QUANTUM FLUCTUATIONS IN GAS LASER RADIATION

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Quantum fluctuations of the radiation from a gas laser are considered. An exact solution of the quantum problem is obtained for a simple laser model with a single relaxation time and a broad Doppler contour. Photon diffusion coefficients are obtained for the two most important cases—traveling and standing waves. Nonmonochromaticity of the atomic radiation in the field of a standing wave is the reason for additional "shot" noise, which is absent in the traveling wave field. These radiation fluctuations are of about the same magnitude as the quantum fluctuations. The limit of a strong radiation field is investigated in greatest detail.

THIS paper is concerned with the fluctuations of the radiation of a gas laser due to spontaneous emission of the atoms. Two approaches are possible in describing "quantum" noise in a laser. In the framework of the Langevin noise theory, one can introduce random outside forces into the classical equations of motion of the electromagnetic field and radiating medium.[1,2] It is possible to find the intensity of random forces in the vicinity of the classical generation threshold from thermodynamic relations. However, in the region far from generation threshold, the noise level cannot be found from thermodynamic considerations. In this case it is necessary, generally speaking, to use the quantum approach developed in[3-8].

Below we shall follow the method proposed in[6], in which the exact solution of the problem of fluctuations of laser radiation was obtained in the quasiclassical limit. The simplest possible laser model with motionless atoms was considered. Now we shall study another limiting case, characteristic of gas lasers, in which the Doppler linewidth is much greater than the natural one:

$$kv_0\tau \gg 1, \tag{1}$$

where k is the wave number, v_0 is the thermal velocity of the atoms, and τ is the lifetime of an excited atom. In addition, the lifetime of a photon in the resonator $1/\nu$ will be considered long:

$$\nu\tau \ll 1. \tag{2}$$

In this case the radiation fluctuations are Markovian, and the distribution function of the photons obeys the Fokker-Planck equation.

Note that cases (1), (2) were considered approximately in[8-11]. Willis[8] and Klimontovich and Landa[9] did not assume the radiation to be weak; however, only the first harmonic was taken into account in the generation of a standing wave.

In a strong radiation field higher harmonics become important. A moving atom emits in the field of a standing wave a set of frequencies that are multiples of twice the Doppler frequency (see Eq. (20)). The nonmonochromaticity of the atomic radiation is the source of the additional "shot" noise, which is absent in the traveling wave case. These fluctuations are statistically independent of quantum fluctuations and have approximately the same magnitude.

In this paper we shall obtain an exact solution (for certain values of the laser parameters) of the quantum problem, with fluctuations from the spatial modulation of the medium taken into account. The case of a strong radiation field is treated in the greatest detail.

FUNDAMENTAL EQUATIONS

We start from the single-mode model of a laser in the form of a quantum oscillator of frequency Ω interacting with N spins with transition frequency ω_{ab}. The Hamiltonian of this system as the form ($\hbar = 1$)

$$H = H_0 + H_1, \quad H_0 = \Omega a^+a + {}^1/_2\omega_{ab}\sum \sigma_3^i,$$
$$H_1 = a^+\sum_i g_i(t)\sigma_-^i + \sum_i \bar{g}_i(t)\sigma_+^i. \tag{3}$$

Here a^+, a are the Bose photon creation and annihilation operators, $\sigma_\pm^i$ are matrices that flip the i-th spin up and down. The summation is over all spins. A bar over a symbol indicates the complex conjugate.

The time dependence of the coupling constants is associated with the thermal motion of the atoms and has a different form for different modes of oscillation of the electromagnetic field. In the two most important cases of standing and traveling waves we have respectively

$$g_i(t) = g\cos\varphi_i(t), \tag{4a}$$
$$g_i(t) = ge^{i\varphi_i(t)}, \tag{4b}$$

$$g = d_{ab}\sqrt{\omega_{ab}/2V}, \quad \varphi_i(t) = \varphi_{0i} + kv_it, \tag{A}$$

where d_{ab} is the dipole moment of the transition ab, V is the resonator volume, v_i is the velocity of the i-th atom, and φ_{0i} is a random phase associated with the position of the atom at the initial moment of time.

