GAUGE INVARIANT DIAGRAM TECHNIQUE IN A MAGNETIC FIELD

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It is shown that the Green's function of an electron in a homogeneous and isotropic system, placed in a magnetic field, is diagonal with respect to all quantum numbers in the Landau representation and only depends on those quantum numbers on which the electron energy depends. A diagram technique can be developed for this Green's function in such a way that it does not contain any quantum numbers which depend on the gauge of the vector potential.

INTRODUCTION

THE diagram technique for the calculation of the electron Green's function simplifies appreciably in spatially-homogeneous systems (after changing to the momentum representation). However, if the system is in a magnetic field **H**, then even in the case of a homogeneous field the dependence of the vector potential on the coordinates formally makes the system inhomogeneous. The Landau representation, in which the zero-order Green's function G^0 is diagonal, is the most natural representation to use in the presence of a homogeneous magnetic field; the diagram technique is therefore formulated in this representation.[1,2] In this connection the gauge-invariant quantum numbers l and p_z appear along with a single nongauge-invariant quantum number depending on the gauge of the vector potential, for example, p_x.

In the present article a gauge-invariant formulation of the diagram technique is proposed in which only the quantum numbers l and p_z appear. This technique is based on considerations about translational invariance in a magnetic field, which were investigated in an article by one of the authors.[3] It is shown below that in a system which is homogeneous and axially symmetric around **H**, not only the zero-order Green's function G^0 but also the exact Green's function G is diagonal in the Landau representation. This important property makes it possible to preserve the rules of the diagram technique during the transition from "thin" lines to "fat" lines in the same way in which this occurs in the momentum representation in a homogeneous system.

The diagram technique is formulated for electrons interacting with phonons; however, the technique is obviously also applicable to the electron-electron interaction after replacing the phonon lines by Coulomb lines.

1. DIAGONAL NATURE OF THE GREEN'S FUNCTION

Let us demonstrate that if the system of electrons, placed in a magnetic field, is found in a state which is spatially homogeneous and axially symmetric around the direction of **H**, then the electron Green's function is diagonal with respect to all quantum numbers λ in the Landau representation and only depends on those quantum numbers on which the energy depends, i.e., on the number l of the level and on the longitudinal momentum p_z:

$$G_{\lambda\lambda'}(\varepsilon) = \delta_{\lambda\lambda'} G_{lp_z}(\varepsilon). \tag{1.1}$$

For the proof, first let us change to the coordinate representation

$$G_{\lambda\lambda'}(\varepsilon) = \int d\mathbf{x} \int d\mathbf{x}' \psi_\lambda^*(\mathbf{x}) \psi_{\lambda'}(\mathbf{x}') G(\varepsilon; \mathbf{x}, \mathbf{x}'), \tag{1.2}$$

where the $\psi_\lambda(\mathbf{x})$ are the wave functions of the electron in the presence of the magnetic field. Let us replace the integration variables $\mathbf{x}$ and $\mathbf{x}'$ by $\mathbf{r} = (1/2)(\mathbf{x}' + \mathbf{x})$ and $\mathbf{s} = \mathbf{x}' - \mathbf{x}$, and in G let us take the Fourier transform with respect to $\mathbf{s}$. Then we find

$$G_{\lambda\lambda'}(\varepsilon) = \int d\mathbf{r} \int d\mathbf{s} \psi_\lambda^*(\mathbf{r} - {}^1/_2\mathbf{s}) \psi_{\lambda'}(\mathbf{r} + {}^1/_2\mathbf{s}) \int d\mathbf{p} e^{-i\mathbf{p}\mathbf{s}} G(\varepsilon; \mathbf{r}, \mathbf{p}). \tag{1.3}$$

Here $G(\epsilon; \mathbf{r}, \mathbf{p})$ is the Green's function in the Wigner representation; its connection with the coordinate representation is the same as for the density matrix.[3]

