RADIAL FOCUSING OF A CHARGED-PARTICLE RING BEAM EXPOSED TO COHERENT SYNCHROTRON RADIATION

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The possibility of radial stabilization of a charged-particle ring beam as a result of the reaction of coherent magnetic bremsstrahlung radiation is demonstrated. The stability of a relativistic beam of finite thickness is discussed and the growth rates are determined. This is followed by the solution of the problem in the nonlinear approximation which yields the size of the focused beam, the maximum amplitude of the radiation field, and the characteristic time for the process.

1. It is well known[1] that in the self-stabilization method the radial focusing is produced by radiation deceleration force due to the scattering of electrons by ions. Since the characteristic time for this process is governed by electron-ion collisions, the time necessary for the beam to contract is quite considerable (of the order of a few seconds or more[1]). The effectiveness of the focusing process can be substantially increased by the use of the coherent radiation from a charged-particle beam. In the case of a straight beam, one can use the two-stream instability[2] for this purpose, which is accompanied by substantial coherent radiation from the beam.[3]

In this paper we propose a method for the stabilization of charged-particle ring beams in cyclic accelerators and storage rings, based on the use of coherent synchrotron radiation.1) The processes giving rise to this radiation can be conventionally divided into (1) the negative mass instability[4-6] (when the frequencies of the perturbation fields are very different from the natural frequencies of the chamber) and (2) the radiation instability which may occur when the harmonics of the cyclotron frequency are close enough to the natural frequencies of the resonator. The latter effect was discovered by Gaponov and Schneider[7,8] and is also referred to as induced cyclotron radiation. The collective instability of ring beams was subsequently investigated in[9-11], where the instability growth rates were determined for different relationships between beam and resonator parameters. In particular, it is shown in[10] that the radiation instability will occur in a chamber of infinite radius when, in effect, the resonator is a vacuum and all the harmonics of the cyclotron frequency are resonance harmonics. The basic mechanism for this emission of radiation is synchrotron radiation of charged particles which occurs in the absence of the resonator.[12,13]

We shall consider radial self-focusing of a ring beam which is in the form of a cylindrical layer of relativistic charged particles of finite thickness which circulates in a magnetic field in vacuum. The radiation field due to such a beam is a superposition of H waves propagating azimuthaly with a velocity equal to that of the beam, i.e., a velocity less than the velocity of light. This wave does not penetrate the plasma but propagates along the boundary, so that the radiation field decreases from the outer beam boundary toward its center (surface wave). At the same time, the beam is located in the high-frequency radial potential well,[14] whose depth and wall slope increase as the stability develops and the field amplitude increases. The net result of all this is the focusing of the beam.2)

In the first part of this paper we shall determine the time-independent state of the beam and this will be followed by an analysis of its stability. The dispersion relation ensuing from this, which takes account of the particle energy spread (as a function of radial distance) and the precise geometry of the radiation fields enables us to investigate both the case of a "thin" ring, when the beam is monoenergetic, and the case of a beam with an energy spread. In the latter case, the instability develops only in a narrow layer on the outer boundary between the beam and the vacuum, and the growth rate turns out to be much smaller than in the case of a monoenergetic beam. In the second part of this paper we discuss the nonlinear stabilization of a beam by a radiation field.

2. Consider a charged-particle ring beam in the form of a cylindrical layer of thickness σ (the radii of the bounding surfaces are, respectively, $a < b$) placed in a constant magnetic field H_0 which is parallel to the axis of the cylinder. The set of equations describing the state of the beam consists of the relativistic equations of motion for the beam with velocity $v = (v_r, v_\varphi, 0)$, the continuity equations, and the Maxwell equations for the radiation fields $E = (E_r, E_\varphi, 0)$ and $H = (0, 0, H_z)$. If we take the axis of the cylinder as the z axis of the cylindrical set of coordinates, we can determine the time-independent state of the beam from the conditions $\partial/\partial t = \partial/\partial\varphi = v_r = 0$:

$$v_\varphi = \frac{\omega_H r}{\gamma}, \quad \gamma = \left(1 + \frac{\omega_H^2 r^2}{c^2}\right)^{1/2}, \quad \bar{H}_z = -\frac{\omega_0^2}{\omega_H^2}(\gamma - 1), \qquad (1)$$

where $\omega_0^2 = 4\pi e^2 n_0/m$, $\omega_H = eH_0/mc$, and n_0 is the

1)The basic idea of this method, whereby ring beams of charged particles are stabilized as a result of collective processes, was first suggested by Ya. B. Faĭnberg and independently by O. I. Yarkov.