As was shown in[6], in the quasiclassical region all calculations are conveniently done in the representation of coherent states in which the annihilation operator is diagonal:

$$a|z\rangle = z|z\rangle, \; z = x + iy. \tag{5}$$

We shall also use a diagonal representation in which the density matrix of the quantum oscillator R(t) is of the form

$$R(t) = \int d^2z|z\rangle\langle z|\rho(z,t), \quad \int d^2z\rho(z,t) = 1. \tag{6}$$

We restrict attention now to the case of exact resonance[1)] $\Omega = \omega_{ab}$ and go over to the interaction representation. Then in the diagonal representation the Hamiltonian is converted to a differential non-Hermitian operator:

$$H \to \mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1, \quad \mathscr{H}_0 \equiv \sum_i \mathscr{H}_{0i} = \bar{z} \sum_i g_i(t)\sigma_-^i + \text{herm. conj.}$$

$$\mathscr{H}_1 = -\nabla \sum_i g_i(t)\sigma_-^i, \quad \nabla = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{7}$$

The equations of motion for the distribution function of the electromagnetic field $\rho(z, t)$ (continuity equation) and the density matrix of the i-th spin $r_i(z, t)$, according to[6], have the following form:

$$i\frac{\partial\rho}{\partial t} = \bar{\nabla}(V_+\rho) - \text{c.c.}, \tag{8}$$

$$V_+ = \sum_i g_i(t)\,\mathrm{Sp}(\sigma_+^i r_i) + i\nu\bar{z}, \tag{9}$$

$$\frac{\partial r_i}{\partial t} + \frac{r_i - r_{0i}}{\tau} = \frac{1}{i}[\mathscr{H}_{0i}, r_i] + \frac{1}{i}S_i, \tag{10}$$

$$S_i = \bar{g}_i(t) r_i(\sigma_+^i - \mathrm{Sp}(\sigma_+^i r_i))\nabla \ln\rho. \tag{11}$$

Here r_{i0} is the spin density matrix in the absence of radiation; for an excited atom $r_{i0} = \frac{1}{2}(1 + \sigma_3^i)$, for an unexcited one $r_{i0} = \frac{1}{2}(1 - \sigma_3^i)$. The total number of atoms is N, and $\Delta N > 0$ means an overpopulation of atoms. Equation (10) without quantum corrections describes the behavior of the spin in the classical field z. The quantum correction S_i which determines the dispersion of the photon distribution function is of order $1/|z| \ll 1$. Note that Eq. (10) maintains the normalization of the spin density matrix: $\mathrm{Sp}\, r_i(z, t) = 1$, since $\mathrm{Sp}\, S_i = 0$. Since relaxation in the spin system takes place much faster than the photon distribution function changes (in a time of order $1/\nu$), the problem reduces in essence to finding the diffusion coefficients in the Fokker-Planck equation for $\rho(z, t)$:

$$\frac{\partial\rho}{\partial t} = 2\nu\left\{\frac{\partial}{\partial\xi}\left(\xi A(\xi)\rho + B(\xi)\frac{\partial\rho}{\partial\xi}\right) + \frac{C(\xi)}{\xi^2}\frac{\partial^2\rho}{\partial\alpha^2}\right\}. \tag{12}$$

Here cylindrical coordinates are used: $z = \xi^{1/2}e^{i\alpha}$, $\bar{z} = \xi^{1/2}e^{-i\alpha}$; $A(\xi)$ is the radiation gain coefficient, $B(\xi)$ and $C(\xi)$ are the radial and azimuthal diffusion coefficients.

