DAMPING OF MAGNETIC SURFACE LEVELS IN A SUPERCONDUCTOR

A. Ya. BLANK and L. A. FAL'KOVSKIĬ

L. D. Landau Institute of Theoretical Physics, U.S.S.R. Academy of Sciences

Submitted August 21, 1970

Zh. Eksp. Teor. Fiz. 60, 797–805 (February, 1971)

The damping of the magnetic surface levels in a superconductor associated with scattering of the excitations by surface roughnesses is calculated. In the quasiclassical region of the spectrum, the damping increases with the glancing angle.

In recent theoretical articles[1-4] it has been shown that discrete quantum levels, located below the energy gap, may arise in the single-particle excitation spectrum of a superconductor. The discrete levels are due to the finite motion of the excitations in the potential well created by the magnetic field near the surface of the superconductor. These levels are in many respects analogous to the discrete levels in a normal metal, corresponding to the periodic motion of the electrons along the surface in the magnetic field.[5] The specific properties of quantization in a superconductor consist in the fact that in the latter a pair of excitations undergo periodic motion—namely, an electron and a "hole" which are bound to one another by the potential Δ and which possess a nonquadratic dispersion law which is characteristic of a superconductor. The motion of the pair is confined between the surface of the superconductor and the turning point in the magnetic field. The turning point, which is determined from the condition that the velocities of the quasiparticles vanish, is simultaneously a point of total reflection—at this point the mutual transformation of the electron and "hole" takes place. A typical trajectory of the motion of the excitations near the surface of a superconductor in the presence of a magnetic field is shown in Fig. 1. In the quasiclassical approximation the corresponding area in phase space is quantized. The discrete levels are characterized by the following set of quantum numbers: the number n and the components of the momentum in the plane of the surface, $\mathbf{p} = (p_y, p_z)$.

The existence of quantized surface levels shoud lead to a number of resonance effects, in particular, to the absorption of the energy of an electromagnetic field at frequencies below the energy gap. Experimentally, in a number of metals a sharp increase in the absorption has been observed at the frequency $\Delta' = \Delta - \Omega\delta p_0$, corresponding to the energy gap Δ being shifted by an amount which is of the order of the interaction energy of the excitations with the magnetic field[6,7] (here $\Omega = eH/mc$ is the cyclotron frequency, H denotes the value of the field on the surface of the metal, δ denotes the penetration depth, and p_0 is the Fermi momentum). This bump (in the absorption curve) is caused by excitations moving in parallel with the surface of the metal. The existence of trajectories of the type indicated on the figure and the corresponding quantum levels might appear in the fine structure of the resonant bump. As the calculations show, the conditions for the existence of discrete levels are rather stringent and include the

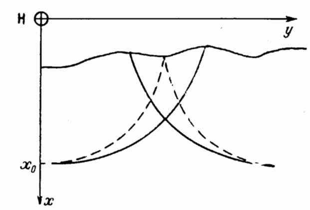

FIG. 1

requirement of a sufficiently large penetration depth (see [3]). The absorption of a high-frequency field corresponding to the surface levels can apparently be observed in thin superconducting films, corresponding to this requirement.

A calculation of the damping of the surface excitations is of natural interest. At low temperatures and in sufficiently pure samples, the collisions of the quasiparticles with surface irregularities are the major mechanism for broadening of the levels. Although collisions with the surface are elastic in nature, they lead to a violation of the periodicity of the motion of the excitations, and by the same token create a finite probability for a transition to other states.

The surface roughness is characterized by an average height a and by an average length d of the planar segments. It is difficult to say anything definite about the magnitude of these parameters at the present time, but in any case for specular samples in the optical region the parameter a is smaller than the corresponding wavelength of light and, probably, is of atomic dimensions. The characteristic parameter of length in a superconductor, determining the wavelength of the surface states—is the penetration depth δ. The small parameter a/δ appearing here guarantees almost complete specular nature and makes it possible to carry out an expansion of all physical quantities. Such an investigation was carried out in article [8] in the case of a normal metal, and below its generalization to the case of the Gor'kov equations will be developed.

