QUASICLASSICAL QUANTIZATION OF NONLINEAR SYSTEMS

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The rules for the quantization of a system described by a nonlinear second-order equation with slowly varying coefficients are derived. Physical problems in which a need for such quantization rules arises are discussed.

1. FORMULATION OF THE PROBLEM

A number of physical problems lead to the necessity to determine the spectrum of the equation

$$d^2w / dz^2 + f(w, \zeta, \lambda) = 0. \tag{1}$$

In this equation $\zeta = \alpha z$, f is some arbitrary function, and α is assumed to be small because the problem is considered in the quasiclassical approximation.

By spectrum is understood the set of admissible values λ. This set is determined by the form of the function f and by the presence of points at which the quasiclassical nature is violated, for example, turning points or boundary conditions. In this connection the spectrum λ may turn out to be discrete. We shall call the rule for determining the spectrum λ the rule of quantization.

We note that Eq. (1) may be regarded as the equation of motion of a certain hypothetical particle. Below we shall use a corresponding terminology. In what follows we shall be interested in periodic solutions of Eq. (1). The periodicity conditions together with the boundary conditions or the conditions at the turning points determine the quantization rule.

In this article we shall use the following as boundary conditions: either a boundary condition of the first kind

$$w|_{z=a,b} = 0, \tag{2}$$

or of the second kind

$$\left.\frac{\partial w}{\partial z}\right|_{z=a,b} = 0. \tag{2'}$$

The contents of the following sections contain a derivation of the rules of quantization; here as an example let us consider a problem in which it is necessary to find the spectrum of Eq. (1).

The partial differential equation

$$\sum_{i=1}^{3} \frac{\partial^2 F}{\partial x_i^2} + k^2 \Phi\left(\frac{x_3}{l}, |F|^2\right) F = 0 \tag{3}$$

describes the nonlinear propagation of electromagnetic waves in a plasma.[1] The phenomenological Ginzburg-Landau equations[2] of the theory of superconductivity have the same form. This same equation appears in the theory of superfluidity,[3] and also in certain other problems.[4]

In investigating the propagation of electromagnetic waves in a plasma, we shall assume that the propagation takes place along the x_2 axis, and the electric field has a single component directed along the x_1 axis, $k = \omega/c$ is the wave number, ω is the frequency of the electromagnetic field, and Φ is the dielectric constant.

If the propagation takes place in a waveguide with perfectly conducting walls, then F vanishes on its boundaries, i.e., a boundary condition of the first kind exists. The dependence of the dielectric constant on x_3 may be due, for example, to a nonuniform distribution of the electron concentration. The quantity l is a measure of the distance over which the dielectric constant varies.

In the phenomenological theory of superconductivity Φ is defined by the following formula:[2]

$$\Phi\left(\frac{x_3}{l}, |F|^2\right) = A\left(\frac{x_3}{l}\right) - B\left(\frac{x_3}{l}\right)|F|^2.$$

In this case F is related to the energy gap by a known relationship, k will be of the order of the inverse coherence length, and the dependence of A and B on x_3 may be caused by the presence of a temperature gradient in the superconductor. A boundary condition of the second kind exists at the boundary between the superconductor and the normal metal.[2]

We shall seek a solution of Eq. (3) in the form

$$F = F_0 e^{i[\lambda k x_2 - \sigma]} w,$$

where σ is a constant. In this connection let us take $z = kx_3$ and $\alpha = 1/kl$. We arrive at Eq. (1) for w, where

$$f(w, \zeta, \lambda) = [\Phi(z/kl, |F_0|^2 w^2) - \lambda^2] w. \tag{4}$$

Here it is assumed that w is a real quantity. In order to find the field in a waveguide filled with plasma, it is necessary to solve the one-dimensional equation (1) with the boundary condition (2) and to determine the admissible values of λ from the requirement of periodicity. The situation is analogous for superconductivity. The difference consists in the utilization of the boundary condition (2′) instead of the boundary condition (2). From the condition $\alpha \ll 1$ it follows that $kl \gg 1$.