CALCULATION OF DIFFUSION COEFFICIENTS

1. We consider first the more complex case of a laser generating a standing wave. Leaving out for brevity's sake the index i in Eqs. (10) and (11), we write a system of equations for $P = 2ie^{2i\alpha}\,\mathrm{Sp}(\sigma_+ r)$ and $Q = \mathrm{Sp}(\sigma_3 r)$:

$$\frac{dP}{dt} + \frac{P}{\tau} = \omega_0\cos\varphi(t)Q + f_+(t), \quad f_+ = 2e^{2i\alpha}\,\mathrm{Sp}(\sigma_+ r), \tag{13}$$

$$\frac{dQ}{dt} + \frac{Q - q_0}{\tau} = -\omega_0\cos\varphi(t)\,\mathrm{Re}\,P + f(t), \quad f = \frac{1}{i}\mathrm{Sp}(\sigma_3 r). \tag{14}$$

The quantity $\omega_0 = 2g\xi^{1/2}$ is the characteristic frequency of the oscillations in the population of the upper and lower states of the atom in an applied field. For an excited atom $q_0 = +1$, for an unexcited one $q_0 = -1$.

We shall solve Eqs. (13) and (14) by perturbation theory. Let $P = p + \delta p$ and $Q = q + \delta q$, where p and q are classical quantities (polarization and overpopulation) that do not depend on the gradient of ρ, and δp and δq are small quantum additions of order $\xi^{-1/2}$ proportional to f_+ and f. In calculating f_+ and f, we may assume that $P \approx p$ and $Q \approx q$. Then, according to (11), we have

$$f = \frac{1}{2}\omega_0 p(1+q)\cos\varphi\frac{\partial\ln\rho}{\partial\xi}, \tag{15}$$

$$f_+ = f_1 + if_2, \quad f_1 = -\frac{1}{2}\omega_0(1 + q - p^2)\cos\varphi\frac{\partial\ln\rho}{\partial\xi}, \tag{16}$$

$$f_2 = \frac{1}{4}\omega_0(1+q)\cos\varphi\frac{\partial\ln\rho}{\xi\,\partial\alpha}. \tag{17}$$

The solution to Eqs. (14) and (15) can be found in the following fashion.[2)] Isolating first the time dependence in the form $e^{-t/\tau}$, we then transform from the variable t to the variable

$$\psi(t) = \frac{\omega_0}{\omega}\sin\varphi(t), \quad \omega = kv. \tag{18}$$

After this (14) and (15) are transformed to a system of equations with constant coefficients and a certain variable right hand side. From this we find

$$P(t) = \int_0^\infty dt_1 e^{-t_1/\tau}\left\{\left[\frac{q_0}{\tau} + f(t - t_1)\right]\sin[\psi(t) - \psi(t - t_1)] + f_1(t - t_1)\cos[\psi(t) - \psi(t - t_1)] + if_2(t - t_1)\right\}. \tag{19}$$

Here we consider only the stationary behavior of the spins. Hence the upper limit of integration over t_1 is replaced by infinity and the decaying solution of the homogeneous system (13), (14) is left out.

To find the classical radiation current p(t) is suffices to keep only the first term on the right of (19), which is proportional to q_0; the other terms give a contribution in the quantum correction to the polarization $\delta p(t)$. It is obvious that the function P(t) is periodic with period $2\pi/\omega$.

2. Let us now consider the classical radiation current more in detail. The single-particle contribution of interest to us in (9) has the form

$$gp\cos\varphi = a_0(\omega) + \sum_{n=1}^\infty (a_n(\omega) + a_{n-1}(\omega))e^{2in\varphi} + \text{c.c.}, \tag{20}$$

$$a_n(\omega) = \frac{gq_0}{2\tau}(-1)^n\int_0^\infty dt\, e^{-t/\tau}J_{2n+1}\left[\frac{2\omega_0}{\omega}\sin\left(\frac{\omega t}{2}\right)\right]e^{-i(n+1/2)\omega t}, \tag{21}$$

where $J_n(x)$ is a Bessel function of order n.

