MOTION OF VORTICES IN TYPE II SUPERCONDUCTORS

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The motion of Abrikosov vortex lines in type II superconductors is considered phenomenologically for $\kappa \gg 1$ and $T = 0°K$ in the presence of a transport current which in the general case depends on the time. An effective Ohm's law is introduced for the transport current and the effective resistance in the mixed state is determined by means of this law. A phenomenological generalization of the results is proposed, in which elastic properties of the vortex line are taken into account, and the problem of the vibrational spectrum of the vortex line is solved. The existence of helicon oscillations is demonstrated. An expression is obtained for the relaxation time of such oscillations. The effective ac resistance arising in the mixed state as a result of the excitation of vibrational degrees of freedom of the vortex lines is calculated. The resistance exhibits a resonance behavior which depends on the frequency of the external harmonic current.

1. The present work is an attempt to consider from a phenomenological point of view the problem of the motion of Abrikosov vortex lines, which are nonstationary in the general case, in type II superconductors. Well-known experiments[1,2], which reveal the effective resistance in type II superconductors in the mixed state (the resistance effect), have led to the development of representations in which the resistance is associated with the motion of the vortex lines under the action of the external current. Experiments on the induced motion of vortices[3,4] evidently confirm the existence of such a connection. Therefore, it is important to establish the dependence of the rate of migration of the vortices on the external current. On this basis, several phenomenological models have been proposed which, however, contain, in our opinion, a number of invalid assumptions.[5-7] In particular, Bardeen and Stephen,[5] and also Nozières and Vinen[6], used the representation that, in the motion of a vortex with a constant velocity, the external current penetrates into the normal center of the vortex without change. This assumption undoubtedly gives a lucid interpretation of the nature of the losses in the mixed state; however, as we shall see below, there is no necessity of bringing in this assumption to explain the resistance effect. Furthermore, as has been pointed out in[6,7], the assumption of the existence of local equilibrium of the normal electrons with the lattice in the core of the vortex is doubtful. Further, it has been required by the authors of[5,6] that the condition $\varphi_0 = 2\pi\xi^2 H_{c2}$, which is given by the Ginzburg-Landau theory for fields $H \ll H_{c2}$, be extended over the entire range of fields up to $H \sim H_{c2}$. This continuation seems arbitrary to us.

Besides the instances noted, there are a number of circumstances which stimulated this research to a significant degree. First, the phenomenological model should lead to equations which would determine the electric and magnetic fields in each element of volume of the superconducting phase, and any macroscopic effects for the superconductor as a whole should be obtained as a result of appropriate averagings. Second, the motion of the vortex leads of necessity to the induction of an electric field in each element of volume of the superconducting phase. The presence of such a field is the reason for the resistance effect. Third, the empirical relation for the effective resistance $\rho_f = \rho_n H/H_{c2}$ in the mixed state is applicable over such a wide range of conditions[7] that it would be desirable to have such a scheme for the calculation of the resistivity which would not depend on the specific model of the vortex or on the conditions of its motion.

In Sec. 2 of this paper, assuming axial symmetry of the magnetic field of the vortex, a temperature $T = 0°K$ and $\kappa = \lambda/\xi \gg 1$, we introduce the relation (2), which connects the velocity of motion of the vortex, the magnetic induction and the external current in each element of volume of the s phase. This equation is equivalent to an effective Ohm's law for the external current. An expression is derived for the effective resistance in the mixed state by means of this equation.

In Sec. 3, we propose a phenomenological generalization of relation (2) for an account of the elastic properties of the vortex line. On the basis of this generalization, the problem of the vibrational spectrum of the vortex line is solved and the existence is shown of helicon oscillations, the n-th harmonics of which decay with the relaxation time:

$$\tau_n = Ne\varphi_0\omega_{c2}\tau / cTk_n^2.$$

In addition, the effective resistance arising as the result of the excitation of the vibrational degrees of freedom of the vortex line is calculated. An analysis of the resonance behavior of the effective resistance as a function of the frequency of the external harmonic current is carried out.