By the substitution $v = p_1^{-1/2}(\zeta)w$, Eq. (1) reduces to the following form:

$$\frac{d}{dz} p_1(\zeta) \frac{dv}{dz} + p_2(v, \lambda, \zeta) = 0, \tag{5}$$

where

$$f(w, \zeta, \lambda) = \frac{p_2(\zeta, \lambda, p_1^{-1/2} w)}{p_1^{1/2}(\zeta)} - p_1^{1/2}(\zeta) \frac{d^2 p_1(\zeta)}{dz^2}.$$

An equation of the type (5) is obtained from Eq. (3) in cases of spherical or cylindrical symmetry.

In concluding this section, we note that the examples cited by no means exhaust the problems leading to Eq. (1).

2. QUANTIZATION DUE TO THE TURNING POINTS

As already indicated above, quantization of a system may be related to the presence of turning points, which will be investigated in this section. Here we shall confine our attention to only the case of small nonlinearity, i.e., we assume

$$f(w, \zeta, \lambda) = q^2(\zeta, \lambda) w - \beta\varphi(w, \zeta, \lambda), \tag{6}$$

where $\beta \ll 1$. Physically this may be realized in a plasma when the frequency of the propagating wave is substantially larger than the Langmuir frequency, or when the fields are sufficiently small. A similar situation may occur in superconductors when the temperature is close to the critical temperature.

Substituting (6) in (1), we will have the following equation for w:

$$d^2w / dz^2 + q^2(\zeta, \lambda) w = \beta\varphi(w, \zeta, \lambda). \tag{7}$$

Let us consider the motion of a hypothetical particle in the potential well $q(\zeta, \lambda)$. The turning points at z = a and z = b are determined by the roots of the equation

$$q(\zeta, \lambda) = 0. \tag{8}$$

Let us assume that the quantity $q^2(\zeta, \lambda) > 0$ for $a < z < b$, and for values of z outside of this interval we assume $q^2(\zeta, \lambda) < 0$.

Far away from the turning points inside the well, in order to determine w one can apply Bogolyubov's method[5] for equations with small nonlinearity and coefficients which are slowly varying with time. The criteria for the applicability of the first approximation of this method have the form

$$\beta^2(b-a) \ll 1, \tag{9}$$

$$q^{-3/2}(\zeta, \lambda) \frac{dq(\zeta, \lambda)}{dz} \sim \alpha \ll 1. \tag{10}$$

It is easy to see that the inequality (10) is not satisfied near the turning points.

We shall seek the solution of Eq. (6) far away from the turning points in the form

$$w = u(z) \cos S(z). \tag{11}$$

The standard procedure leads to the following shortened equations:

$$\frac{du}{dz} = -\frac{1}{2q(\zeta, \lambda)} \frac{dq(\zeta, \lambda)}{dz} u,$$
$$\frac{dS}{dz} = q(\zeta, \lambda) - \frac{\beta}{2u_0 q^{1/2}(\zeta, \lambda)} \varphi_1(u, \zeta, \lambda), \tag{12}$$

where φ_1 is the coefficient affiliated with cos S in the expansion of φ(u cos S, ζ, λ) in a Fourier series in cos S:

$$\varphi_1(u, \zeta, \lambda) = \frac{1}{\pi} \int_0^{2\pi} \varphi(u \cos S, \zeta, \lambda) \cos S \, dS.$$

We note that the first equation of the system (12) is valid accurate to terms of arbitrarily high order in β, whereas in the second equation of this system terms of order β^2 have been neglected.

One can easily solve the system (12) in quadratures, and w has the following form:

$$w = u_0 q^{-1/2}(\zeta, \lambda) \cos \left[S_0 + \int_d^z g(\zeta, \lambda) dz \right], \tag{13}$$

where d is some point inside the potential well, and $g(\zeta, \lambda) = q(\zeta, \lambda) - \beta\varphi_1/2u_0 q^{1/2}(\zeta, \lambda)$. The constant u_0 is determined from subsidiary conditions. For example, in electrodynamics u_0 is found from the conditions for excitation of the waves, and S_0 is found by matching the solutions near the turning point.

In the neighborhood of a turning point the solution (13) is not valid since it is obtained under the quasi-classical assumption (10), which is violated in the immediate vicinity of the zeros of $q(\zeta, \lambda)$. However, one can find the solution near the turning point by iterations with respect to the parameter β. The solutions obtained by such a method will have a common region of validity with formula (13). From the condition of identity of the two solutions in the interval where they should agree, S_0 can be determined.