1. SPECTRUM OF THE SURFACE EXCITATIONS ASSOCIATED WITH SPECULAR REFLECTION

Let the superconductor occupy the half-space $x > 0$, the z axis is directed along the magnetic field whose vector potential we take in the form $\mathbf{A} = (0, A, 0)$,

$$A(x) = -\int_x^\infty H(x')\,dx'.$$

The behavior of a superconductor in a magnetic field is, as is well known, determined by a system of homogeneous equations for the two-component wave function $\Psi = \begin{pmatrix} g \\ f \end{pmatrix}$:

$$\partial^2\Psi / \partial x^2 + \hat{P}(x)\Psi = 0,$$

where

$$\hat{P}(x) = \begin{pmatrix} u_+(x) & 1 \\ -1 & u_-(x) \end{pmatrix}, \quad u_\pm(x) = \alpha \pm (\varepsilon + a(x)),$$
$$\alpha = \mu - p^2/2m, \quad a(x) = (e/mc)A(x)p_y. \tag{1}$$

We have neglected the term which is quadratic in the vector potential. The coordinate x is measured in units of $\hbar/\sqrt{2m\Delta}$, the momentum in units of $\sqrt{2m\Delta}$, and the energy in units of Δ. Here and below we shall use the two-dimensional vector **p** with components p_y and p_z: $p^2 = p_y^2 + p_z^2$.

In fields which are smaller than the critical field, one can regard the energy gap Δ as independent of the magnetic field.

Let us write the condition for specular reflection at the boundary of the superconductor for $x = 0$ in the form

$$\Psi(0) = 0. \tag{2}$$

The quasiclassical solution of Eqs. (1), obtained in articles [2,3], has the form

$$\Psi = \frac{c}{\sqrt{v}} \exp\left\{ i \int_{x_0}^{x} p(x)\,dx \right\}, \tag{3}$$

where $p(x)$ takes the four values $\pm p_\pm(x)$:

$$p_\pm^2(x) = \alpha \pm \sqrt{(\varepsilon + a)^2 - 1} \tag{4}$$

and correspondingly the velocity $v = \partial\epsilon/\partial p$. The turning point x_0 is determined from the condition that the square root in formula (4) vanishes.

The solution of Eq. (1) which is finite as $x \to \infty$ may be written down in the following form:

a) in the classically inaccessible region

$$g(x) = \frac{1}{\sqrt{|v_+|}} \exp\left\{ i \int_{x_0}^{x} p_+\,dx \right\}, \tag{5}$$

b) in the oscillatory region

$$g(x) = \frac{1}{\sqrt{|v_+|}} \exp\left\{ i\left(\int_{x_0}^{x} p_+\,dx + \frac{\pi}{4} \right)\right\} + \frac{1}{\sqrt{|v_-|}} \exp\left\{ i\left(\int_{x_0}^{x} p_-\,dx - \frac{\pi}{4} \right)\right\}. \tag{6}$$

A second linearly independent solution, which is also attenuated as $x \to \infty$, is obtained from expressions (5) and (6) by complex conjugation. The functions $f(x)$ are calculated from the functions (5) and (6) with the aid of Eq. (1).

The boundary conditions (2) lead to the following quantization rules:

$$S = \int_0^{x_0} [p_+(x) - p_-(x)]\,dx = \left(n + \frac{1}{2}\right)\pi. \tag{7}$$

For estimates we shall use the London law for the decrease of the field strength inside a superconductor

$$a = \Omega\delta p_y e^{-x/\delta}. \tag{8}$$

In formula (4) the square root changes by an amount of the order of unity, but in the region we are interested in of real values of the momentum $p_\pm$ the parameter α may change from a quantity of the order of unity to $\mu/\Delta \gg 1$. Values of $\alpha \sim 1$ correspond to small glancing angles of the particles along the surface, i.e., $\varphi \sim p_\pm/p_0 \sim (\delta/R)^{1/2}$ ($R = v/\Omega$ denotes the cyclotron radius). The discrete levels are distributed in the interval $1 > \epsilon > 1 - \Omega\delta p_y$. Hence one can see that the region of variation of p_y near p_0 is the most important, and the dependence of ϵ on p_y in formula (7) is basically determined by the parameter α.