2)The mechanism of radial focusing due to forces which are quadratic in the field amplitude has already been discussed in [3] in the case of straight beams.

initial beam density. According to Eq. (1), allowance for relativistic effects leads to a radial dependence of the beam-particle energy, where for $\gamma \gg 1$ we can approximately suppose that $\gamma \approx \omega_H r/c$.

The quantity $\overline{H}_z$ is the addition to the constant magnetic field due to the beam current. We shall restrict our attention to low-density beams with $\gamma\omega_0^2/\omega_{II}^2 \ll 1$, which will enable us to neglect $\overline{H}_z$ in comparison with H_0 and to ignore the beam space charge.

We shall now investigate the stability of the beam against high-frequency perturbations. To do this, we shall write the beam thickness in the form $n = n(r) + n_1(t, r)$ and $v = v_0(r) + v_1(t, r)$. We shall then linearize the equations of motion and will seek the solution in the form $\exp(is\varphi - \omega t)$. The result is

$$v_s = i\frac{eE_s}{m\gamma^3\Delta_s}, \quad n_1 = \frac{e}{mr\Delta_s}\left[i\frac{snE_{\varphi s}}{\gamma^3\Delta_s} + \frac{d}{dr}\left(\frac{rnE_{rs}}{\gamma^3\Delta_s}\right)\right], \tag{2}$$

where $\Delta_s = \omega - sv_0/r$. Calculating the high-frequency current components with the aid of Eq. (2), we obtain

$$j_{\varphi s} = \frac{e^2}{m}r\left[i\frac{\omega nE_{\varphi s}}{\gamma^3\Delta_s^2} + \frac{v_0}{r\Delta_s}\frac{d}{dr}\left(\frac{rnE_{rs}}{\gamma^3\Delta_s}\right)\right], \quad j_{rs} = i\frac{e^2nE_{rs}}{m\gamma^3\Delta_s}, \tag{3}$$

and if we substitute these into the Maxwell equations, we obtain the Bessel equation for the function H_{zs}:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH_{zs}}{dr}\right) + \left(\frac{\omega^2}{c^2} - \frac{s^2}{r^2}\right)H_{zs} = \frac{4\pi}{cr}\left[isj_{rs} - \frac{d}{dr}(rj_{\varphi s})\right] = g_s(r). \tag{4}$$

The solution of this equation is finite at the origin, and has the form of a diverging cylindrical wave at infinity. It can be written in the form

$$H_{zs} = \begin{cases} AJ_s(\omega r/c), & r < a \\ BH_s^{(1)}(\omega r/c), & r > b \end{cases}, \tag{5}$$

where J_s is the Bessel function and $H_s^{(1)} = J_s + iN_s$ is the Hankel function of the first kind. In the region $a < r < b$, occupied by the beam, we can use the method of successive approximations to solve Eq. (4), and this yields

$$H_{zs} = CJ_s + DN_s + \frac{\pi}{2}\int_a^r [N_s(r)J_s(\xi) - J_s(r)N_s(\xi)]g_s(\xi)\xi\, d\xi. \tag{6}$$

The function $g_s(\xi)$ in this result must be expressed in terms of the field for $n = 0$.