If the system is spatially homogeneous, then by applying to the Green's function literally the same considerations which were applied to the density matrix in[3], one can show that

$$G(\varepsilon; \mathbf{r}, \mathbf{p}) = G(\varepsilon; \mathbf{k}), \quad \mathbf{k} = \mathbf{p} - e\mathbf{A}(\mathbf{r})/c, \tag{1.4}$$

that is, one can show that the Green's function does not depend on the coordinate $\mathbf{r}$ and on the canonical momentum $\mathbf{p}$ independently, but only through the kinetic momentum $\mathbf{k}$. Using (1.4), in (1.3) one can change from an integration over $\mathbf{p}$ to an integration over $\mathbf{k}$. After this the integration over $\mathbf{r}$ and $\mathbf{s}$ is not coupled to the Green's function and may be carried out explicitly. For example, using the gauge $\mathbf{A} = (-Hy, 0, 0)$, $\lambda = lp_x p_z$, after direct integration of the known wave functions we find:

$$\begin{aligned} G_{\lambda\lambda'}(\varepsilon) &= (2\pi)^2 \delta(p_x - p_x') \delta(p_z - p_z') \cdot \\ &\cdot 2a(-1)^l \int d\mathbf{k}_\perp G(\varepsilon, \mathbf{k})|_{k_z = p_z} \Lambda_{ll'}^*(\mathbf{k}_\perp). \end{aligned} \tag{1.5}$$

Here $a = (c/|e|H)^{1/2}$ is the magnetic length ($\hbar = 1$), $\mathbf{k}_\perp$ is the component of $\mathbf{k}$ perpendicular to **H**; then

$$\begin{aligned} \Lambda_{ll'}(\mathbf{k}_\perp) &= e^{-i\varphi(l-l')} \left[\frac{2^l l!}{2^{l'} l'!}\right]^{1/2} \varkappa^{l'-l} e^{-1/4\varkappa^2} L_l^{l'-l}\left(\frac{1}{2}\varkappa^2\right) \\ &= (-1)^{l-l'} \Lambda_{l'l}^*(\mathbf{k}_\perp), \end{aligned} \tag{1.6}$$

where $\kappa = k_\perp a$, L_l^α are the Laguerre polynomials defined according to[4], and $k_\perp$ and φ denote the magnitude of $\mathbf{k}_\perp$ and its azimuthal angle with respect to **H**, measured from k_x.

From Eq. (1.5) it is seen that spatial homogeneity leads to the result that G does not depend on p_X, a result which is quite easy to understand because p_x determines the position of the center of the Larmor oscillator, which is not important in a homogeneous system. If we also assume axial symmetry in the system, then $G(\epsilon, \mathbf{k})$ does not depend on φ; in this connection the integration over φ gives $\delta_{ll'}$, which also proves Eq. (1.1).

Thus, for a spatially homogeneous and axially symmetric system we have

$$G_{\lambda\lambda'}(\varepsilon) = \delta(p_z - p_z')\delta(p_x - p_x')\delta_{ll'}G_l(\varepsilon, p_z). \qquad (1.7)$$

The quantum number p_X is related to the gauge of the vector potential. The diagonality with respect to p_X and the fact that the diagonal elements do not depend on p_X means that the diagonal part $G_l(\epsilon, p_z)$ is gauge invariant. It is natural to expect that rules for a diagram technique can be formulated for it, said rules not containing the quantum number associated with the gauge.

We note that although the diagonality with respect to l follows from axial symmetry, this connection is very nontrivial since the quantum number l does not have any direct relation to axial symmetry. The latter is clear even from the fact that in the axial gauge $\mathbf{A} = (1/2)\mathbf{H} \times \mathbf{x}$ the transformation properties of the electron's wave functions associated with a rotation around $\mathbf{H}$ are determined not by the quantum number l but by a different quantum number m which appears instead of p_X.

Let us indicate that considerations about the diagonal nature of the scattering matrix of an electron, interacting with phonons, with respect to the quantum number l are given in article[5], based on a direct calculation according to perturbation theory.