2. We propose to limit ourselves to the case of motion of vortices in homogeneous crystals, and to consider the dissipative effects in the linear approximation. Considering the ensemble of Cooper pairs as a charged liquid, we introduce the velocity of motion of the vortex $v_L(x, y, z, t)$ as the velocity of that underformed contour, the magnetic induction flux through which remains constant during motion of the vortex, i.e.,

$$\partial \mathbf{b}_s / \partial t - \mathrm{rot}[\mathbf{v}_L \mathbf{b}_s] = 0. \qquad (1)^*$$

The connection of the migration velocity of points of the vortex with the velocity $v_T = j_T/Ne$ (where j_T is the external current should be established independently of the system of electrodynamic equations for the fields, and should be determined at each point of the s phase. The conclusive experiments on the resistivity in type II superconductors in the mixed state can be explained only when the given connection is linear both in the velocity and in the value of the magnetic induction (in the axially symmetric case considered). It is not difficult to see that the nonlinearity of dependence of v_L on v_T and (or) b_s would lead to a nonlinear dependence of ρ_f on the external field and (or) the external current, since experiment shows that $\rho_f = \rho_n H/H_{c2}$. Therefore, the desired connection can be written in general form as follows:

$$[\mathbf{v}_L \mathbf{b}_s] + \alpha[\mathbf{v}_T \mathbf{b}_s] + \beta b_s \mathbf{v}_T + \gamma b_s \mathbf{v}_L = 0, \qquad (2)$$

where α, β, γ are unknown parameters which depend perhaps on $\omega_{c2}\tau$ and the ratio of the macroscopic fields (for example, H/H_{c2}). We note that the requirement of linearity eliminates the possibility of the addition to (2) of terms containing the density of the "London" current, which would have led to a nonlinear dependence of ρ_f on the external current.

Equation (2) represents the independent fundamental relation of the theory and can be written without relation to the specific mechanism of interaction of the electrons with the lattice and with the fields. It is equivalent to

$$\mathbf{j}_T = \sigma_f(\mathbf{E} + c^{-1}\delta[\mathbf{v}_T \mathbf{H}]), \qquad (3)$$

where $\mathbf{H}$ is the external field, $\sigma_f = Nec/H\mu$ and, as we shall see below, plays the role of the effective conductivity of the superconductor in the mixed state, $\mu = (\beta - \alpha\gamma)/(1 + \gamma^2)$, $\delta = -(\alpha + \beta\gamma)/(1 + \gamma^2)$ and $\mathbf{E} = -\mathbf{v}_L \times \mathbf{H}/c$ is the effective macroscopic electric field induced by the vortex motion. The coefficients α, β and γ are unique quantities, depending on the specific mechanism of the interaction of the electrons with the lattice and with the fields. We note that the results of the well-known models[5-12] can be obtained by the selection of specific values of these coefficients. Thus, for example, the ratio of the energy loss per period to the energy of oscillation of the vortex, which was introduced in[9] can be obtained from our results (see Eq. (13) below) if we take $\delta = N_s/2N_c$ (where N_s is the concentration of the s electrons and N_c the concentration of normal electrons in the core of the vortex). The dispersion laws set down in[8,12] are obtained from our results (see Eq. (12) below) if we take $\delta = 1$, and the dispersion law obtained in[11] for a choice $\delta = 1 + 0.87/\ln\kappa$.

The solution of (2) has the form

$$v_{Lx} = \delta v_{Tx} + \mu v_{Ty}, \quad v_{Ly} = -\mu v_{Tx} + \delta v_{Ty}, \qquad (4)$$

which gives for the Hall angle

$$\mathrm{tg}\,\theta_H = v_{Lx}/v_{Ly}\big|_{v_{Ty}=0} = -\delta/\mu. \qquad (5)$$

It should be emphasized that, in accord with (2), the relation of Bardeen and Stephen for the Hall angle, $\tan\theta_H = -\omega_{c2}\tau H/H_{c2}$, contradicts the requirement of complete entrainment of the vortex by the external current in the limit of an ideally pure material ($\omega_{c2}\tau \gg 1$). Actually, we have in the given limit, from (2), $\lim \delta = 1$ as $\omega_{c2}\tau \to \infty$. But this contradicts the value $\delta = H/H_{c2}$, which follows from the relation of Bardeen and Stephen (see Eq. (9) below).

