\mathbf{k}} = \omega_{\mathbf{k}} u_{\mathbf{k}} + \sum_{\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3} V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) u_{\mathbf{k}_1}^* u_{\mathbf{k}_2} u_{\mathbf{k}_3}. \quad (1.4)$$

In view of the weakness of the interaction of the waves, we shall seek solutions of this equation in the form

$$u_{\mathbf{k}} = a_{\mathbf{k}}(t)e^{-i\omega_{\mathbf{k}}t}, \quad (1.5)$$

where $a_k(t)$ are slowly changing functions (over the period $2\pi/\omega_k$). For $a_k(t)$ we have, from (1.4) and (1.5),

$$i\dot{a}_{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3} V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times \langle a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \exp\{i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})t\}\rangle, \quad (1.6)$$

where the brackets indicate that only the slowly changing components are included, i.e., those for which

$$|\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}| \ll \omega_{\mathbf{k}}. \quad (1.7)$$

In a number of problems, it is necessary to consider a wave interaction process that is inhomogeneous in space. In this case, under the assumption of the weakness of the interaction of the waves, the wave field can be represented in the form of a superposition of waves with slowly changing amplitudes (in space over the wavelength $2\pi/k$, and in time). The equations for these waves have the following form (see, for example, [6]):

$$i\left(\frac{\partial}{\partial t} - v_{\text{rp}}\frac{\partial}{\partial x}\right) a_{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3} V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times \langle a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \exp\{i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})t\}\rangle. \quad (1.8)$$

2. SELF-MODULATION OF MONOCHROMATIC WAVES

The equations (1.6) have the following solution:

$$a_{\mathbf{k}} = a_0 \exp\{iV|a_0^2|t\}\delta(\mathbf{k} - \mathbf{k}_0), \quad V = V(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0), \quad (2.1)$$

which corresponds to a single monochromatic wave with a frequency that depends on the amplitude. This solution can be shown to be unstable: pairs of waves with initially small amplitudes and frequencies grow exponentially with time in the presence of a finite-amplitude wave (2.1) if $V\omega''(k) < 0$.[1] Here the growth increment of the amplitudes $a_\pm \equiv a_{\omega_0 \pm Q}$ is maximal for frequencies $\omega_\pm$ such that

$$2\omega_0 - \omega_+ - \omega_- \approx 2V|a_0^2|. \quad (2.2)$$

For not very large values of the amplitude a_0 (which is the case assumed here), the frequencies and wave vectors of the excited wave pairs and the monochromatic wave are close to one another, so that this instability appears as a phenomenon of long-wave low-frequency modulation of the monochromatic wave.

We shall now consider the process of self-modulation with the participation of only a single pair of modulating waves with wave vectors $\mathbf{k}_0 \pm \boldsymbol{\kappa}$ and frequencies $\omega_0 \pm \Omega$. Waves with the combination frequencies $\omega_{\pm n} = \omega_0 \pm n\Omega$, $n > 1$ should also appear at the same time with this pair of waves, thanks to the nonlinear interaction between the waves. However, as will be shown below, when the waves $(\mathbf{k}_0 \pm \boldsymbol{\kappa}, \omega_0 \pm \Omega)$ develop from the initially small "priming" amplitudes $|a_\pm(0)| \ll |a_0(0)|$, the amplitudes of the combination waves with the frequencies $\omega_{\pm n}$ always remain negligibly small.

We shall seek a solution of Eq. (1.6) with the initial conditions

$$a_{\mathbf{k}}(0) = a_0^0\delta(\mathbf{k} - \mathbf{k}_0) + a_+^0\delta(\mathbf{k} - \mathbf{k}_0 - \boldsymbol{\varkappa}) + a_-^0\delta(\mathbf{k} - \mathbf{k}_0 + \boldsymbol{\varkappa}), \quad |a_\pm^0| \ll |a_0^0|, \quad (2.3)$$

in the following form:

$$a_{\mathbf{k}} = 0, \quad \text{if} \quad \mathbf{k} \neq \mathbf{k}_n = \mathbf{k}_0 + n\boldsymbol{\varkappa}, \qquad a_{\mathbf{k}_n} = a_n^{(1)} + a_n^{(2)} + \ldots. \quad (2.4)$$