From the continuity of the tangential components of the radiation field H_{zs} and $E_{\varphi s} = -i(c/\omega)dH_{zs}/dr$ across the inner boundary of the layer ($r = a$) we have $D = 0$, and the boundary conditions at $r = b$ yield the following dispersion relation:

$$\frac{2i}{\pi} - \int_a^b H_s^{(1)}\left(\frac{\omega\xi}{c}\right)g_s(\xi)\xi\, d\xi = 0. \tag{7}$$

If we express $g_s(\xi)$ in terms of the field, using Eq. (3), and integrate by parts subject to $n(a) = n(b) = 0$, we can transform Eq. (7) to the form

$$\frac{2i}{\pi} - \frac{4\pi e^2}{m}\int_a^b \left\{\frac{1}{\omega}\frac{s^2}{\xi^2}H_sJ_s + \frac{1}{\Delta_s}\left[H_s'J_s' + \frac{v_0 s}{\omega\xi}\left(\frac{s^2}{\xi^2} - \frac{\omega^2}{c^2}\right)H_sJ_s\right] - \frac{v_0 s}{\Delta_s^2\xi^2}H_s'J_s\right\}\frac{n\xi\, d\xi}{\gamma^3\Delta_s} = 0. \tag{8}$$

In the analogous results obtained in[10], the derivation was based on an expansion in terms of the eigenfunctions of the resonator (Bessel functions), and the fields were assumed averaged over the beam cross section. In contrast, Eq. (8) takes into account the change in the geometry of the field associated with the presence of the beam. Formally, this leads to the appearance of the imaginary part in the Hankel function (Neumann function) on the right-hand side of Eq. (8). It is well known[15] that the asymptotic form of this function increases for the higher harmonics $s \gtrsim \gamma^3$, and this exponential part exceeds the real part:

$$J_s \approx \frac{1}{\pi}\left(\frac{\pi}{2sw}\right)^{1/2}\exp\left(-\frac{sw^3}{3}\right),$$

$$H_s^{(1)} \approx iN_s = -i\left(\frac{2}{\pi sw}\right)^{1/2}\exp\left(\frac{sw^3}{3}\right), \tag{9}$$

where $w^2 = 1 - (\omega r/sc)^2 > 0$. Physically, the presence of this field geometry can be explained by the fact that the phase velocity of the radiation wave is less than the velocity of light, so that we are dealing with a surface wave that can propagate only along the beam boundary. For $r < b$, where $\omega^2 > 0$, the field decreases monotonically with decreasing radius (energy build-up). For $b < r < sc/\omega$ the field decreases with increasing radius, and for $r > sc/\omega$ we have radiation (volume wave). It is important to note that the term proportional to Δ_s^{-3} appears as a result of the integration by parts because of the dependence of Δ_s on r [according to Eq. (1), this is equivalent to a radial energy dependence].

Substituting Eq. (9) in Eq. (8), and transforming to the variable γ under the integral sign, we obtain

$$1 + \frac{2\pi e^2 s}{m\omega}\int_{\gamma_a}^{\gamma_b}\left(\frac{1}{w} + \frac{\omega}{\Delta_s}w^3 + \frac{1}{s}\frac{\omega^2}{\Delta_s^2}\right)\frac{n\,d\gamma}{\gamma^4\Delta_s} = 0, \tag{10}$$

where $\gamma_a = \omega_H a/c$, $\gamma_b = \omega_H b/c$, and the derivatives of the asymptotic functions are $J_s' = \omega w J_s/c$ and $N_s' = -\omega w N_s/c$. Assuming, as a rough approximation, that $s \sim \gamma^2$,[12,13] we arrive at the conclusion that, when $\Delta_s < \omega_H$, the last term in Eq. (10) is the dominant term, whereas for $\Delta_s > \omega_H$ the first term predominates. We shall consider Eq. (10) in these two limiting cases which, respectively, correspond to a narrow and a broad beam.

Before we solve Eq. (10), let us define the skin-layer depth and the beam-energy spread. To do this, let us expand w/r and $\Delta_s(r)$ into a series at the point b ($\gamma = \gamma_b - \Delta\gamma$):

$$w \approx \frac{1}{\gamma_b}(1 + 2\gamma_b\Delta\gamma)^{1/2}, \quad \Delta_s = \omega - \frac{s\omega_H}{\gamma_b} - \frac{s\omega_H}{\gamma_b^2}\Delta\gamma. \tag{11}$$

It follows from this expression that w and Δ_s are independent of energy, and can be taken out from under the integral sign when the following inequalities are respectively satisfied:

$$\gamma_b\Delta\gamma \ll 1, \quad \frac{s\omega_H}{\gamma_b^2}\Delta\gamma \ll \left|\omega - \frac{s\omega_H}{\gamma_b}\right|. \tag{12}$$