Let us estimate the order of magnitude of the distance between the levels. For this we shall use, for example, the approximate expression (7) for the spectrum:

$$2^{3/2}(3\pi)^{-1}\frac{\alpha^{-1/2}}{\Omega p_0}(\varepsilon + \Omega\delta p_0 - 1)^{3/2} = \left(n + \frac{1}{2}\right), \tag{9}$$

which is valid for $\alpha \gg 1$ near the lower boundary of the spectrum $1 - \epsilon \gg \epsilon - 1 + \Omega\delta p_0$. In this region the distance between the levels is given by

$$\frac{\partial\varepsilon}{\partial n} = \pi\Omega p_0 \left(\frac{\alpha/2}{\varepsilon + \Omega\delta p_0 - 1} \right)^{1/2}. \tag{10}$$

For $\alpha \sim \epsilon + \Omega\delta p_0 - 1 \sim 1$ the number of levels is of the order of $(\pi\Omega p_0)^{-1}$, i.e., in dimensional units $n \sim \delta(m\Omega\delta p_0)^{1/2}/\hbar$.

We notice the existence of a certain selection rule for the levels. It is obvious that only those states are realized which correspond to nontrivial eigenfunctions $\Psi_n(x)$. The solution of Eq. (1), which is finite for $x \to \infty$, corresponds to a linear combination $C_1\Psi(x) + C_2\Psi^*(x)$, where $\Psi(x)$ is found with the aid of Eq. (6). This solution satisfies the boundary condition (2) and corresponds to the discrete spectrum (7). If, however, for some eigenvalue not only the eigenfunction itself but also its derivative vanishes on the boundary then, as is well known, the corresponding solution of the second order equation (1) is trivial. Calculating, for example, the value of the derivative $C_1g'(0) + C_2g^{*\prime}(0)$ for the condition $C_1g(0) + C_2g^*(0) = 0$, with Eqs. (6) and (7) taken into account we find

$$C_1g'(0) + C_2g^{*\prime}(0) \sim 1 + \frac{|v_+| + |v_-|}{2\sqrt{|v_+||v_-|}}(-1)^n.$$

The derivative $C_1f'(0) + C_2f^{*\prime}(0)$ is proportional to this same expression. For even values of n this quantity is of the order of unity, for odd values it is small within the bounds of smallness $1/\alpha$. Thus, there is an approximate selection rule by virtue of which the states with even values of n turn out to be preferred. The selection rule, just like the quantization rule, remains exactly the same if one requires $\Psi'(0) = 0$ instead of the boundary condition (2).

2. THE GREEN'S FUNCTION OF AN UNBOUNDED SUPERCONDUCTOR

In the following section we shall find the Green's function G which satisfies the boundary condition on a rough surface and which describes the surface states. For this purpose we need the Green's function G_∞ of Eq. (1), extended over all space. It is convenient to continue the potential $A(x)$ as an even function of x into the region $x < 0$ and to seek the Green's function $G_\infty(\mathbf{r}, \mathbf{r}')$ which vanishes as $x \to \pm\infty$. In view of the vector nature of Eq. (1), the Green's function is given by the matrix

$$\hat{\mathscr{G}}_\infty(\mathbf{rr}') = \begin{pmatrix} G_\infty(\mathbf{rr}') & F_\infty(\mathbf{rr}') \\ F_\infty^+(\mathbf{rr}') & -G_\infty(\mathbf{r}'\mathbf{r}) \end{pmatrix}. \quad (11)$$

See [9] for the definition and determination of the functions G and F.

In view of the homogeneity of the problem with respect to the variable $\mathbf{s} = (y, z)$ the function $\hat{\mathscr{G}}_\infty(\mathbf{r}, \mathbf{r}')$ depends on $\mathbf{s}$ and $\mathbf{s}'$ only in terms of the difference $\mathbf{s} - \mathbf{s}'$. Let us introduce the Fourier transform of the Green's function according to the formula

$$\hat{\mathscr{G}}_\infty(\mathbf{rr}') = \int \hat{\mathscr{G}}_\infty(xx'\mathbf{p}) \exp[i\mathbf{p}(\mathbf{s}-\mathbf{s}')] \frac{d^2p}{(2\pi)^2}. \quad (12)$$

The function $\hat{\mathscr{G}}_\infty(xx'\mathbf{p})$ satisfies the equation

$$\frac{d^2\hat{\mathscr{G}}_\infty(xx'\mathbf{p})}{dx^2} + \hat{P}(x)\hat{\mathscr{G}}_\infty(xx'\mathbf{p}) = -\delta(x-x')\hat{I}, \quad (13)$$

where $\hat{I}$ denotes the unit matrix.