The procedure which has been discussed is ideologically similar to the one which is applied in order to obtain the Bohr-Sommerfeld quantization conditions in a linear theory. We shall construct an approximate solution near the turning point. For this it will be necessary for us to find the fundamental solutions of the equation

$$\frac{d^2w}{dz^2} + q^2(\zeta, \lambda) w = 0. \tag{14}$$

For $|z - a| \gg 1$ we define these solutions w_1 and w_2 in the following way:

$$w_1 = \begin{cases} q^{-1/2}(\zeta, \lambda) \cos \left[\int_a^z q(\zeta, \lambda) dz - \gamma \right], & z > a, \\ |q\zeta, \lambda)|^{-1/2} \exp \left[\int_a^z |q(\zeta, \lambda)| dz \right], & z < a, \end{cases}$$

$$w_2 = \begin{cases} q^{-1/2}(\zeta, \lambda) \sin \left[\int_a q(\zeta, \lambda) dz - \gamma \right], & z > a, \\ |q(\zeta, \lambda)|^{-1/2} \exp \left[-\int_a^z |q(\zeta, \lambda)| dz \right], & z < a. \end{cases}$$

The constant γ depends on the law according to which $q(\zeta, \lambda)$ tends to zero at the point a. For example, if it tends to zero according to a linear law, then $\gamma = \pi/4$.

The fundamental solutions of Eq. (14) in a form suitable for all values of z can be constructed by the standard-equation method; however, for our purposes it is sufficient to construct their asymptotic solutions.

Let us select $u_0 w_1$ as the zero-order approximation. The next approximation is obtained by the method of variation of constants and has the form

$$w = u_0 \Big\{ w_1(z, \lambda) + \frac{\beta}{u_0} \Big[w_2(z, \lambda) \int_{-\infty}^z \varphi(u_0 w_1(z, \lambda), \zeta, \lambda) w_1(z, \lambda) dz$$
$$- w_1(z, \lambda) \int_{-\infty}^z \varphi(u_0 w_1(z, \lambda), \zeta, \lambda) w_2(z, \lambda) dz \Big] \Big\}. \tag{15}$$

The arbitrary constants associated with the construction of the solution (15) were selected such that $w(z, \lambda)$ is exponentially damped as $z \to \infty$. In this connection, it is assumed that $\varphi(w, \zeta, \lambda)/w$ tends to zero for small values of w, since in the opposite case the second integral in formula (15) diverges. This is not an additional restriction. In fact, let

$$\lim_{w\to 0}\frac{\varphi(w,\zeta,\lambda)}{w}=T(\zeta,\lambda)\neq 0.$$

Let us assume $\varphi(w, \zeta, \lambda) = T(\zeta, \lambda)w + \varphi'(w, \zeta, \lambda)$. By definition $\varphi'(w, \zeta, \lambda)/w$ tends to zero as $w \to 0$. Having substituted $\varphi(w, \zeta, \lambda)$ in the form given above into Eq. (1) and having introduced the notation $q_1^2(\zeta, \lambda) = q^2(\zeta, \lambda) - T(\zeta, \lambda)$ we again arrive at an equation of the type (1) with $\varphi'(w, \zeta, \lambda)$ on the right-hand side, which satisfies the required condition. It is also considered that $\varphi(w, \zeta, \lambda)$ increases more slowly than the exponential of ζ.

Let us consider the solution of Eq. (15) in the region $z > a$, $|z - a| \gg 1$, for values of z sufficiently large so that over the major portion of the interval of integration one can replace w_1 and w_2 by their asymptotic forms. The contribution from the region where it is impossible to do this is small. One can easily see that for such large values of z the second integral inside the square brackets can be neglected.

This is associated with the fact that on the major portion of the interval of integration w_2 behaves like a sine function, and $\varphi(u_0w_1, \zeta, \lambda)$ is a cosine function of the same argument. Expanding φ in a Fourier series with respect to the cosine and performing the integration, it is not difficult to verify that as a consequence of the orthogonality of the sines and cosines one can neglect this integral. Estimates based on these considerations lead to the same result.