The definition of $\Lambda_{ll'}(\mathbf{k}_\perp)$ given in Eq. (1.6) holds in the case of an electron ($e < 0$). For a positive particle (hole) it is necessary to change the direction of rotation of the azimuthal angle and measure it from the $-k_x$ axis, i.e., make the substitution $\varphi \to \pi - \varphi$.

2. ELIMINATION OF THE QUANTUM NUMBERS THAT ARE NOT GAUGE INVARIANT

The rules of the diagram technique in the Landau representation for the quantity $G_l(\epsilon, p_z)$ can be obtained by a transition from the coordinate representation to the Landau representation just like this is done for spatially homogeneous systems by changing to the momentum representation. The only difference consists in the fact that upon integration over the spatial coordinates of a site, a delta-function of the sum of the momenta does not arise, but instead the matrix element

$$\int dx \psi_\lambda^*(\mathbf{x}) e^{i\mathbf{q}\mathbf{x}} \psi_{\lambda'}(\mathbf{x}) = \delta(p_z - p_z' - q_z)\delta(p_x - p_x' - q_x)$$

$$\times \exp\left\{-i\frac{1}{2}\frac{c}{eH}q_y(p_x + p_x')\right\}\Lambda_{ll'}(\mathbf{q}_\perp). \qquad (2.1)$$

Here the unprimed quantum numbers of the electron correspond to incoming electron lines, and the primed quantum numbers correspond to outgoing electron lines. The phonon line is assumed to be outgoing; for an incoming phonon line it is necessary to change the sign of $\mathbf{q}$.

The formulation of the gauge-invariant technique consists in the fact that one can perform the integration over p_X, corresponding to internal electron lines, in general form. For this we note that the electron lines of diagram G form several "paths." One of the paths begins and ends on the two free ends of the diagram, the remaining paths form closed loops. It is obvious that one can perform the integration over p_X, which includes the delta-function and phase factors from (2.1), for each path separately. It is convenient to separate the phonon lines into two groups: the lines u which begin and end on a single path, and the lines v which begin and end on different paths.

First let us consider a loop without any u lines (see Fig. 1). Its contribution is given by

$$C = \int dp_{1x} \ldots \int dp_{nx}\delta(p_{1x} - p_{2x} - v_{1x})\delta(p_{2x} - p_{3x} - v_{2x})$$
$$\ldots\delta(p_{n-1x} - p_{nx} - v_{n-1x})\delta(p_{nx} - p_{1x} - v_{nx})e^{i\alpha\chi}$$
$$\alpha = c/2eH, \qquad (2.2)$$

$$\chi = -v_{1y}(p_{1x} + p_{2x}) - v_{2y}(p_{2x} + p_{3x})$$
$$\ldots - v_{n-1y}(p_{n-1x} + p_{nx}) - v_{ny}(p_{nx} + p_{1x}). \qquad (2.3)$$

Using the delta-function, one can transform the phase to the form

$$\chi = -2p_{1x}\sum_{i=1}^{n} v_{iy} + 2\sum_{i=1}^{n-1}\sum_{j=1}^{i-1} v_{iy}v_{jx} + \sum_{i=1}^{n-1} v_{iy}v_{ix} - v_{ny}v_{nx}. \qquad (2.4)$$

After this one can carry out the integration and one finds

$$C = 2\pi\frac{|e|H}{c}\delta\left(\sum_{i=1}^{n} v_{ix}\right)\delta\left(\sum_{i=1}^{n} v_{iy}\right)e^{i\alpha\chi'}, \qquad (2.5)$$

where χ' denotes the last three terms in expression (2.4) for χ. Using the delta-function which appears in (2.5), one can transform the phase one more time:

$$\chi' = \sum_{j<i=1}^{n}[\mathbf{v}_j, \mathbf{v}_i], \qquad (2.6)$$

where the notation

$$[\mathbf{v}_1, \mathbf{v}_2] = v_{1x}v_{2y} - v_{2x}v_{1y}. \qquad (2.7)$$

has been introduced. It is clear that a sum over all pairs of phonon lines, emerging from a given loop, appears in χ', where the order of the sequence inside the square brackets is determined by the direction of the electron lines along the loop.