The first term on the right of (20) describes the stationary radiation current from one particle and is independent of the random phase φ_0. The remaining terms in (20) oscillate in time with frequency 2ω and depend on the initial phase. The reason for these oscillations is that the standing wave is the sum of two oppositely traveling waves, and the contribution to the radiation current from each traveling wave has the fre-

[1)]Under the conditions of the inequality (1) this limitation is in fact significant only for the standing wave.

[2)]Integration of the gas laser equations in the field of a standing wave was examined by Rautian. [12] An approximate method for the case of three different relaxation times was proposed by Stenholm and Lamb. [13] The classical theory of a gas laser in a standing wave field has been considered recently by Stenholm. [14]

quency of the wave for forward scattering and is shifted in frequency by 2ω for backward scattering. The total contribution to the radiation current on account of nonlinear effects is thus modulated with frequency 2ω. Clearly, this kind of effect is absent in the traveling wave case, since then we are only interested in forward scattering.

Summing (20), we see that the first term on the right of (21) gives a contribution of order ΔN, and the remaining ones, because of the random phases, of order $N^{1/2}$. The fluctuations arising in this way lead to an additional increase in the photon diffusion coefficient compared to the traveling wave case. To estimate this effect, we proceed as follows. We separate the total radiation current into two parts $V_+ = j_+ + \delta j_+$, with the classical polarization $p^i(t)$ contributing to j_+ and the quantum polarization $\delta p^i(t)$ to δj_+. Further, we set

$$j_+ = j_{0+} + \Delta j_+, \quad \Delta j_+ \ll j_{0+}, \quad j_{0+} = \frac{\bar{z}}{2i|z|}\sum (a_0(\omega_i) + \text{c.c.}) + i\nu\bar{z}, \tag{22}$$

$$\Delta j_+ = \frac{\bar{z}}{2i|z|}\sum_i\sum_{n=1}^{\infty}[(a_n(\omega_i) + a_{n-1}(\omega))e^{2in\varphi_i} + \text{c.c.}]. \tag{23}$$

The characteristic frequency of the fluctuations in the radiation current Δj_+ is of order $\omega \sim 1/\tau$ in weak field (near generation threshold, $\omega_0\tau \ll 1$) and $\omega \sim \omega_0$ far from the generation threshold ($\omega_0\tau \gg 1$).

Thus, because of the condition $\nu\tau \ll 1$, the frequency of radiation current fluctuations is large compared to the frequency of photon relaxation: $\omega \gg \nu$. Hence the photon distribution function can be written

$$\rho = \rho_0 + \Delta\rho; \ \Delta\rho \ll \rho_0, \ \langle\Delta\rho\rangle = \langle\Delta j_+\rangle = 0, \tag{24}$$

where ρ_0 is the quasistationary part of the photon distribution function, which varies in a time of the order $1/\nu$, and $\Delta\rho$ is an addition fluctuating with the frequency of the radiation current Δj_+. The angular brackets signify the average over the position of the particles (over the phases φ_{0i}), as well as over a time that is long relative to the period of radiation current fluctuations but short compared to the photon lifetime.

Linearizing (8) with respect to Δj_+, we find

$$i\frac{\partial\Delta\rho}{\partial t} = \overline{\nabla}(\Delta j_+\rho_0) - \text{c.c.} \tag{25}$$

We now average the initial equation (8) over phases and time and use the solution of equation (25). Omitting the zero index for the quasistationary distribution function, we have

$$i\frac{\partial\rho(t)}{\partial t} = \overline{\nabla}\left\{(j_{0+} + \langle\delta j_+\rangle)\rho(t) + \frac{1}{i}\int_{-\infty}^{t}dt'[\langle\Delta j_+(t)\Delta j_+(t')\rangle\overline{\nabla}\rho(t) - \langle\Delta j_+(t)\overline{\Delta j_+(t')}\rangle\nabla\rho(t)]\right\} - \text{c.c.} \tag{26}$$