Let us transform the first integral inside the square brackets. We represent this integral in the form of a sum of two integrals: the integral with the limits of integration $[-\infty, d]$ and the integral with the limits $[d, z]$. Let $d - a \gg 1$; then in the integral with the limits $[d, z]$ one can replace w_1 by its asymptotic form, and then expand $\varphi(u_0w_1, \zeta, \lambda)$ in a Fourier series with respect to the cosine. Neglecting the integrals of rapidly oscillating functions, we obtain

$$\int_d^z \varphi(u_0w_1,\zeta,\lambda)\,w_1\,dz \cong \frac{1}{2}\int_d^z \varphi_1(u_0q^{-1/2}(\zeta,\lambda),\zeta,\lambda)\,q^{-1/2}(\zeta,\lambda)\,dz.$$

Taking account of this property, and also the asymptotic behavior of w_1 and w_2 one can rewrite formula (15) in the following form:

$$w=u_0q^{-1/2}(\zeta,\lambda)\left\{\cos\left[\int_a^z q(\zeta,\lambda)\,dz-\gamma\right]+\frac{\beta}{u_0}\sin\left[\int_a q(\zeta,\lambda)\,dz-\gamma\right]\right.$$
$$\times\left[\int_{-\infty}^d \varphi(u_0w_1(z,\lambda),\zeta,\lambda)\,w_1(z,\lambda)\,dz+\frac{1}{2}\int_d \varphi_1(u_0q^{-1/2}(\zeta,\lambda),\zeta,\lambda)\right.$$
$$\left.\left.\times\, q^{-1/2}(\zeta,\lambda)\,dz\right]\right\}\cong u_0q^{-1/2}(\zeta,\lambda)\cos\left\{\int_d q(\zeta,\lambda)\,dz\right.$$
$$-\frac{\beta}{u_0}\int_{-\infty}^d \varphi(u_0w_1(z,\lambda),\zeta,\lambda)\,w_1(\zeta,\lambda)\,dz$$
$$\left.-\frac{\beta}{2u_0}\int_d \varphi_1(u_0q^{-1/2}(\zeta,\lambda),\zeta,\lambda)\,q^{-1/2}(\zeta,\lambda)\,dz\right\}+O(\beta^2),$$

After elementary transformations we finally obtain

$$w=u_0q^{-1/2}(\zeta,\lambda)\cos\left\{\int_d^z g(\zeta,\lambda)\,dz-\gamma+\int_a^d q(\zeta,\lambda)\,dz\right.$$
$$\left.-\frac{\beta}{u_0}\int_{-\infty}^d \varphi(u_0w_1(z,\lambda),z,\lambda)\,w_1(z,\lambda)\,dz\right\}. \tag{16}$$

There is a common region of validity of the approximate solutions (13) and (16) of Eq. (6), namely: $1 \ll |z - a| \ll \beta^{-2}$. The right-hand side of this inequality guarantees the applicability of the iterations, and the left-hand side ensures the possibility of using the asymptotic forms. In this region the solutions (13) and (16) must agree. Comparing these two formulas, we obtain the following expression for S_0:

$$S_0=\int_a^d q(\zeta,\lambda)\,dz-\frac{\beta}{u}\int_{-\infty}^d \varphi(u_0w_1(z,\lambda),\zeta,\lambda)\,w_1(z,\lambda)\,dz-\gamma. \tag{17}$$

As an application of the obtained result, let us find the coefficient of reflection of a weakly nonlinear electromagnetic wave from a turning point.

Let a generator emitting a wave with amplitude E_0 be located at the point d. With reflection taken into consideration, the field at the point d has the form

$$E=E_0(1+e^{i\sigma}).$$

On the other hand, with the aid of relation (13) E at the point d is written as:

$$E=E_0e^{i\xi}u_0q^{-1/2}(d,\lambda). \tag{18}$$

Comparing these two formulas, we find

$$u_0=2q^{1/2}(d,\lambda),\quad \sigma=2S_0,$$
$$\xi=-S_0,\quad R=e^{-2iS_0}. \tag{19}$$

The coefficient of reflection is denoted by R. If the coefficient of reflection in the absence of nonlinearity (for $\beta = 0$) is denoted by R_0, then

$$R/R_0=\exp\left\{2i\frac{\beta}{u_0}\int_{-\infty}^d \varphi(u_0w_1(z,\lambda),z,\lambda)\,w_1(z,\lambda)\,dz\right\}. \tag{20}$$

Notwithstanding the smallness of β, for large propagation paths the advance of the phase of the coefficient of reflection may be comparable with or larger than unity.