The first of these inequalities may be interpreted as the ratio of the beam thickness σ to the skin layer depth $\delta = c/\omega_H\gamma_b$. In fact, since differentiation of the asymptotic function given by Eq. (9) results in multiplication by $\omega_H\gamma/c$, we obtain $\gamma\Delta\gamma = \sigma/\delta$. The second inequality is stronger, and is a measure of the extent to which the beam is monoenergetic. Assuming that Eq. (12) is satisfied, we find that

$$\omega = \frac{s\omega_H}{\gamma_b} + \frac{1 + i\sqrt{3}}{2^{4/3}}\left(\frac{4\pi e^2\omega_H s}{m\gamma_b^5}\int_{\gamma_a}^{\gamma_b} n\,d\gamma\right)^{1/3}. \tag{12'}$$

Substituting Eq. (12) into Eq. (12′), we obtain the maximum energy spread in the monoenergetic beam, and the maximum growth rate:

$$s\Delta\gamma_s \lesssim \frac{\omega_0}{\omega_H}\gamma_b^{1/2}, \quad \mu_{max} \lesssim \frac{\omega_0}{\gamma_b^{3/2}}. \qquad (13)$$

Let us now consider the case of a diffuse beam whose linear dimensions exceed the skin-layer depth: $\gamma\Delta\gamma > 1$. In this case, $\Delta_s > \omega_H$ and it is sufficient to retain only the first term under the integral sign in Eq. (10). Substituting $\omega = \Omega + i\mu_s$ in Eq. (10), and separating the real and imaginary parts with the aid of the formula

$$\frac{1}{\Delta_s} = \frac{1}{\Omega_s - s\omega_H/\gamma} - i\pi\frac{\gamma^2}{s\omega_H}\delta(\gamma - \gamma_0), \quad \gamma_0 = \frac{s\omega_H}{\Omega_s} \qquad (14)$$

which automatically takes into account the phase of the logarithm, we find that

$$\mu_s = \frac{\pi}{2}\frac{\omega_0^2}{\omega_H\gamma_0}, \quad \gamma_a \leqslant \gamma_0 \leqslant \gamma_b;$$

$$\Omega_s^2 \approx -\frac{1}{2}\frac{s\omega_0^2}{\gamma_b^2}\ln\left|\frac{\gamma_b - \gamma_0}{\gamma_a - \gamma_0}\right|. \qquad (15)$$

In deriving the second equation, which determines the radiation frequency Ω_s, we have used the condition $\Delta\gamma \ll 1$ and retained only the function $(\gamma - \gamma_0)^{-1}$ under the integral sign. Since on the right-hand side of this relation we have a small parameter (beam density), and according to the first equation in Eq. (15) the instability develops only for frequencies such that $s\omega_H/\gamma_b < \Omega_s < s\omega_H/\gamma_0$, it follows that this equation is satisfied only for $\gamma_b - \gamma_0 \ll 1$ when the value of the logarithm is sufficiently large. Hence, it follows that only the thin charged-particle layer on the outer boundary $(r = b)$ with the vacuum is unstable. It would appear that this can be explained by the fact that the boundary between the beam and the vacuum is, in fact, a waveguide for the radiated wave and, therefore, the radiation condition is not satisfied for the interior of the beam. The reduction in the growth rate by the factor $\gamma^{-1/2}\omega_H/\omega_0 \gg 1$ in comparison with the maximum value [cf. Eq. (13)] can be explained by the fact that the phase velocity of the perturbation wave $v_{ph} \approx \omega_H b/\gamma_b$ exceeds the velocity of all the particles in the interior of the beam and, therefore, the energy of this wave should be efficiently absorbed by the nonresonance region of the beam as a result of the acceleration of the nonresonance particles.

3. Let us now consider the nonlinear focusing of the beam by the radiation field.[3)] While the instability is developing, the beam experiences the radial force $F_r = -m(v_1\nabla\gamma v_1)$, which is quadratic in the field amplitude.[4)] Using Eq. (2), we can rewrite this in the form

$$F_r = -\frac{3}{4}\frac{e^2}{m\gamma^5}\sum_s\frac{1}{\mu_s^2}\exp(2\mu_s t)\frac{d}{dr}|E_{rs}|^2 \qquad (16)$$

where, as before, we are assuming that $s \gg 1$, and retain only those terms which are proportional to s.