Let $\hat{Y}_1(x)$ and $\hat{Y}_2(x)$ be linearly independent solutions of the homogeneous equation corresponding to (13) such that $\hat{Y}_1(x) \to 0$ as $x \to +\infty$ and $\hat{Y}_2(x) \to 0$ as $x \to -\infty$. We shall seek the solution of Eq. (13) in the form

$$\hat{\mathscr{G}}_\infty(xx'p) = \begin{cases} \hat{Y}_1(x)\hat{C}_1(x'), & x > x' \\ \hat{Y}_2(x)\hat{C}_2(x'), & x' > x \end{cases}. \quad (14)$$

As is evident from Eq. (13) the function $\hat{\mathscr{G}}_\infty(xx')$ is continuous at the point $x = x'$, but its first derivative undergoes a unit discontinuity:

$$\hat{Y}_1\hat{C}_1 - \hat{Y}_2\hat{C}_2 = 0, \quad \hat{Y}_1'\hat{C}_1 - \hat{Y}_2'\hat{C}_2 = -\hat{I} \quad \text{for } x = x'. \quad (15)$$

Solving Eqs. (15) with respect to $\hat{C}_1$ and $\hat{C}_2$, we find

$$\hat{\mathscr{G}}_\infty(xx') = \begin{cases} \hat{Y}_1(x)[\hat{Y}_2'\hat{Y}_2^{-1}\hat{Y}_1 - \hat{Y}_1']^{-1}(x'), & x > x' \\ -\hat{Y}_2(x)[\hat{Y}_1'\hat{Y}_1^{-1}\hat{Y}_2 - \hat{Y}_2']^{-1}(x'), & x' > x \end{cases}. \quad (16)$$

Let us choose the solutions $\hat{Y}_1(x)$ and $\hat{Y}_2(x)$ in the form (see Eqs. (5) and (6))

$$\hat{Y}_1(x) = \begin{pmatrix} g(x) & g^*(x) \\ f(x) & f^*(x) \end{pmatrix}, \quad \hat{Y}_2(x) = \hat{Y}_1(-x). \quad (17)$$

With the aid of Eqs. (16) and (17) it is not difficult to calculate the explicit form of the functions $\hat{C}_1(x)$ and $\hat{C}_2(x)$. For example, the matrix $\hat{C}_1(x)$, which is the only one we need later on, turns out to be given by

$$\hat{C}_1(x) = -W^{-1}\hat{D}(x), \quad (18)$$

where the elements of the matrix D_{ik} are the cofactors to the elements of the matrix $(Y_1')_{ki}$ in the determinant

$$W = \begin{vmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_1' & \hat{Y}_2' \end{vmatrix} = \begin{vmatrix} g(x) & g^*(x) & g(-x) & g^*(-x) \\ f(x) & f^*(x) & f(-x) & f^*(-x) \\ g'(x) & g^{*\prime}(x) & g'(-x) & g^{*\prime}(-x) \\ f'(x) & f^{*\prime}(x) & f'(-x) & f^{*\prime}(-x) \end{vmatrix} \quad (19)$$

(compare with [10]).

3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

As is well known, the spectrum and the damping of the surface excitations are determined by the poles of the single-particle Green's function $\hat{\mathscr{G}}(xx'\mathbf{p})$ which satisfies the required boundary conditions.

Let the rough surface of the superconductor be given by the equation $x = \xi(\mathbf{s})$. We shall assume that the wave function of the particle and correspondingly its Green's function vanish on the surface of the superconductor,

$$\hat{\mathscr{G}}(\xi(\mathbf{s}), \mathbf{s}; x'\mathbf{s}') = 0. \quad (20)$$

Since the characteristic distance over which the wave functions of the surface excitations vary is of the order of the penetration depth, one can expand the boundary condition in powers of ξ/δ:

$$\hat{\mathscr{G}}(0\mathbf{s}; x'\mathbf{s}') + \xi(\mathbf{s})\frac{\partial\hat{\mathscr{G}}(0\mathbf{s}; x'\mathbf{s}')}{\partial x} = 0. \quad (21)$$

The quantity $\xi(\mathbf{s})$ is a random function of the coordinates of the surface, and in what follows an averaging will be carried out over the set $\xi(\mathbf{s})$. In this connection all observable quantities will be expressed in terms of the binary correlation function

$$\langle \xi(\mathbf{s})\xi(\mathbf{s}')\rangle = \xi_2(\mathbf{s}-\mathbf{s}'). \quad (22)$$