Now let us find out the quantization rules. We determine the constants appearing in Eq. (13) from the conditions for matching the solutions at the point b (in this connection we shall denote them by u_0' and S_0'). The procedure, which is analogous to the one which was carried out above, leads to the formula

$$S_0'=-\int_d^b q(\zeta,\lambda)\,dz+\frac{\beta}{u_0'}\int_d^\infty \varphi(u_0'w_1(z,\lambda),\zeta,\lambda)\,w_1(z,\lambda)\,dz+\gamma. \tag{21}$$

In order for the solutions obtained by matching at the points a and b to coincide in the region $a < z < b$, the following equations must be satisfied:

$$S_0-S_0'=l\pi,\quad u_0'=(-1)^l u_0, \tag{22}$$

where l is an integer. Substituting S_0 and S_0' from formulas (17) and (21) into (22) and taking into consideration that the integral of φ is an odd function of u_0',[1)] we finally obtain

$$\int_a^b q(\zeta,\lambda)\,dz - \frac{\beta}{u_0}\int_{-\infty}^{d} \varphi(z, u_0 w_1(z,\lambda))\,w_1(z,\lambda)\,dz = l\pi + 2\gamma. \quad (23)$$

This relation, regarded as an equation with respect to λ, is the desired quantization condition. The corrections to the eigenvalues λ may be calculated if it is taken into account that in the quasiclassical approximation the distance $\delta\lambda$ between two eigenvalues is small in comparison with the eigenvalue itself.

Let us denote the eigenvalue of the linear problem, satisfying the equation

$$\int_a^b q(\zeta,\lambda)\,dz = l\pi + \gamma. \quad (24)$$

by λ_l^0. It is obvious that $\delta\lambda_l = \lambda_{l+1}^0 - \lambda_l^0$ is determined by the following formula:

$$\delta\lambda_l^0 = \pi \Big/ \int_a^b \frac{dq(\zeta,\lambda_l^0)}{d\lambda}\,dz. \quad (25)$$

Now let us calculate the change of the eigenvalue due to the nonlinearity. We shall seek λ_l^β—the eigenvalue of the nonlinear problem—in the form $\lambda_l^0 + \delta\lambda_l^\beta$ assuming that $\delta\lambda_l^\beta \ll \lambda_l^0$. From Eq. (23) one obtains the following formula for $\delta\lambda_l^\beta$:

$$\delta\lambda_l^\beta = \frac{\beta}{\pi u_0}\,\delta\lambda_l^0 \int_{-\infty}^{d} \varphi(u_0 w_1(z,\lambda_l^0), \zeta, \lambda_l^0)\,w_1(z,\lambda_l^0)\,dz. \quad (26)$$

A formula equivalent to (26) may also prove to be useful. If one sets $\lambda_l^0 = \Lambda(\pi l)$, then

$$\lambda_l^\beta = \Lambda\left(\pi l + \frac{\beta}{u_0}\int_{-\infty}^{d} \varphi(u_0 w_1(z,\lambda), \zeta, \lambda_l^0)\,w_1(z,\lambda_l^0)\,dz\right). \quad (27)$$

We note that it follows from Eq. (26) that

$$\frac{\delta\lambda_l^\beta}{\delta\lambda_l} \sim \frac{\beta}{\pi}(b-a)\,|\varphi w_1|_{max},$$

and since $b - a \gg 1$ in virtue of the quasiclassical nature of the problem, then in spite of the smallness of β the quantity $\delta\lambda_l^\beta$ may be comparable with and even exceed $\delta\lambda_l^0$, i.e., lead to a substantial rearrangement of the spectrum. Starting from the conditions of quasiclassical character, one can also easily show that $\delta\lambda_l^\beta \ll \lambda_l^0$.

3. QUANTIZATION RESULTING FROM THE BOUNDARY CONDITIONS

In this Section first the quantization conditions will be obtained for an arbitrary nonlinear homogeneous system (in Eq. (1) f does not depend on ζ), and then the conditions of quantization will be found for an inhomogeneous weakly nonlinear system, and finally we consider a weakly inhomogeneous system having an arbitrary nonlinearity.