Focusing ends when the beam leaves the state of resonance with the radiation wave. Denoting by $\mathscr{E}_{rs}$ the maximum field amplitude, and equating F_r to the defocusing force $-(T/n)dn/dr$, where T is the beam temperature, we obtain the following radial density distribution in the focused beam:

$$n(r) = A\exp\left(-\frac{3}{4}\frac{e^2}{m^2\gamma^5 v_T^2}\sum_s\frac{1}{\mu_s^2}|\mathscr{E}_{rs}|^2\right), \quad v_T^2 = \frac{T}{m}, \qquad (17)$$

where A is an arbitrary constant. We shall determine the width of this focused layer by expanding $\mathscr{E}_{rs}$ in Eq. (17) into a series at r = a, where the field reaches its minimum. Retaining only the first expansion terms, and expressing A in terms of the particle density n_{max} on the inner beam boundary, we find that

$$n(x) = n_{max}\exp\left(-\frac{x}{\sigma_{min}}\right), \quad \frac{1}{\sigma_{min}} = \frac{e^2}{m^2\gamma_a^5 v_T^2}\sum_s\frac{1}{\mu_s^2}\frac{d|\mathscr{E}_{rs}|^2}{da}. \qquad (18)$$

Let us estimate the order of magnitude of σ_{min}. It follows from the Maxwell equations that for the higher-order harmonics $\mathscr{E}_{rs} \sim \gamma\mathscr{E}_{\varphi s}$ and, consequently,

$$\frac{d}{da}|\mathscr{E}_{rs}|^2 \sim \frac{s\omega_H}{c}|\mathscr{E}_{\varphi s}|^2.$$

At the same time, the quantity $\mathscr{E}_{\varphi s}$ is determined by the condition

$$\frac{e\mathscr{E}_{\varphi s}}{m\gamma^3\mu_s^2} \lesssim \frac{v_0\gamma}{s\omega_H}, \qquad (19)$$

which presupposes that the particle shift relative to the wave under the action of the longitudinal field component $\mathscr{E}_{\varphi s}$ is comparable with the resonance wavelength at the end of the instability process. Substituting Eq. (19) into Eq. (18), we obtain

$$\sigma_{min} \sim \frac{v_T^2\omega_H}{c\gamma_a^3\mu^2}\sum_s{}^{-1}\frac{1}{s} \sim \frac{v_T^2}{\omega_0^2}\frac{\omega_H}{c}\ln^{-1}s_{max}, \quad s_{max} \sim \gamma_a^3. \qquad (20)$$

Since the compression of the beam is one-dimensional, we have $n_{max} \sim n_0\sigma/\sigma_{min}$, where n_0 is the initial beam density.

When the beam dimensions exceed the skin-layer depth, we can only speak of the stabilization of the outer boundary of the beam.

4. We have noted that the maximum radiation-field amplitude is restricted by the nonlinear shift of the beam in phase relative to the wave. This effect is determined only by the azimuthal motion of the beam. In this section, we shall develop a nonlinear theory which allows for this effect in the case of a modulated beam in the form of a sequence of q charged-particle bunches distributed at equal distances on a circle of radius R.

We shall suppose that the current produced by the beam is

$$j_\varphi = eNv_\varphi\frac{1}{r}\delta(r - R)\sum_{s=1}^{q}\delta\left[\varphi - \varphi(t) - s\frac{2\pi}{q}\right], \qquad (21)$$

where R is the mean beam radius.[10,11]. Moreover, we shall suppose that the beam is placed in a conducting chamber of radius b, so that the oscillation spectrum is discrete. Let us substitute Eq. (21) in Eq. (4), and seek a solution of the form

$$H_z = H(t)J_q\left(\lambda_p\frac{r}{b}\right)e^{-iq\varphi}. \qquad (22)$$

Integrating with respect to φ between $-\pi/q$ and π/q,

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3)Since the field inside the beam decreases exponentially with radial distance, at a distance δ from the outer boundary r = b, effective focusing can occur only for $\sigma \ll \delta$.