Let us go on to loops containing u lines. The appearance of a single u line can be represented in the following manner. Let us single out two v lines, $\mathbf{v}_a$ and $\mathbf{v}_b$ ($a < b$), we set $\mathbf{v}_a = \mathbf{u}$, $\mathbf{v}_b = -\mathbf{u}$, and we "close" them (see Fig. 2). In this connection $\mathbf{v}_a$ and $\mathbf{v}_b$ drop out of the sums in the delta-functions in (2.5). In connection with the computation of the phase χ', the bracket $[\mathbf{v}_a, \mathbf{v}_b] = [\mathbf{u}, -\mathbf{u}] = 0$, and the brackets containing $\mathbf{v}_a$ and $\mathbf{v}_i$

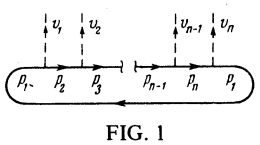

FIG. 1

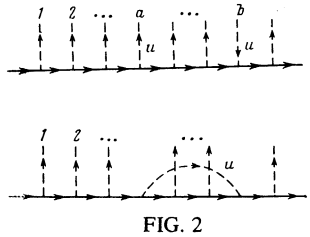

FIG. 2

with $i < a$ or $i > b$ are cancelled by the corresponding brackets containing v_b and v_i; on the other hand, the brackets containing v_a and v_i with $a < i < b$ are doubled. Thus, as is evident from the figure, the contribution of the u line to the phase is determined by its intersection with the other lines. If the process of "closure" is continued for the formation of new u lines, then one can verify that the phase has the following form:

$$\chi_c = \sum [v, v'] + 2\sum [u, u'] + 2\sum [u, v]. \qquad (2.8)$$

Here the first term is the sum over pairs of all the v lines of a given loop, the second term is the sum over the intersections of two u lines (see Fig. 3 a), and the third term is the sum over the intersections of u lines with v lines (Fig. 3 b). It is assumed that all phonon lines are drawn from one side of the electron lines. It is also assumed that the u lines go in the same direction as the electron loop, but the v lines emerge from the loop; if the direction of the phonon lines is different, then the corresponding u or v appears in the brackets with a minus sign.

Finally the contribution from the loop is obtained in the following form:

$$C = \frac{2\pi}{a^2}\delta\Big(\sum \mathbf{v}_\perp\Big) e^{i\alpha\chi_C}, \qquad (2.9)$$

where the summation inside the argument of the delta-function is taken over all v lines of the loop.

The contribution from unclosed paths (see Fig. 4) remains to be considered; first let us consider those without u lines

$$L = \int dp_{2x} \ldots \int dp_{nx}\delta(p_{1x} - p_{2x} - v_{1x}) \ldots \delta(p_{n-1x} - p_{nx} - v_{n-1x}) \times \delta(p_{nx} - p_{n+1x} - v_{nx}) e^{i\alpha\sigma}, \qquad (2.10)$$

$$\sigma = -v_{1y}(p_{1x} + p_{2x}) - \ldots - v_{n-1y}(p_{n-1x} + p_{nx}) - v_{ny}(p_{nx} + p_{n+1x}). \qquad (2.11)$$

The quantity L differs from C by the absence of $\int dp_{1X}$ and by the substitution $p_{1X} \to p_{n+1X}$ in the last delta-function and in the last term of the phase. Further, according to Eq. (2.9) the sum of the transverse phonon momenta originating from a single loop is equal to zero. If such equalities are put together for all loops, then the momenta of the lines beginning on one loop and ending on another cancel. Only the momenta of the lines joining the loops with unclosed paths are left, that is, $\Sigma v_x = 0$ and $\Sigma v_y = 0$. From here it follows that $p_{1X} = p_{n+1X}$, i.e., not only the total function G is diagonal in p_X but also each component part of its diagram. Taking all of these diagonalities into consideration, we have $\sigma = \chi = \chi'$. Now performing the integration, we find

$$L = \delta(p_{1x} - p_{n+1x}) e^{i\alpha\chi'}. \qquad (2.12)$$

If an unclosed path contains u lines, then one can treat them just like in a loop. Therefore we finally obtain

$$L = \delta(p_{1x} - p_{n+1x}) e^{i\alpha\chi_L}, \qquad (2.13)$$

where χ_L is calculated in the same way as χ_C.