We now calculate the current correlators. In doing this it is natural to assume that the initial phases φ_{0i} are distributed statistically independently of one another. Going from summation to integration over frequency, we have for quantities that are independent of the initial population of atoms q_0

$$\sum_i \ldots \to \frac{N}{\sqrt{\pi}kv_0}\int_{-\infty}^{+\infty}d\omega\ldots \tag{27}$$

Here the exponential in the Maxwellian distribution may be replaced by unity because of the inequality (1). For quantities containing q_{0i}, N is replaced by ΔN on the right of (27). Thus we obtain

$$\langle\Delta j_+(t)\Delta j_+(t')\rangle = -e^{-2i\alpha}\frac{N}{2\sqrt{\pi}kv_0}\int_{-\infty}^{+\infty}d\omega\sum_{n=1}^{\infty}|a_n(\omega) + a_{n-1}(\omega)|^2\cos[2n\omega(t-t')]. \tag{28}$$

In weak field ($\omega_0\tau \ll 1$) the correlator (28) decays like $e^{-2|t-t'|/\tau}$; in strong field ($\omega_0\tau \gg 1$) the decay frequency is determined by the characteristic frequency of the field ω_0. The diffusion coefficient corresponding to these fluctuations can be calculated for arbitrary field intensity:

$$\int_{-\infty}^{t}dt'\langle\Delta j_+(t)\Delta j_+(t')\rangle = -e^{-2i\alpha}\frac{\sqrt{\pi}Ng^2}{4kv_0}\sum_{n=1}^{\infty}\frac{|a_n(0) + a_{n-1}(0)|^2}{n}$$
$$= -e^{-2i\alpha}\frac{\sqrt{\pi}Ng^2(\mu-\mu^{-1})^2}{16kv_0(1+\omega_0^2\tau^2)}\ln\frac{1}{1-\mu^4},$$
$$\mu = \omega_0\tau/(1+\sqrt{1+\omega_0^2\tau^2}). \tag{29}$$

From this expression we see that the diffusion coefficient corresponding to "shot" noise due to spatial modulation of the medium is small in two limiting cases: low and high radiation intensity. In the first case ($\omega_0\tau \ll 1$) it is of order $(\omega_0\tau)^2$, and in the second ($\omega_0\tau \gg 1$) of order $(\omega_0\tau)^{-4}\ln\omega_0\tau$. When $\omega_0\tau \sim 1$ the contribution from (29) to the diffusion coefficient is of order of unity.

3. To find the gain coefficient $A(\xi)$ we go over to a calculation of the stationary classical radiation current. We have

$$j_{0+} = i\bar{z}\nu A(\xi), \quad A(\xi) = 1 - \frac{\Delta N}{\sqrt{\pi}kv_0\nu\sqrt{\xi}}\int_{-\infty}^{+\infty}\frac{d\omega\,\mathrm{Re}\,a_0(\omega)}{q_0}. \tag{30}$$

It is clear that $\mathrm{Re}\,a_0(\omega)$ is the frequency (velocity) distribution function of the radiating atoms.

In a weak radiation field we have to do mainly with the Lorentzian distribution

$$\mathrm{Re}\,a_0(\omega) = \frac{1}{2}\frac{gq_0\omega_0\tau}{1+\omega^2\tau^2}\left(1 - \frac{3\omega_0^2\tau^2}{1+4\omega^2\tau^2}\right). \tag{31}$$

The second term on the right of (31) is a small addition due to saturation effects. After integration over frequency we find

$$A(\xi) = 1 - \eta(1-\omega_0^2\tau^2), \quad \eta = \sqrt{\pi}g^2\Delta N/kv_0\nu. \tag{32}$$