We shall assume that $|a_n^{(k+1)}| \ll |a_n^{(k)}|$ and that the expansion (2.4) for the amplitudes a_0 and $a_{\pm 1}$ begins with the first approximation, the expansion for the amplitudes $a_{\pm 2}$ with the second, and so on. In this connection, we must set the amplitudes of all waves in (1.6) except a_0 and $a_{\pm 1}$ equal to zero in the first approximation; for a_0 and $a_{\pm 1}$, we get the following set of equations, with account of (2.4):[1)]

$$i\dot{a}_0 = 2Va_+a_-a_0^*e^{-i\Delta t} + V(|a_0^2| + 2|a_+^2| + 2|a_-^2|)a_0, \quad (2.5)$$
$$i\dot{a}_\pm = Va_0^2a_\mp^*e^{i\Delta t} + V(|a_\pm^2| + 2|a_\mp^2| + 2|a_0^2|)a_\pm;$$

$$\Delta = \omega(\mathbf{k}_0 + \boldsymbol{\varkappa}) + \omega(\mathbf{k}_0 - \boldsymbol{\varkappa}) - 2\omega(\mathbf{k}_0) \approx \omega''(\mathbf{k}_0\boldsymbol{\varkappa}/k_0)^2. \quad (2.6)$$

In view of the smallness of κ here, we neglect the dependence of the coupling coefficients on $\mathbf{k}$:

$$V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \approx V(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0) = V.$$

It is not difficult to establish the fact that Eqs. (2.5) have the following integrals of motion:

$$I = |a_0^2| + |a_+^2| + |a_-^2|, \quad (2.7)$$

$$D = |a_-^2| - |a_+^2|. \quad (2.8)$$

With account of this fact, it is convenient to transform from Eqs. (2.5) to the following set:

$$i\frac{d}{d\tau}A = (I - |A_0|)A_0 + (I + |A_0| + \Delta')A, \quad (2.9)$$
$$i\frac{d}{d\tau}A_0 = 4|A_0|A + 2(I - |A_0|)A_0,$$

where

$$\tau = Vt, \quad \Delta' = \Delta V^{-1}, \quad A = a_+a_-e^{i(2I-\Delta')\tau}, \quad A_0 = a_0^2e^{i2I\tau}.$$

The following relation is valid for the quantities A_0 and A on the basis of the conservation laws (2.7), (2.8):

$$|A^2| = {}^1/_4[(I - |A_0|)^2 - D^2], \quad (2.10)$$

When this is taken into consideration, it is convenient to represent Eq. (2.9) in the form:

$$\frac{d}{d\tau}|A_0| = -4J, \quad \frac{d}{d\tau}R = (3|A_0| - I + \Delta')J, \quad (2.11)$$
$$\frac{d}{d\tau}J = |A_0|[(I - |A_0|)^2 - D^2] - |A_0^2|(I - |A_0|) - (3|A_0| - I + \Delta')R,$$

where

$$J = \operatorname{Im}(A_0A^*) = |A_0A|\sin\theta,$$
$$R = \operatorname{Re}(A_0A^*) = |A_0A|\cos\theta, \quad \theta = \arg(A_0A^*). \quad (2.12)$$

Eliminating the quantity J from the first two equations, we find yet another integral of motion

$$F = R + {}^1/_4(I - \Delta')|A_0| - {}^3/_8|A_0^2|, \quad (2.13)$$

after which the system (2.11) reduces to a second-order differential equation

$$\frac{d^2}{d\tau^2}|A_0| + 4(3|A_0| - I + \Delta')\left[F + \frac{1}{4}(\Delta' - I)|A_0| + \frac{3}{8}|A_0^2|\right] + 4|A_0^2|(I - |A_0|) + 4|A_0|[(I - |A_0|)^2 - D^2] = 0, \quad (2.14)$$

the solution of which can be represented in the form of an integral. This solution depends on the integrals of motion I, D, and F, which are determined from the initial conditions, and also on the value of the frequency interval Δ'.

[1)] Equations of the same form are encountered in problems of nonlinear optics, [7] but without solutions.