We shall seek the Green's function $\hat{\mathscr{G}}(\mathbf{rr}')$ satisfying the boundary condition (21) in the form

$$\hat{\mathscr{G}}(\mathbf{rr}') = \hat{\mathscr{G}}_\infty(\mathbf{rr}') + \int \hat{\mathscr{G}}_\infty(x\mathbf{s}; 0\mathbf{s}'')\hat{\mu}(0\mathbf{s}''; x'\mathbf{s}')\, d^2s'', \quad (23)$$

where the integration is carried out over the plane $x = 0$. Changing to Fourier components, we obtain

$$\hat{\mathscr{G}}(0\mathbf{p}, x'\mathbf{p}') + \int \frac{d^2q}{(2\pi)^2}\xi(\mathbf{p}-\mathbf{q})\hat{\mathscr{G}}'(0\mathbf{q}x'\mathbf{p}') = 0, \quad (24)$$

$$\hat{\mathscr{G}}(x\mathbf{p}x'\mathbf{p}') = \hat{\mathscr{G}}_\infty(x\mathbf{p}x'\mathbf{p}') + \frac{1}{(2\pi)^2}\int \hat{\mathscr{G}}_\infty(x\mathbf{p}0\mathbf{q})\hat{\mu}(0\mathbf{q}x'\mathbf{p}')\, d^2q, \quad (25)$$

$$\hat{\mathscr{G}}_\infty(x\mathbf{p}x'\mathbf{p}') = \hat{\mathscr{G}}_\infty(xx'\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')(2\pi)^2. \quad (26)$$

Expression (25) automatically satisfies Eq. (13) (for $x > 0$), and the function $\hat{\mu}(0\mathbf{p}x'\mathbf{p}')$ has been selected in order to satisfy the boundary condition (21).

Let us introduce a new function $\hat{\nu}(0\mathbf{p}x'\mathbf{p}')$ which is connected with $\hat{\mu}$ by the relation

$$\hat{\mu}(0\mathbf{p}x'\mathbf{p}') = \hat{\mathscr{G}}'^{-1}(00\mathbf{p})\hat{\nu}(0\mathbf{p}x'\mathbf{p}') - \hat{\mathscr{G}}_\infty'^{-1}(00\mathbf{p})\hat{\mathscr{G}}_\infty'(0x'\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')(2\pi)^2. \quad (27)$$

Substituting (27) and (25) into (24) and having made one more substitution

$$\hat{\nu}(0\mathbf{p}x'\mathbf{p}') = \hat{m}(0\mathbf{p}0\mathbf{p}')\hat{\psi}(0x'\mathbf{p}), \quad (27')$$

we obtain the following equation for the function $\hat{m}(0\mathbf{p}0\mathbf{p}')$:

$$\hat{\varphi}^{-1}(00\mathbf{p})\hat{m}(0\mathbf{p}0\mathbf{p}') = (2\pi)^2\delta(\mathbf{p}-\mathbf{p}')\hat{I} + \int \frac{d^2q}{(2\pi)^2}\xi(\mathbf{p}-\mathbf{q})\hat{m}(0\mathbf{q}0\mathbf{p}'). \quad (28)$$

Here

$$\hat{\varphi}^{-1}(00\mathbf{p}) = -\hat{\mathscr{G}}_\infty(00\mathbf{p})\hat{\mathscr{G}}_\infty'^{-1}(00\mathbf{p}), \quad (29)$$

$$\hat{\psi}(0x'\mathbf{p}) = \hat{\mathscr{G}}_\infty(0x'\mathbf{p}) - \hat{\mathscr{G}}_\infty(00\mathbf{p})\hat{\mathscr{G}}_\infty'^{-1}(00\mathbf{p})\hat{\mathscr{G}}_\infty'(0x'\mathbf{p}); \quad (30)$$

the primes denote the derivative with respect to the first argument, evaluated at zero.