Here we have introduced the generation parameter η; the condition $\eta \geq 1$ defines the region of classical generation, and $\eta = 1$ corresponds to threshold. Under stationary conditions the radiation intensity ξ_0 (without fluctuations) is determined from the condition $A(\xi_0) = 0$. Close to threshold ($\eta - 1 \ll 1$) we obtain from this

$$\xi_0 = (\eta - 1)/4g^2\tau^2. \tag{33}$$

In a strong radiation field ($\eta \gg 1$) we find from the formula (21) the following frequency distribution function for the atoms:

$$\mathrm{Re}\,a_0(\omega) = \frac{q_0 g}{4\omega_0\tau}\left(1 - J_0^2\left(\frac{\omega_0}{\omega}\right)\right). \tag{34}$$

We see from this that the frequency distribution is not monotonic; only at high frequencies $\omega \gg \omega_0$ do we have the Lorentzian asymptote Re $a_0(\omega) \sim \omega^{-2}$. Integrating (34) over ω, we obtain the gain coefficient and radiation intensity

$$A(\xi)=1-\frac{4\eta}{\pi^2\omega_0\tau}, \quad \xi_0=\frac{4\eta^2}{\pi^4 g^2\tau^2}. \tag{35}$$

The quadratic dependence of the radiation energy on the number of excited atoms ($\xi_0 \sim \eta^2$) comes about because in a strong radiation field the effective number of radiating atoms depends on the radiation intensity and is of order $\Delta N\omega_0/kv_0$.

4. Now we calculate the diffusion coefficients in Eq. (26) associated with the quantum radiation current $\langle\delta j_+\rangle$. In the general case it is impossible to calculate these analytically. Hence we examine the least studied limiting case of a high intensity field: $\eta \gg 1$. In this case the approximate solution of Eqs. (13) and (14) for p and q can be represented in the form of an expansion in $(\omega_0\tau)^{-1}$

$$p=p^{(0)}+p^{(1)}+\ldots, \quad q=q^{(0)}+q^{(1)}+\ldots,$$
$$p^{(0)}=q_0\sin\psi\cdot J_0(\omega_0/\omega), \quad q^{(0)}=q_0\cos\psi\cdot J_0(\omega_0/\omega),$$
$$p^{(1)}=\frac{q_0}{\omega\tau}[\sin\psi F-\cos\psi G], \quad q^{(1)}=\frac{q_0}{\omega\tau}[\cos\psi F+\sin\psi G],$$
$$F(t)=\int_0^{\varphi(t)} d\varphi(\cos\psi(\varphi)-J_0(\omega_0/\omega)), \quad G(t)=\int_{\pi/2}^{\varphi(t)} d\varphi\sin\psi(\varphi). \tag{36}$$

Substituting (16) and (17) into (19) for p(t), we use expansion (36). In this it is also necessary to expand the integral operator (19) in a series in $1/\tau$.

The contribution to δp from the radial probability flux in first order with respect to $1/\tau$ from terms in which only $p^{(0)}$ and $q^{(0)}$ enter, is zero. This is natural, since the average overpopulation of the atom in this approximation goes to zero. Hence to find the radial diffusion coefficient in (19) it is necessary to retain the higher orders of $1/\tau$. To calculate the azimuthal probability flux, one can set $q(t) \approx 0$ in expression (17).

Thus, the contribution to the radiation current from one particle with velocity $v = \omega/k$ we have, after averaging over the angle φ_0:

$$\langle\delta p\cos\varphi\rangle=\frac{1}{2\omega_0\tau}\Big[(1-J_0^2(x))\Big(q_0-1-J^2(x)+\frac{1}{2}J_0^2(x)(1-J_0(2x))\Big)$$
$$-2q_0xJ_1(x)J_0(x)\Big]\frac{\partial\ln\rho}{\partial\xi}+\frac{i\omega_0\tau/8}{1+\omega^2\tau^2}\frac{\partial\ln\rho}{\xi\,\partial\alpha}, \quad x=\omega_0/\omega. \tag{37}$$