We shall ascertain how the behavior of the amplitudes $a_\pm$ at the initial stage of the process for the initial conditions (2.3) depend on the value of the quantity Δ'. For this purpose, we observe that, since the amplitudes $a_\pm$ are negligibly small at the initial instant, then, because of (2.7), (2.8), and (2.13),

$$I \approx |A_0(0)|, \quad D = 0, \quad F \approx -{}^1/_8 I(I + 2\Delta'). \tag{2.15}$$

By linearizing Eq. (2.14) in the vicinity of the point $|A_0| = I$, we get, with account of (2.15),

$$\ddot{\alpha} - [4I^2 - (2I + \Delta')^2]\alpha = 0, \qquad \alpha = I - |A_0| = 2|A|. \tag{2.16}$$

It is seen from this equation that the exponential growth of amplitudes takes place if

$$-4I < \Delta' < 0 \tag{2.17}$$

and the growth increment is maximal for $\Delta' = -2I$.

Thus, those pairs of waves for which $\Delta' = -2I$ are excited more rapidly than the others at the initial instant. The wave vectors of these waves are determined with the help of Eq. (2.6), in which one must set $\Delta = -2IV$:

$$-2IV = \omega''(k_0)\,(k_0\varkappa / k_0)^2. \tag{2.18}$$

The solution of this equation is not unique in the general case; consequently, a whole set of waves with wave vectors $\mathbf{k}_0 \pm \boldsymbol{\kappa}_i$, where $\boldsymbol{\kappa}_i$ are the solutions of Eq. (2.18) have the maximum rate of increase at the initial moment.

We now consider in more detail the behavior of the amplitudes a_0 and $a_\pm$ in the case when $\Delta' = -2I$. In this case, Eqs. (2.13) and (2.14) take the following form:

$$|A_0|\,(I - |A_0|)\cos\theta + {}^3/_2 I|A_0| - {}^3/_4|A_0^2| = 2F, \tag{2.19}$$

$$\frac{d^2}{d\tau^2}|A_0| + \frac{1}{2}(I - |A_0|)\,[\,|A_0|\,(10I + 7|A_0|) - 24F] = 0 \tag{2.20}$$

(here we have set $D = 0$).

The first of these equations determines the integrated curves over the phase surface $(|A_0|, \theta)$ or, what amounts to the same thing, on the surface $(|A|, \theta)$. The phase picture, which is determined by these equations, is represented in Fig. 1. The abscissas represent the phase θ and the ordinates $|a_\pm^2|/I$. Inasmuch as the phase picture is periodic in θ with period 2π, as is seen from (2.19), it is drawn in Fig. 1 for the interval $[0, 2\pi]$. The letter O denotes the focus (its coordinates are $\theta_0 = \pi$, $|a_\pm^2|_0 = I/7$), while S ($\theta_{1S} = \pi/2$, $\theta_{2S} = 3\pi/2$, $|a_\pm^2|_S = 0$) denotes the saddle points. The integral curve $|a_\pm^2| = 0$ passes through them (this curve corresponds to an unstable state of motion), as does the separatrix (in Fig. 1, this is represented by the heavy line) which divides the integral curve into two parts.

The regime of self-modulation corresponds, as is seen from (2.3), (2.15), to curves lying close to the separatrix. In the self-modulation process, in accord with the phase picture, the amplitudes $a_\pm$ change periodically (see Fig. 1). They increase from small values to some maximum values $|a_\pm^2|_{max} = 2I/7$, and then decrease. The phase θ here is measured either under bounded conditions (when the integral curve lies inside the separatrix) or, in the opposite case, unbounded. It is seen from Fig. 1 and Eqs. (2.11) that the increase in the amplitudes $a_\pm$ takes place for $\theta = \pi/2$, while the phase θ changes very little. The increase in the amplitudes $a_\pm$ ceases when the changes in the phase θ become significant; when this phase reaches the value $3\pi/2$, the amplitudes $a_\pm$ decrease. It is thus seen that the slowness of change of the phase θ in the region $\theta = \pi/2$ is a condition for increase in the amplitudes, in addition to (2.17).