Iteration of Eq. (28) with respect to ξ with a simultaneous averaging over the random inhomogeneities leads to the standard diagram theory of scattering by impurities.[9] This procedure was discussed in detail in article [8] and the generalization to the matrix equations (13) and (28) can be carried out without any difficulty. Using this technique we arrive at the following equation for the averaged function $m(0\mathbf{p}0\mathbf{p}')$:

$$\langle \hat{m}(0\mathbf{p}0\mathbf{p}')\rangle = \overline{\hat{m}(\mathbf{p})}\,\delta(\mathbf{p}-\mathbf{p}')(2\pi)^2, \tag{31}$$

$$\overline{\hat{m}(\mathbf{p})} = \hat{\varphi}(\mathbf{p}) + \hat{\varphi}(\mathbf{p})\int \frac{d^2q}{(2\pi)^2}\,\xi_2(\mathbf{p}-\mathbf{q})\,\overline{\hat{m}}(\mathbf{q})\,\overline{\hat{m}}(\mathbf{p}), \tag{32}$$

whose solution has the form

$$\overline{\hat{m}}(\mathbf{p}) = [\hat{\varphi}^{-1}(\mathbf{p}) - \hat{\Sigma}(\mathbf{p})]^{-1}, \tag{33}$$

$$\hat{\Sigma}(\mathbf{p}) = \int \frac{d^2q}{(2\pi)^2}\,\xi_2(\mathbf{p}-\mathbf{q})\,[\varphi^{-1}(\mathbf{q}) - \hat{\Sigma}(\mathbf{q})]^{-1}. \tag{34}$$

From the found solution we obtain the following expression for the averaged Green's function

$$\langle \hat{\mathscr{G}}(x\mathbf{p}x'\mathbf{p}')\rangle = (2\pi)^2\,\delta(\mathbf{p}-\mathbf{p}')\,\hat{\mathscr{G}}(xx'\mathbf{p}), \tag{35}$$

$$\hat{\mathscr{G}}(xx'\mathbf{p}) = \hat{\mathscr{G}}_\infty(xx'\mathbf{p}) - \mathscr{G}_\infty(x0\mathbf{p})\{[\hat{\mathscr{G}}_\infty(00\mathbf{p}) + \hat{\Sigma}(\mathbf{p})\mathscr{G}_\infty'(00\mathbf{p})]^{-1}\cdot$$
$$\times[\hat{\mathscr{G}}_\infty(0x'\mathbf{p}) - \hat{\mathscr{G}}_\infty(00\mathbf{p})\hat{\mathscr{G}}_\infty'^{-1}(00\mathbf{p})\hat{\mathscr{G}}_\infty'(0x'\mathbf{p})] + \hat{\mathscr{G}}_\infty'^{-1}(00\mathbf{p})\hat{\mathscr{G}}_\infty'(0x'\mathbf{p})\}. \tag{36}$$

Introducing the operator $[\hat{\mathscr{G}}_\infty + \hat{\Sigma}\hat{\mathscr{G}}_\infty']^{-1}$ after the curly bracket sign and neglecting the regular terms proportional to $\hat{\Sigma}$ in the numerator, we write down expression (36) in a form which is quite analogous to that obtained for a normal metal (compare with [8]):

$$\hat{\mathscr{G}}(xx'\mathbf{p}) = \hat{\mathscr{G}}_\infty(xx'\mathbf{p}) - \hat{\mathscr{G}}_\infty(x0\mathbf{p})[\hat{\mathscr{G}}_\infty(00\mathbf{p}) + \hat{\Sigma}(\mathbf{p})\hat{\mathscr{G}}_\infty'(00\mathbf{p})]^{-1}\hat{\mathscr{G}}_\infty(0x'\mathbf{p}). \tag{37}$$

The spectrum and the damping of the surface excitations are determined by the poles of the second term in (37), which are related to the vanishing of the determinant

$$|\hat{\mathscr{G}}_\infty(00\mathbf{p}) + \hat{\Sigma}(\mathbf{p})\hat{\mathscr{G}}_\infty'(00\mathbf{p})| = 0. \tag{38}$$

For $\hat{\Sigma} = 0$ formula (38) gives the spectrum of the surface levels in a superconductor which were obtained in article [2].

Finally, having substituted the explicit form (18) and (14) of the function $\mathscr{G}_\infty(00\mathbf{p})$ into formula (38), we obtain

$$|\hat{Y}_1(0) + \hat{\Sigma}\hat{Y}_1'(0)| = 0. \tag{39}$$

The expression for $\hat{\Sigma}$ also acquires a relatively simple form. Substituting (14) and (18) into (34) and using the definition (29) we find

$$\hat{\Sigma}(\mathbf{p}) = -\int \frac{d^2q}{(2\pi)^2}\,\xi_2(\mathbf{p}-\mathbf{q})\,[\hat{T}^{-1}(\mathbf{q}) + \hat{\Sigma}(\mathbf{q})]^{-1}, \tag{40}$$

where

$$\hat{T}(\mathbf{p}) = \hat{Y}_1'(0\mathbf{p})\,\hat{Y}_1^{-1}(0\mathbf{p}). \tag{41}$$

4. DAMPING OF THE SURFACE LEVELS

Let us find the correction to the spectrum and the damping of the levels due to the imperfect nature of the surface, assuming the quantity $p_\pm\Sigma$ to be small.