It is seen from this that the diffusion coefficients have a different "spectral composition": in the radial coefficient the main contribution is borne by a frequency of order ω_0; in the azimuthal it is of order $1/\tau$. This is a characteristic of the generation of a standing wave at the line center. After integration over frequency in (37) we find the coefficients $B(\xi)$ and $C(\xi)$ for the standing wave (case a)

$$B(\xi)\approx\xi\frac{N}{\Delta N}, \quad C(\xi)=\frac{\eta\xi}{16}\frac{N}{\Delta N}. \tag{38a}$$

Note that the coefficient $B(\xi)$ is obtained by numerical integration of the square bracket on the right side of (37) (with 10% accuracy). Also, in the strong field region we are considering ($\eta \gg 1$) the contribution to the diffusion coefficient from noise due to modulation of the radiating medium (Eq. (29)), turns out to be small, of order $\eta^{-2}\ln\eta$.

5. We go now to a calculation of the coefficients A, B, and C for the case of a traveling wave (case b). Here (13) and (14) are easily solved exactly, since they are reduced by a simple unitary transformation to equations with constant coefficients. By averaging the diffusion coefficients found in[6] over frequency, we find for case b

$$A(\xi)=1-\eta/\omega_0\tau, \quad B(\xi)=\xi N/2\Delta N,$$
$$C(\xi)=\xi\eta^2N/16\Delta N. \tag{38b}$$

Knowing A, B, and C, we can easily find the fluctuations in the number of photons and the width of the laser line.

PHOTON DISTRIBUTION FUNCTION AND RADIATION LINE WIDTH

We expand the distribution function of a quantum oscillator in Fourier series:

$$\rho(z,t)=\frac{1}{\pi}\sum_{m=-\infty}^{+\infty}\rho_m(\xi,t)e^{-im\varphi}, \quad \rho_m=-\bar{\rho}_{-m}.$$

Restricting attention to the case of stationary generation, we observe that the distribution function for the number of photons $\rho_0(\xi)$ is a Gaussian in the quasiclassical region. Hence it is sufficient to indicate the average number $n = \xi_0$ and the dispersion Δn of the photons:

$$n=\int_0^\infty d\xi\xi\rho_0(\xi), \quad (\Delta n)^2=\int_0^\infty d\xi\xi^2\rho_0(\xi)+n-n^2.$$

In the quasistationary approximation ($\rho_{\pm1}$ = const $e^{-\Delta\nu t}\rho_0(\xi)$) we find the decay of the average field and thereby the radiation linewidth $\Delta\nu$.

Thus, for case a (standing wave) we find from (35) and (38a)

$$\left(\frac{\Delta n}{\sqrt{n}}\right)^2=1+\frac{2N}{\Delta N} \quad \frac{\Delta\nu}{\nu}=\frac{\eta N}{8n\Delta N}. \tag{39a}$$

The corresponding formulas for the traveling wave are

$$\left(\frac{\Delta n}{\sqrt{n}}\right)^2=1+\frac{N}{\Delta N}, \quad \frac{\Delta\nu}{\nu}=\frac{\eta^2N}{8n\Delta N}. \tag{39b}$$

In case b the linewidth is η times greater than in case a. And the linewidth of the traveling wave does not depend on the power ($n \sim \eta^2$). Note that this same result is derived from the model of a laser with motionless atoms.[6]

A comparison with the results of[9] can only be made in the strong field region. From this comparison it is seen that eliminating the higher harmonics of the spatial modulation of the medium leads to a radiation power that is too high—a factor $32/\pi^4 \sim 0.3$ is missing in the expression for power. According to[9], $(\Delta n)^2/n \sim \eta^2$ for $\eta \gg 1$, whereas from (39) we have $(\Delta n)^2/n \sim 1$. This discrepancy is apparently due to the use in[9] of an inexact correlator for the random forces. The expressions for $\Delta\nu$ in the case of the standing wave differ by the power of η but agree in the traveling wave case.

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