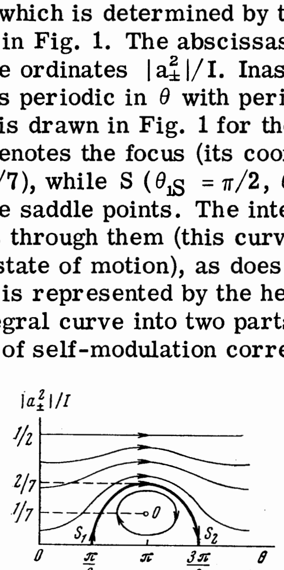

FIG. 1

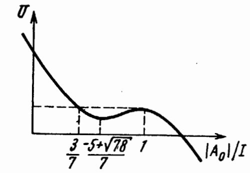

FIG. 2

The changes in the amplitude a_0 along the integral curves, which are determined by the condition (2.19), obey Eq. (2.20) in which we must set $F = 3I^2/8$ for the self-modulation regime, by virtue of (2.3) and (2.19); after this, the equation formally coincides with the equation of motion of a particle in a potential well, which is shown graphically in Fig. 2. Here it is seen that the point $|A_0| = I$ is a stopping point, so that at this point, at $d|A_0|/d\tau = 0$, the system is in a state of unstable equilibrium (the line $|a_\pm^2| = 0$ corresponds to such a state in Fig. 1). By virtue of the conservation law (2.7), $|A_0|$ does not exceed the value I (for this reason, deviations of $|A_0|$ from I are possible only in the direction of smaller values) and, consequently, there is a periodic change in the self-modulation process, while the minimum achievable values are $|A_0|_{min} = 3I/7$ and $|a_0|_{min} \approx 0.65|a_0(0)|$. The period of change of $|A_0|$ (the self-modulation period T_a) depends essentially on the initial condition: when $|A_0|_{max}$ is near I, then $T_a \approx 2\ln[I/(I - |A_0|_{max})]$, and if $I - |A_0|_{max} \sim I$, then $T_a \sim 2\pi/I$.

In accord with (2.4), the equations (2.5), on the basis of which the investigation of self-modulation of a monochromatic wave of finite amplitude has been carried out, are the first-approximation equations. In the succeeding approximation, the amplitudes $a_{\pm 2}$ must be taken into account along with the amplitudes $a_{\pm 1}$. On the right-hand side of the equation for $a_{\pm 2}^{(2)}$ one should substitute the amplitudes $a_0^{(1)}$ and $a_{\pm 1}^{(0)}$ found in the first approximation, after which, as the result of simple estimates, one can find that $\max|a_{\pm 2}^{(2)}| < 0.1|a_0(0)|$ for the initial conditions (2.3). This estimate is very rough; however, it permits us to estimate the accuracy of the three-wave analysis of the self-modulation process.

We note that all the results are valid only for the initial conditions of the form (2.3). In the opposite case (for example, when one of the amplitudes a_+ or a_- is not small at the initial instant), the amplitudes of the combination waves can become comparable with $|a_0|$ and the three-wave description ceases to be valid.

3. REGION OF APPLICABILITY

In the preceding section we obtained a description of the self-modulation process with effective excitation of only a single pair of waves. It is important to ascertain the region of applicability of these results.

It is evident that the results obtained describe the self-modulation process over a long period of time $t \gg 1/\gamma$ only when the solutions that are found are stable relative to excitation of other pairs of waves. Such a situation occurs, for example, when the wave frequency spectrum is discrete and the instability condition (2.17) is satisfied for no more than one pair of waves.

In the case of a continuous spectrum, the condition (2.17) is satisfied for a continuum of wave pairs, so that a very large number of wave pairs exist which can be excited simultaneously with comparable growth increments. In this case, the solutions found in the preceding section are realized for the initial conditions (2.3) when the amplitudes of all the wave pairs save one are equal to zero; however, these solutions are unstable. Thus, if waves of arbitrary small amplitudes are excited in the medium other than the selected pair, these additional pairs can grow with time and come into play. It is impossible to ignore them, beginning from that moment when their energy becomes comparable with the energy of the selected wave pair. Consequently, in order for the description of the self-modulation process given in the preceding section to be applicable, it is necessary to select at the initial instant a pair of waves (or a pair of wave packets $\Delta\omega \ll \Omega$) whose initial rate of amplitude growth (the rate of amplitude growth is equal to the product of the amplitude and the increment) is greater than the amplitude growth rates of the other pairs. This description is valid so long as the energy of the selected pair exceeds the energy of the other wave pairs.