For an f which does not depend on ζ, Eq. (1) has the first integral

$$W = (dw/dz)^2 + U(w,\lambda). \quad (28)$$

The constant W is obtained from some kind of additional conditions (which were mentioned above), and $U(w,\lambda) = 2\int f(w,\lambda)\,dw$. From Eq. (28) one can easily obtain the solution of Eq. (1), determining w (z) in implicit form:

$$z - a = \int_{w(a)}^{w} \frac{dw}{\sqrt{W - U(w,\lambda)}}. \quad (29)$$

In order for w to periodically depend on z, the existence of two nondegenerate roots of the equation[6]

$$U(w,\lambda) = W. \quad (30)$$

is necessary. Let us denote these roots by w_1 and w_2. From Eq. (28) it is seen that w_1 and w_2 are extremal values of w. We shall assume that w_1 corresponds to a minimum of w and w_2 corresponds to a maximum. The period D of the function w is, as is well known, determined by the formula[6]

$$D(W,\lambda) = 2\int_{w_1(\lambda,W)}^{w_2(\lambda,W)} \frac{dw}{\sqrt{W - U(w,\lambda)}}. \quad (31)$$

First let us consider the boundary condition (2). In order to satisfy this boundary condition at the point a, it is necessary to set $w(a) = 0$ in formula (29). Let us require fulfilment of the boundary condition at the point b. At the beginning of the period we shall regard the extremum of the function w (z) nearest to the point a as a maximum. Let us investigate expression (29) at the point b.

Here several cases may be represented. Let the extrema nearest to the points a and b be minima. Then one can easily see that from Eq. (28) it follows that

$$\frac{b-a}{D(W,\lambda)} = \frac{l}{2} + \frac{2}{D(W,\lambda)}\left|\int_0^{w_1(\lambda,W)} \frac{dw}{\sqrt{W - U(w,\lambda)}}\right|, \quad (32)$$

where l is an integer indicating how many half-periods fit into the segment [a, b]. The integral in formula (32) describes the "remainder" adjacent to the points a and b. If U is an even function of w, then $w_2 = -w_1$ and the integral in formula (32) is equal to D/2. Under this assumption we simply have

$$(b-a)/D(W,\lambda) = l/2. \quad (33)$$

Formulas (32) and (33), regarded as equations with regard to λ, are the desired rules of quantization. If the extrema nearest to the points a and b are maxima, then the quantization rule for an arbitrary $U(w,\lambda)$ is given by formula (32) in which it is necessary to replace w_1 by w_2. For even U formula (33) is valid. This formula also describes the quantization of the system if on the one hand the nearest extremum is a maximum, and if on the other hand it is a minimum.

The same kind of discussions lead to the quantization rule (33) if a problem having boundary conditions of the second kind is solved.

The quantization condition (33) can be given in a more customary form if we change from the period $D(W,\lambda)$ to the frequency $\Omega(W,\lambda) = 2\pi D^{-1}(W,\lambda)$. In terms of the new notation (33) is rewritten as

$$\Omega(W,\lambda) = \pi l/(b-a). \quad (34)$$

[1)]Even though the proof of this assertion for arbitrary ϕ is elementary, it is nevertheless somewhat cumbersome. For problems described by Eq. (3) this assertion is obvious.

We note that for a boundary condition of the second kind the quantization rule was first given in article [7].

Let us go on to an inhomogeneous system with a small nonlinearity, described by Eq. (7). Since by assumption no points are present where the quasiclassical conditions are not satisfied, the solution is given by formula (13).

The quantization condition is obtained by the same method as in Sec. 2, and for boundary conditions of both the first and the second kinds it has the form

$$\int_a^b g(\zeta, \lambda)\, dz = \pi l. \tag{35}$$

Formula (35) is applicable for arbitrary values of l. Substituting q (ζ, λ) we finally obtain

$$\int_a^b q(\zeta, \lambda)\, dz - \frac{\beta}{2u_0}\int_a^b \varphi_1(u_0 q^{-1/2}(\zeta, \lambda), \zeta, \lambda)\, q^{-1/2}(\zeta, \lambda)\, dz = \pi l. \tag{36}$$

The general case of arbitrary nonlinearity of an inhomogeneous system may be considered with the aid of the method of averaging in the form proposed by Volosov.[8] The essence of this method consists in the fact that the integration constants in the solution of Eq. (29) for the case of an explicit dependence of f on ζ are treated as functions of ζ, for which equations are found by using the method of averaging.