We note that the presence of a delta-function in (2.9) agrees with the results of article[6] because the loop is the contribution to the average value of a certain number of phonon operators (without electron lines).

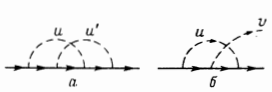

FIG. 3

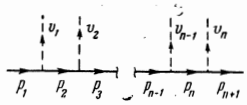

FIG. 4

Now let us turn our attention to the "vertex" diagram Γ containing two external electron lines—the entering line $\epsilon l p_Z p_X$ and the outgoing line $\epsilon' l' p'_Z p'_X$, and containing an arbitrary number of external (outgoing) phonon lines $\omega_1 q_1$, $\omega_2 q_2$, ... It is not difficult to verify that if in such a diagram one carries out the integration over p_X of the internal electron lines, then from each loop appears the same contribution as in the diagram for G, but the contribution from an unclosed path will contain an additional phase factor associated with the presence of external phonon end points. If this factor is separated out, and the vertex function is written in the following form:

$$\Gamma(\varepsilon l p_x, p_z, \varepsilon' l' p_x' p_z'; \ \omega_1 q_1, \omega_2 q_2, \ldots) = \delta(p_x - p_x' - q_{1x} - q_{2x} - \ldots) \exp\{-i\alpha(q_{1y} + q_{2y} + \ldots)(p_x + p_x')\} \times \Gamma_{ll'}(\varepsilon p_z, \varepsilon' p_z'; \ \omega_1 q_1, \omega_2 q_2, \ldots) \qquad (2.14)$$

then the quantity $\Gamma_{ll'}$ does not depend on p_X or p'_X and it is gauge invariant; for it, the result of the integration over p_X of the internal lines will be calculated according to the same rules as for G.

Physically it is easy to understand the fact that the dependence of Γ on p_X and p'_X turns out to be unimportant (only in terms of the phase factor). The quantities p_X and p'_X determine the positions y_0 and y'_0 of the centers of the Larmor oscillator before and after the scattering of the electron by the phonons q_1, q_2, ... It is obvious that in view of the homogeneity of space the probability of this scattering can only depend on the relative position of the centers. Therefore $|\Gamma|^2$ may depend on $y_0 - y'_0$, that is, on $p_X - p'_X$ but it cannot depend on $(1/2)(y_0 + y'_0)$, i.e., it cannot depend on $p_X + p'_X$. On the other hand, $p_X - p'_X = q_{1X} + q_{2X} + \ldots$. Therefore, one can assume that $|\Gamma|^2$ depends only on q_1, q_2, ... and does not depend on p_X or p'_X.

However, if we turn our attention to diagrams containing a large number of external electron lines, then one can easily see that the dependence of these diagrams on p_X of the external lines is essential. One can also understand this physically. Let us consider a diagram with two incoming electron lines (p_{1X} and p_{2X}) and two outgoing lines (p'_{1X} and p'_{2X}). Such a diagram describes the scattering of two electrons one against the other with their centers at y_{01} and y_{02} before scattering and with their centers at y'_{01} and y'_{02} after scattering. It is obvious that the probability of this scattering essentially depends on the relative position $y_{01} - y_{02}$ and $y'_{01} - y'_{02}$ before and after scattering, i.e., on $p_{1X} - p_{2X}$ and $p'_{1X} - p'_{2X}$. Therefore for diagrams containing more than two external electron lines, i.e., describing the many-particle properties of an electron gas, it is impossible to isolate the gauge-invariant factors by the simple method which was used for G and Γ.

3. DIAGRAM TECHNIQUE

From the preceding section it follows that in order to evaluate diagrams containing not more than two external electron lines, one may formulate the following rules, which do not contain any quantum numbers that are not gauge invariant.