We now estimate this time interval. For this purpose, we consider two wave pairs with such initial amplitudes that

$$|A_2(0)| \ll |A_1(0)| \ll I, \tag{3.1}$$

where $A_{1,2} = a_{k_0+\kappa_{1,2}} a_{k_0-\kappa_{1,2}}$, and with growth increments γ_1 and γ_2, $\gamma_1 \gtrsim \gamma_2$. At the beginning, thanks to the effect of the original wave, the growth of the amplitudes A_1 and A_2 is exponential:

$$A_1(t) = A_1(0)\exp(\gamma_1 t), \quad A_2(t) = A_2(0)\exp(\gamma_2 t).$$

This holds so long as the growing wave pairs do not exert any influence on the original wave and on one another, i.e., so long as $A_1(t)$ does not become comparable with $A_0(t)$. It is not difficult to estimate from the relation $A_1(t_1) \approx I/2$ the time t_1 at which this takes place, i.e.,

$$t_1 \sim \frac{1}{\gamma_1}\ln\frac{1}{2|A_1(0)|}. \tag{3.2}$$

At this moment A_2 reaches the value $A_2(0)\exp(\gamma, t_1)$, and $|A_2(t_1)| \ll |A_1(t_1)|$ because of (3.1). After this, intensive energy exchange takes place between the first pair and the original wave, on which the second pair cannot exert any influence because of the smallness of its amplitude. As a result of this exchange, as was shown in the previous section, the amplitude of the first pair reaches a maximum value at a time $\sim\pi/\gamma$, and then decreases, tending toward its initial value. So far as the amplitude of the second pair A_2 is concerned, its exponential growth ceases at this time for the following reasons.

First, as was noted previously, the condition for the growth of A_2 is not only (2.17), but also the slowness of the change of the relative phase $\theta = \arg(A_0A_2^*)$ in the region $\theta_2 = \pi/2$. However, this condition is violated because, as a result of the interaction of the original wave with the first pair, its phase θ_0 begins to change relatively more rapidly, which hinders the growth of A_2. In fact, we have from (2.9)

$$\frac{d}{d\tau}\theta_0 = 2|A_1|(\cos\theta_1 + 1), \quad \theta_0 = \arg A_0, \quad \theta_1 = \arg(A_0A_1^*),$$

so that when $|A_1| \sim I$ we have $\dot{\theta}_0 \sim \gamma_1$.

Second, the first pair affects the second pair in addition to the effect of the original wave. This influence retards the growth of amplitude of the second pair. Thus, in the interval $[t_1, t_1 + \pi/\gamma_1]$, only a single pair of waves plays a role in the self-modulation process. After this, the amplitude of the first pair A_1 again becomes small, tending to its initial value, just as A_0, and does not interfere with the growth of the amplitude of the second pair. As a result, we have, after the time interval $t \approx 2t_1 + \pi/\gamma_1$,

$$A_0(2t_1 + \pi/\gamma_1) \sim A_0(0), \quad A_1(2t_1 + \pi/\gamma_1) \sim A_1(0),$$
$$A_2(2t_1 + \pi/\gamma_1) \sim A_2(0)\exp(2\gamma_2 t_1) = A_2(0)\left(\frac{I}{2|A_1(0)|}\right)^{2\gamma_2/\gamma_1}$$

Evidently, the amplitudes of the first and second pairs become comparable within such time intervals, where the number n is determined by the equation

$$|A_2[n(2t_1 + \pi/\gamma_1)]| > |A_1(0)|,$$

i.e.,

$$n > \frac{\gamma_1}{2\gamma_2}\left[\frac{\ln(2|A_2(0)|/I)}{\ln(2|A_1(0)|/I)} - 1\right]. \tag{3.3}$$

It is seen from this expression that $n \geq 1$ and depends very weakly on the initial conditions.