We shall seek a solution of Eq. (1) in the form

$$\mu(z) - \mu(a) = \frac{1}{D(W, \zeta, \lambda)} \int_{w(a)} \frac{dw}{\sqrt{W(\zeta, \lambda) - U(w, \zeta, \lambda)}}, \tag{37}$$

where D is determined by formula (32); w, which is thus determined, is a periodic function of μ. If f does not depend on ζ, then $\mu(z) = z/D$, and we again arrive at formula (29).

First let us find the equation for W. In order to do this we differentiate formula (28) with respect to z, after which with Eq. (1) taken into consideration we obtain

$$dW / dz = \alpha \partial U / \partial \zeta. \tag{38}$$

Let us average Eq. (38) over the period. After simple transformations (see [8]) we obtain

$$\frac{d\overline{W}}{d\zeta} = \frac{2}{D(\overline{W}, \zeta, \lambda)} \int_{w_1(\overline{W}, \zeta, \lambda)}^{w_2(\overline{W}, \zeta, \lambda)} \frac{dw}{\sqrt{\overline{W} - U(w, \zeta, \lambda)}}. \tag{39}$$

This equation has the integral

$$\int_{w_1(\overline{W}, \zeta, \lambda)}^{w_2(\overline{W}, \zeta, \lambda)} \sqrt{\overline{W}(\zeta, \lambda) - U(w, \zeta, \lambda)}\, dw = \text{const}. \tag{40}$$

From expression (40) one can find $\overline{W}$ as a function of ζ. One can also show that $W = \overline{W} + 0(\alpha)$. Similarly the following differential equation is obtained for μ:

$$d\mu / dz = D^{-1}(\overline{W}, \zeta, \lambda). \tag{41}$$

Thus, μ is determined with the aid of a quadrature, and the substitution of $\overline{W}$ found from Eq. (40) into Eq. (41).

From Eq. (41) it follows that

$$\mu = \int D^{-1}(\overline{W}(\zeta, \lambda)\zeta, \lambda)\, dz. \tag{42}$$

Now, by starting from Eqs. (41) and (37), one can obtain the quantization rules by reasoning in the same way as at the beginning of this Section. For a boundary condition of the first kind, these conditions have the form

$$\int_a^b D^{-1}(\overline{W}, \zeta, \lambda)\, dz = \frac{l}{2} + D^{-1}(\overline{W}(\alpha a, \lambda), \alpha a, \lambda) \times \left| \int_0^{w_i(\alpha a, \lambda)} \frac{dw}{\sqrt{\overline{W}(\alpha a, \lambda) - U(w, \alpha a, \lambda)}} \right| + D^{-1}(\overline{W}(\alpha b, \lambda), \alpha b, \lambda) \times \left| \int_0^{w_k(\alpha b, \lambda)} \frac{dw}{\sqrt{\overline{W}(\alpha b, \lambda) - U(w, \alpha b, \lambda)}} \right|; \tag{43}$$

i and k take the values 1 or 2 depending on whether the extrema nearest to the boundaries a and b are minima or maxima.

For boundary conditions of the second kind the quantization condition looks like:

$$\int_a D^{-1}(\overline{W}(\zeta, \lambda)\zeta, \lambda)\, dz = \frac{l}{2}. \tag{44}$$

Changing from the period to the frequency, one can write the quantization rule in the form

$$\int_a^b \Omega(\overline{W}(\zeta, \lambda)\zeta, \lambda)\, dz = \pi l. \tag{45}$$

4. SPECTRUM OF ELECTROMAGNETIC WAVES IN AN INHOMOGENEOUS AND CONFINED PLASMA

As already indicated in Section 1, the propagation of waves in a plasma under definite conditions, which are specified there, is described by Eq. (1) in which the function f is given by formula (4).

The dielectric constant Φ for a plasma has the following form:

$$\Phi = 1 - \frac{\omega_0^2}{\omega^2} n\left(\zeta, \frac{F_0^2}{F_1^2} w^2\right). \tag{46}$$

Here $\omega_0^2 = 4\pi e^2 N_0/m$, e is the electron charge, m is its mass, N_0 is the equilibrium concentration of electrons at the point z = 0, ω is the frequency of the electromagnetic field,

$$n = N_0^{-1} N(\zeta, F_0^2 w^2 / F_1^2),$$

N is the concentration of electrons in the plasma, and F_1 is the characteristic field defined by the mechanism which leads to a dependence of the concentration on the field.