To the zero-order approximation in $\hat{\Sigma}$, Eq. (39) gives

$$|\hat{Y}_1(0\mathbf{p})| = \left|\begin{matrix} g & g^* \\ f & f^* \end{matrix}\right|_{x=0} = \frac{2i}{(|v_+||v_-|)^{1/2}}(p_-^2 - p_+^2)\sin\{\varphi_+ - \varphi_-\} = 0,$$

$$\varphi^\pm = \int_0^{x_0} p^\pm\,dx \pm \frac{\pi}{4}, \tag{42}$$

which also determines the spectrum (7) in the case of an ideal surface. Taking the roughness of the surface into account leads to a shift and to an attenuation which are proportional in the linear approximation in Σ. We shall characterize the shift and the damping by a correction δn which is added to the number n of the corresponding level. The correction δn is given by

$$\partial n = -\left.\frac{|\hat{Y}_1|\,\mathrm{Sp}\,\hat{\Sigma}\hat{T}(0\mathbf{p})}{\partial|\hat{Y}_1(0\mathbf{p})|/\partial n}\right|_{|\hat{Y}_1(0\mathbf{p})|=0}. \tag{43}$$

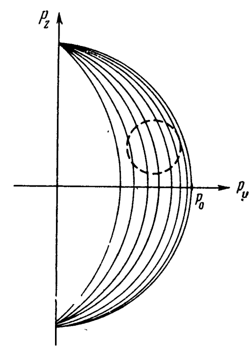

FIG. 2

The matrix $\hat{T}$, defined by formula (41), has the form

$$\hat{T}(0\mathbf{p}) = \frac{2i}{|\hat{Y}_1|}\,\mathrm{Im}\begin{pmatrix} g'f^* & gg^{*\prime} \\ f'f^* & gf^{*\prime} \end{pmatrix}. \tag{44}$$

As is evident from formulas (43) and (40), the correction to the level is expressed in terms of the binary correlation function $\xi_2(\mathbf{p})$ whose explicit form is unknown. Therefore it makes sense to estimate the order of magnitude of the shift and damping of the level. In the calculation we shall use expression (9) for the spectrum for $\alpha \gg 1$, keeping in mind that the estimates remain valid even for $\alpha \sim 1$. In the case $\alpha \gg 1$ one can set v_+ equal to v_-, and with the aid of (6) and (44) we obtain the following result for even n (which appear in virtue of the selection rule which was mentioned in Section 1):

$$\hat{T} = [(p_- - p_+)\sin(\varphi_+ - \varphi_-)]^{-1}\begin{pmatrix} u_- - u_+ & -2 \\ (u_- - u_+)(u_- - p_+p_-) & -2(u_- - p_+p_-) \end{pmatrix} \tag{45}$$

For $\alpha \sim 1$ expression (45) remains valid in order of magnitude, just like all of the subsequent expressions.

Let us estimate the integral (40). As usual the principal value of the integral determines the level shift, and by-passing the poles of the integrand leads to the damping. It is convenient to estimate the integral (40) by using the schematic picture of the distribution of the levels in the plane of p_y and p_z which is depicted in Fig. 2 in the form of a narrow crescent.

As one can see from formula (9), the circle $\alpha \sim 1$ serves as the external boundary of the region ($n \gg 1$), and the internal boundary, corresponding to small values of the number n, corresponds to $\alpha \sim 2m\Delta\delta^2$ (in dimensional units) for p_y close to p_0. The points of the crescent appear because the width of the energy band, within the limits of which the levels are located, is proportional to the quantity $\Omega\,\delta p_y$ and this quantity becomes small as $p_y \to 0$. The external boundary of the spectrum (for $p_z = 0$) corresponds to glancing angles of the quasiparticles $\varphi \sim [(p_0 - p_y)/p_0]^{1/2} \sim \sqrt{\delta/R} \sim 10^{-2}$, but the internal boundary corresponds to significantly larger angles $\varphi \sim \delta/\xi_0 \sim 10^{-1}$ to 10^{-2} (ξ_0 denotes the correlation radius of the pair).