If not one but a large number of wave pairs are excited at the initial instant, along with the selected wave pair, then the self-modulation process with effective excitation of only a single pair occurs in a time interval $t \sim n(2t_1 + \pi/\gamma_1)$, and

$$n > \frac{1}{2}\left[\frac{\ln\Sigma}{\ln(2|A_1(0)|/I)} - 1\right], \quad \Sigma = 2\sum_{\varkappa\neq\varkappa_1}\frac{|A_\varkappa(0)|}{I}. \tag{3.4}$$

The consideration just given determines the conditions for the realization of the self-modulation process with the effective excitation of a single wave pair and also the region of applicability of the results.

4. SELF-MODULATION OF GRAVITATIONAL WAVES ON THE SURFACE OF A LIQUID

As an example, it is useful to consider gravitational waves on the surface of a liquid. The instability of monochromatic gravitational waves has been widely discussed[1-3] and has been observed experimentally.[3]

The deviations of the surface of a heavy liquid from the mean level $\eta(x, t)$, in the linear approximation of the equations of hydrodynamics, are represented in the form of a superposition of plane waves, as is well known:[8]

$$\eta(\mathbf{x}, t) = \sum_k a_k \exp\{-i\omega_k t + i\mathbf{kx}\} \tag{4.1}$$

with the dispersion law

$$\omega^2(\mathrm{k}) = gk \operatorname{th} kh, \tag{4.2}$$

where g is the acceleration due to gravity, and h is the depth of the liquid.

With account of four-wave interactions, the equations for the slowly changing amplitudes a_k can be represented in the form (1.6). The expression for the interaction coefficients $V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ for the case of a deep liquid $kh \gg 1$ can be found in [1]. In the approximation used in the present work, $\mathbf{k} \approx \mathbf{k}_1 \approx \mathbf{k}_2 \approx \mathbf{k}_3$, these coefficients have the following form for the case of a liquid of arbitrary depth:[3]

$$V(\mathbf{k}, \mathbf{k}, \mathbf{k}, \mathbf{k}) = 2\omega k^2 X(k),$$

where

$$X(k) = \frac{9 - 10\operatorname{th}kh + 9\operatorname{th}^4 kh}{8\operatorname{th}^4 kh} + \frac{4 + 2\operatorname{sech}^2 kh + 3\operatorname{cth} kh \operatorname{sech}^4 kh}{1 - 2kh \operatorname{th} kh}. \tag{4.3}$$

We note that for $kh \gg 1$ the function $X(k) \approx 1$, so that (4.3) is materially simplified for the case of a deep liquid:

$$V(\mathbf{k}, \mathbf{k}, \mathbf{k}, \mathbf{k}) = 2\omega k^2. \tag{4.4}$$

Substituting (4.2)–(4.4) in (1.6), (2.5), (2.6), etc., we obtain the description of the self-modulation process of gravitational waves of finite amplitude.

The necessary condition for self-modulation of monochromatic waves of finite amplitude is $V\omega''(k) < 0$. As seen from (4.2) this condition is satisfied for the case $kh > 1.363$. (Experimental investigations (see [3]) have confirmed this result.) The published experimental results indicate that only a single pair of waves appeared during the self-modulation process, independently of whether an external initial low-frequency modulation of the wave existed or not; when the modulated wave was first excited, the frequency of the modulation did not change thereafter in the self-modulation process. When the same self-modulation was developed "spontaneously," then the modulation frequency corresponded to the excitation of a pair with maximum increment. In other words, the dominant role is played by only a single pair of waves with the greatest growth rate—that pair whose seed amplitudes are greater than the others or, for equal seed amplitudes, the one whose increment is greater.

Unfortunately, the published experimental results refer to the initial stage of the self-modulation process, a sufficiently complete theoretical description of which the authors had available. On the basis of these results, however, it is not possible to decide whether the self-modulation process is periodic or whether other pairs come into play after the first period, and the wave becomes chaotic. It is also not clear whether the amplitudes $a_\pm$ are bounded above by the values $\sqrt{2/7}\,|a_0(0)|$ and whether the contribution of the harmonics $\omega_{\pm n}$ $n > 1$, are negligible. An experimental test of these results would be of interest.

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