We proceed to the problem of the power dissipated. Substituting (1) in the Maxwell equation curl $\mathbf{e}_s = -c^{-1}\partial \mathbf{b}_s/\partial t$, we get

$$\mathbf{e}_s = -c^{-1}[\mathbf{v}_L \mathbf{b}_s] + \mathrm{grad}\, u,$$

where u is a certain potential. It is not difficult to see that the requirements of a linear dependence of ρ_f on the external field, axial symmetry of the magnetic field of the vortex, an ideal conductivity for the vortices at rest, all lead to u = const. In this connection, the electric field induced in every element of the volume of s phase by the motion of the ith vortex is determined by the relation

$$\mathbf{e}_s^{(i)} = -c^{-1}[\mathbf{v}_L \mathbf{b}_s^{(i)}]. \qquad (6)$$

Therefore, the dissipated power density in a given volume element in a phase, due to motion of the i-th vortex, is

$$w^{(i)} = \mathbf{j}_T \mathbf{e}_s^{(i)} = \frac{b_s^{(i)}\mu}{Nec} j_T^2.$$

Averaging this over the entire volume of the superconductor, and taking it into account that the flux $\varphi^{(i)}$ of the magnetic field of the i-th vortex is

$$\varphi^{(i)} = \int_{(\Sigma)} \mathbf{b}_s^{(i)}\, d\Sigma,$$

we get for the mean power density dissipated throughout the specimen, after summing over all vortices, with account of the fact that $\Sigma_i \varphi^{(i)} = HS$ (where S is the cross-section area of the sample perpendicular to the magnetic field)

$$\langle w \rangle = \rho_n \omega_{c2}\tau\mu j_T^2 H / H_{c2}. \qquad (7)$$

By identifying the coefficient of j_T^2 with the effective specific resistance ρ_f, we get (compared with (3))

$$\rho_f = \rho_n \omega_{c2}\tau\mu H / H_{c2}. \qquad (8)$$

We note that there are but two parameters in the proposed phenomenology: μ and δ. Using these results of experiments on the resistivity,[1,2] we fix one of them, say

$$\mu = 1/\omega_{c2}\tau. \qquad (9)$$

The second parameter left arbitrary by us, δ, can be established, for example, from experiments on the Hall effect in a mixed state (see Eq. (5)) or from experiments which could clarify the dispersion law for helicon oscillations of the vortex line (see Eq. (12) below). We have deliberatly not fixed the value of δ since it is quite well known that the choice of the specific model in a number of works (see, for example,[5,6]) has not been uniquely verified by the most recent experiments.[13-16]

3. We proceed to the problem of taking into account the elastic properties of the vortex line. An attractive idea (see the reference in[6] to the unpublished work[17]) on the effective replacement of the tensile stress acting in

*rot $[\mathbf{v}_L \mathbf{b}_s] \equiv$ curl $[\mathbf{v}_L \times \mathbf{b}_s]$.

a vortex displaced from equilibrium by the Lorentz force (here and below, we shall be concerned with small displacements). This substitution is possible if we introduce the equivalent field of the velocities v_ρ of circular current of the s fluid, so selected that

$$\frac{Ne}{c}[v_e\varphi_0] = T\frac{\partial^2 s}{\partial z^2}, \tag{10}$$

where $\mathbf{s}(\mathbf{r}, t)$ is the vector displacement of the given point of the vortex from the equilibrium position, i.e., the x axis, and the stress $T = (\varphi_0/4\pi\lambda)^2\ln$.[18] The displaced vortex must then be considered as absolutely inelastic. On it will act the Lorentz force, which is exactly equal to the restoring force which would have acted on exactly the same displaced vortex with given elastic properties. The introduction of v_ρ leads to the induction of an additional electric field (inasmuch as the character of the vortex motion is changed which, in turn, leads to additional energy dissipation of the external current, which is associated with the excitation of vibrational degrees of freedom of the vortex line. We note that the hypothesis (10) is equivalent to an assumption on the result (but of course not on the mechanism) of the action of the external current on the vibrational degrees of freedom of the vortex.

Taking into account all that has been said, and replacing the velocity v_T in (2) corresponding to the external current by the sum $v = v_T + v_\rho$, where v_ρ is determined from (10), it is not difficult to obtain

$$\gamma\frac{\partial \mathbf{s}}{\partial t} + \left[\frac{\partial \mathbf{s}}{\partial t}\mathbf{z}\right] + \alpha[\mathbf{v}\mathbf{z}] + \beta\mathbf{v} = \frac{cT}{Ne\varphi_0}\left(\beta\left[\frac{\partial^2 \mathbf{s}}{\partial z^2}\mathbf{z}\right] - \alpha\frac{\partial^2 \mathbf{s}}{\partial z^2}\right), \tag{11}$$

where $\partial \mathbf{s}/\partial t = v_L$.