For a heating mechanism the quantity F_1^2 is determined by the following formula:[1]

$$F_1^2 = \frac{6m^2}{e^2 M} \vartheta(\zeta)\, [\omega^2 + \nu^2(\zeta)], \tag{47}$$

where $\vartheta(\zeta)$ denotes the temperature, $\nu(\zeta)$ is the frequency of collisions in an equilibrium plasma, and M is the mass of a molecule. For a striction mechanism, the dependence of the concentration on the field[2] has the form:

$$F_1^2 = 8\vartheta(\zeta) m\omega^2 / e^2. \tag{48}$$

In the present section we are only considering small nonlinearities because for them very simple results are obtained. One of the reasons for the smallness of the nonlinearity may be the weakness of the field ($F_0^2 w^2 / F_1^2 \ll 1$). Expanding expression (46) for Φ in a series in powers of the square of the field and substituting this expansion into Eq. (4), and then substituting (4) into (1), we arrive at Eq. (7), where

$$q^2(\zeta,\lambda)=1-\lambda^2-\frac{\omega_0^2}{\omega^2}n(\zeta,0),$$
$$\beta=\frac{F_0^2}{F_{10}^2},\quad \varphi(\zeta,w)=\psi(\zeta)w^3, \tag{49}$$

where F_{10}^2 denotes the value of $F_1^2(\zeta)$ at the point $\zeta = 0$,

$$\psi(\zeta)=\frac{\omega_0^2}{\omega^2}n'(\zeta,0)\frac{F_{10}^2}{F_1^2(\zeta)},$$

where a prime indicates differentiation with respect to $F_0^2w^2/F_1^2$. The smallness of β corresponds to the weakness of the field.

The smallness of the parameter ω_0^2/ω^2 (the frequency of the electromagnetic field is larger than the Langmuir frequency) may be another reason for the smallness of the nonlinearity. In this case we again arrive at Eq. (7) with the following values for the functions and parameters appearing in it:

$$q^2(\zeta)=1-\lambda^2,\quad \beta=\omega_0^2/\omega^2,\quad \varphi(\zeta,w)=n(\zeta,w). \tag{50}$$

Now in order to determine the spectrum of the system, one can use the formulas singled out above, and also obtaining the final answer reduces to the evaluation of integrals (formulas (23) and (36)). This occurs both for quantization due to turning points as well as for quantization determined by boundary conditions.

Specific calculations can be carried out if the explicit dependence of F_1 on ζ is known. We shall confine our attention to the very simplest example, when F_1 does not depend on ζ, and the boundary conditions are the reason for quantization. In this case the calculation is carried out according to formula (36).

If the nonlinearity is related to weakness of the field, then the expression for λ has the following form:

$$\lambda=\left\{1-\frac{\omega_0^2}{\omega^2}-\left[\frac{\pi l}{2(b-a)}+\sqrt{\frac{\pi^2l^2}{4(b-a)^2}+\frac{3F_0^2\omega_0^2n'(0)}{8F_1^2w^2}}\right]^2\right\}^{1/2} \tag{51}$$

Now let us consider a small nonlinearity associated with the smallness of ω_0^2/ω^2. If the electrons are characterized by the Boltzmann distribution and a striction mechanism plays the major role, then (see the second article cited in [1])

$$n\left(\frac{|F|^2}{F_1^2}\right)=\exp\left\{-\frac{F_0^2}{F_1^2}w^2\right\} \tag{52}$$

and the following formula holds for λ:

$$\lambda=\left\{1-\left[\frac{\pi l}{2(b-a)}+\left(\frac{\pi^2l^2}{4(b-a)^2}+\frac{\omega_0^2}{\omega^2}\exp\left\{-\frac{F_0^2}{2F_1^2}\right\}\times\left[I_0\left(\frac{F_0^2}{2F_1^2}\right)+I_1\left(\frac{F_0^2}{2F_1^2}\right)\right]\right)^{1/2}\right]^2\right\}^{1/2}. \tag{53}$$

Here I_n are the cylindrical functions of imaginary argument. Formula (53) is applicable for arbitrary values of $F_0^2/2F_1^2$. For $F_0^2/2F_1^2 \gg 1$ the asymptotic formula for λ is obtained by replacing the coefficient affiliated with the term ω_0^2/ω^2 in formula (53) by $F_1/\sqrt{\pi}\,F_0$.

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42