The dotted circle of radius $1/d$ with its center at the point $\mathbf{p}$ indicates the region in which the function $\xi_2(\mathbf{p}-\mathbf{q})$ is different from zero. Its order of magnitude here is a^2d^2.

The width of the region with respect to p_y for $p_z = 0$ is of order $\delta p_y = (m\Delta\delta)^2/p_0$, and the distance between the levels is

$$\partial p_y / \partial n \sim \sqrt{2m\Delta}\,\alpha^{3/2}/p_0\delta.$$

It is obvious that in a typical case $1/d \gg \partial p_y/\partial n$, i.e., a large number of levels fall in the region where the correlation function $\xi_2(p)$ does not vanish. As is shown in article [8], to within the accuracy $p_\pm \Sigma \ll 1$ adopted here, one can omit the self-energy part in the denominator of (40). In this case the damping and the shift turn out to be of the same order, and in the expression for the damping one can change from a summation over n to an integration. The correction to the energy level reduces to the expression

$$\partial n \sim \frac{(m\Delta)^2}{p_+ - p_-} \int d^2q\, \xi_2 \frac{1}{p_+(0) - p_-(0)}. \tag{46}$$

Its magnitude is determined by the ratio of the dimensions of the regions $1/d$ and ∂p_y.

For $1/d \ll \partial p_y$ we obtain

$$\partial n \sim \frac{a^2 (m\Delta)^2}{(p_+ - p_-)^2} \sim a^2 m\Delta\alpha \sim \frac{a^2 p_0}{\xi_0}\alpha. \tag{47}$$

For small angles the correction is small, but for maximum angles of the slope of the trajectories $\partial n \sim (ap_0\delta/\xi_0)^2$.

If $1/d \gtrsim (m\Delta\delta^2/p_0$, then

$$\partial n \sim m\Delta\sqrt{\alpha}\int d^2q\xi_2\sqrt{\alpha} \sim a^2 d(m\Delta p_0\alpha)^{1/2}(\partial p_y)^{3/2} \sim a^2 d\delta^3 p_0^{5/2} \xi_0^{-7/2} \alpha^{1/2}. \tag{48}$$

For the maximum value of α, $\partial n \sim a^2 dp_0^3(\delta/\xi_0)^3$.

Thus, for a sufficiently rough surface the damping of the level associated with a fixed value of ϵ increases with the angle like $\sqrt{\alpha}$, i.e., it is proportional to the glancing angle of the quasiparticles, whereas in the case of a smooth surface with large values of d it is proportional to the square of the glancing angle. For a fixed value of the glancing angle, the damping is maximum for small values of the quantum number n, and decreases with increasing n. As a result the intermediate part of the spectrum, where the distance between the levels is still not too small, turns out to be preferred.

[1] P. Pincus, Phys. Rev. **158**, 346 (1967).

[2] M. Ya. Azbel' and A. Ya. Blank, ZhETF Pis. Red. **10**, 49 (1969) [JETP Lett. **10**, 32 (1969)].

[3] A. Ya. Blank, Zh. Eksp. Teor. Fiz. **58**, 1862 (1970) [Sov. Phys.-JETP **31**, 999 (1970)].

[4] M. Ya. Azbel', Zh. Eksp. Teor. Fiz. **59**, 295 (1970) [Sov. Phys.-JETP **32**, 159 (1971)].

[5] R. E. Prange and Tsu-Wei Nee, Phys. Rev. **168**, 779 (1968).

[6] J. F. Koch and P. A. Pincus, Phys. Rev. Lett. **19**, 1044 (1967).

[7] J. F. Koch and C. C. Kuo, Phys. Rev. **164**, 618 (1967); J. R. Maldonado and J. F. Koch, Phys. Rev. B, Vol. 1, 1031 (1970).

[8] L. A. Fal'kovskiĭ, Zh. Eksp. Teor. Fiz. **58**, 1830 (1970) [Sov. Phys.-JETP **31**, 981 (1970)].

[9] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962 (English Transl. Prentice-Hall, 1963).

[10] M. A. Naĭmark, Lineĭnye differentsial'nye operatory (Linear Differential Operators), Nauka, 1969, p. 116 (English Transl., Frederick Ungar Publishing Co., 1967).

Translated by H. H. Nickle
85