Equation (11) is in fact the generalized Ohm's law for the external field with account of the elastic properties of the vortex line. It must be emphasized that the hypothesis (10) allowed us to fix the coefficients of the terms containing the vector $\mathbf{s}$ in the general equation for the connection of the external current with the fields, with account of the vibrational degrees of freedom of the vortex, which one could have written down by making use of the considerations of linearity and translational invariance. Thus, (10) allows us to proceed from this general equation to Eq. (11). We consider the solution of this equation in the presence of a harmonic external current and for the boundary conditions

$$\mathbf{s}(\mathbf{r}, t)\,|_{z=0;\, z=l} = v_L{}^0 t,$$

where l is the length of the vortex and v_L^0 is the transport velocity of the vortex when it is considered as absolutely rigid, under the action of the constant components of the external current.

The eigenfrequency part of the solution has the form

$$u = \sum_{k=0}^{\infty} C_{2k+1}(x, y)\exp(i\Omega_{2k+1}t)\sin\frac{(2k+1)\pi}{l}z,$$

where

$$u = s_x + is_y, \quad \Omega_n = \frac{cT}{Ne\varphi_0}\left(\frac{n\pi}{l}\right)^2(\delta + i\mu).$$

Thus we arrive at the existence of helicon oscillations of the vortex line with eigenfrequencies

$$\omega_n = \frac{cT}{Ne\varphi_0}\left(\frac{n\pi}{l}\right)^2\delta, \tag{12}$$

which decay with relaxation time

$$\tau_n = \frac{Ne\varphi_0}{cT}\left(\frac{l}{n\pi}\right)^2\omega_{c2}\tau. \tag{13}$$

(As was mentioned above, for a definite choice of δ the results transform into the results of[8-12]). The solution which describes the stimulated oscillations of the vortex under the action of the variable component of the external current has the form

$$u^\sim = \sum_{n=1,\, k=0}^{\infty}\delta_{n;\,2k+1}\frac{4(i\delta - \mu)w^\sim\omega_0\sin(\omega_0 t + \varphi_n)}{n\pi[\omega_0{}^2 + (i\delta - \mu)^2(cT/Ne\varphi_0)^2k_n{}^4]}\sin k_n z,$$

where

$$\operatorname{tg}\varphi_n = [Ne\varphi_0 l^2\omega_0(i\delta - \mu)]\,/\,(n\pi)^2cT(\mu^2 + \delta^2)$$

and ω_0 is the frequency of the variable part of the external current. Thence, in accord with the methodology set forth in Sec. 2, we find that the variable and constant components of the external current dissipate independently of one another in the same measure as each of these components contribute to the vortex motion. The dissipation produced by the constant component of the external current is connected with the motion of each vortex as a whole and for it the effective resistance is given as before by Eq. (8). The dissipation of the energy of the variable component of the external current takes place with an effective resistivity

$$\rho_f^\sim = \frac{8H\mu}{\pi^2 Nec}\sum_{n=1,\, k=0}^{\infty}\delta_{n;\,2k+1}\frac{\omega_0{}^2(\omega_0{}^2 + \tilde\omega_n{}^2 + \omega_n{}^2)}{n^2[(\omega_0{}^2 + \tilde\omega_n{}^2 - \omega_n{}^2)^2 + 4\tilde\omega_n{}^2\omega_n{}^2]}, \tag{14}$$

where the notation $\tilde\omega_n = 1/\tau_n$ has been introduced.

The resonance behavior of $\tilde\rho_f$ as a function of frequency of the external current is obvious. One can show that the resonance frequencies are equal to

$$\omega_0^{\text{res}} = \frac{cTk_n{}^2}{Ne\varphi_0\sin\theta_H\sqrt{2\sin\theta_H - 1}}. \tag{15}$$

As is seen from this formula, resonances in the harmonics can take place only for materials with $\theta > \pi/6$ and $\omega_0^{\text{res}} \gtrsim 10$ Hz for specimens with $l \sim 1$ mm. The analogy should be noted between the expressions obtained for ω_n and τ_n in our model and the theory of helicons in normal metals.[19] We note that the results obtained permit us to carry out the calculation of experiments which have as their aim the investigation of the character of the oscillations of vortex lines from measurements of the resistivity which accompanies such oscillations.

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73