1. An electron line $\epsilon l p_z$ corresponds to $iG^0_l(\epsilon p_z)$.
2. A phonon line $\omega \mathbf{q}$ corresponds to $iD^0(\omega \mathbf{q})$.
3. For internal lines in addition one has

$$\int_{-\infty}^{+\infty}\frac{d\varepsilon}{2\pi}\int_{-\infty}^{+\infty}\frac{dp_z}{2\pi}\sum_{l=0}^{\infty}, \qquad \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{\infty}\frac{d\mathbf{q}}{(2\pi)^3}.$$

4. A vertex at which the line $\epsilon l p_z$ comes in and the lines $\epsilon' l' p_z'$ and $\omega \mathbf{q}$ go out corresponds to

$$(-i)(2\pi)^2\delta(\varepsilon-\varepsilon'-\omega)\delta(p_z-p_z'-q_z)\Lambda_{ll'}(\mathbf{q}_\perp);$$

if the phonon line is incoming, then the sign of $\mathbf{q}$ is changed.

5. An electron loop corresponds to

$$(-1)\frac{2\pi}{a^2}\delta\Big(\sum \mathbf{q}_\perp\Big)e^{i\alpha\chi_C}, \qquad \alpha = c/2eH,$$

where the summation goes over all phonon lines which are external with respect to the given loop.

6. An unclosed electron path corresponds to $e^{i\alpha\chi_L}$.

The phases χ_C and χ_L are determined according to Eq. (2.8).

As an example let us consider the diagram shown in Fig. 5. Since the rules 1 through 4 corresponds to the usual diagram technique, then we shall only write down the phases which appear according to rules 5 and 6:

$$\chi_C = [-\mathbf{v}_1, -\mathbf{v}_3]+[-\mathbf{v}_1, \mathbf{v}_2]+[-\mathbf{v}_3, \mathbf{v}_2] + 2[\mathbf{u}_1, -\mathbf{v}_1],$$
$$\chi_L = [\mathbf{v}_1, -\mathbf{v}_2]+[\mathbf{v}_1, \mathbf{v}_3]+[-\mathbf{v}_2, \mathbf{v}_3]+ 2[\mathbf{u}_2, -\mathbf{u}_3] + 2[\mathbf{u}_2, \mathbf{v}_3].$$

An important question is, to what extent is the diagram technique according to the formulated rules a Feynman technique, i.e., can it be used to obtain equations of the Dyson type? It is obvious that in the formulated technique one can only obtain equations in which quantities corresponding to diagrams containing more than two external electron lines do not appear. In addition, one must take into consideration that the phase factor, according to rules 5 and 6, depends on the topology of the entire diagram as a whole; therefore, upon cutting the diagram in two, in general it cannot be factored into factors corresponding to the separate parts of the diagram. From the methods by which the rules for calculating the phases were derived, it is clear that upon cutting the phonon lines, the phase factor may be factorized. Factorization does not occur upon cutting the electron path. However, it is important that the "deficit" phase δ does not depend on the internal structure of the parts which are produced, but only on the external lines of these parts. The additional inclusion of this phase makes it possible to formulate a rule for "cutting a diagram in two":

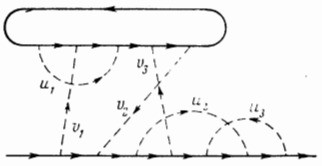

FIG. 5

$$\Gamma = S\Gamma'\Gamma'' e^{i\alpha\delta}, \qquad \delta = (\Sigma q', \Sigma q'').$$

Here Γ' and Γ'' are the two diagrams which are formed after cutting the diagram Γ with respect to the electron path (and also, perhaps, cutting it with respect to a certain number of phonon lines); Γ' contains the initial portion of the path and Γ'' contains the final portion; $\mathbf{q}'$ and $\mathbf{q}''$ correspond to the external phonon lines of diagrams Γ' and Γ''. The symbol S denotes summation and integration over the cut lines according to rule 3.

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