4)We note that this expression loses its meaning near resonance, i.e. when $eE/mc\omega \lesssim \mu/\omega$. [16] However, in our case, $eE/mc\omega \sim (\mu/\omega)^2$ [cf. Eq. (19)], so that the gradient approximation is still valid.

and with respect to r between 0 and b, we obtain the following ordinary differential equation for $H(t)$:

$$\frac{d^2H}{dt^2} + \omega_p{}^2 H = -4\pi e n v_\varphi \omega_p J_q' \left(\lambda_p \frac{R}{b}\right) e^{iq\varphi(t)}. \tag{23}$$

where $\varphi(t)$ is the running coordinate of each bunch, $J_q' = 0$, $\omega_p = c\lambda_p/b$, and n is the mean density

$$n = N\frac{q}{\pi}(\lambda_p{}^2 - q^2)^{-1} J_q{}^{-2}(\lambda_p)\frac{\lambda_p}{b^2}$$

Equation (23), together with the equation of motion for each bunch

$$\frac{d}{dt}\gamma v_\varphi = \frac{e}{m}\operatorname{Re} E_\varphi[t, R, \varphi(t)], \qquad v_\varphi = R\frac{d\varphi}{dt} \tag{24}$$

form a set of nonlinear equations with time as the independent variable. We shall seek the solution in the form

$$H(t) = -i\mathscr{E}(t)\exp(i\omega_p t + i\vartheta), \tag{25}$$

where $\mathscr{E}$ and ϑ vary slowly during the period of the high-frequency field, and the modulation frequency $q\varphi_0 \equiv q\omega_H/\gamma_0$ is close to one of the natural frequencies of the resonator $\omega_p = \Omega$, which allows us to restrict our attention to the resonance harmonic. Substituting Eq. (25) in Eqs. (23) and (24), and recalling that

$$E_\varphi(r, t) = i\frac{c}{\omega} J_q'\left(\lambda_p \frac{r}{b}\right) H(t),$$

we obtain

$$\begin{aligned} &\frac{d}{dt}\gamma v_\varphi = \frac{e}{m}\mathscr{E} J_q'(R)\cos(\psi - \vartheta), \\ &\dot{\mathscr{E}} = -2\pi e n v_0 J_q'(R)\cos(\psi - \vartheta), \\ &\dot{\vartheta} = -2\pi e n v_0 J_q'(R)\frac{1}{\mathscr{E}}\sin(\psi - \vartheta), \end{aligned} \tag{26}$$

where $\psi = q\varphi(t) - \Omega t$. The first and second equations in Eq. (26) lead to the conservation of momentum in the beam-field system:

$$m v_\varphi \gamma + \frac{\mathscr{E}^2}{4\pi n v_0} = m v_0 \gamma_0, \qquad v_0 = \frac{\omega_H R}{\gamma_0} \equiv v_\varphi(0). \tag{27}$$

If we now transform to the variable $\Phi = \vartheta - \psi$ with the aid of Eq. (27), we can reduce the order of the set of equations given by Eq. (26):

$$\begin{aligned} &\dot{\mathscr{E}} = -2\pi e n v_0 J_q'(R)\cos\Phi, \\ &\dot{\Phi} = 2\pi e n v_0 J_q'(R)\frac{1}{\mathscr{E}}\sin\Phi + \Omega\left\{1 - \frac{\gamma_0 - w^2}{[1 + v_0{}^2(\gamma_0 - w^2)^2/c^2]^{1/2}}\right\} \end{aligned} \tag{28}$$

Integrating this subject to the initial conditions $\mathscr{E}(0) = \Phi(0) = 0$, we obtain Φ as a function of $\mathscr{E}$:

$$\frac{\omega_0}{\Omega} w \sin\Phi = w^2 + \frac{c^2}{v_0{}^2}\left\{\left[1 + \frac{v_0{}^2}{c^2}(\gamma_0 - w^2)^2\right]^{1/2} - \gamma_0\right\}, \tag{29}$$

where $\omega_0^2 = 4\pi e^2 n/m$ and $w^2 = \mathscr{E}^2/4\pi n m v_0^2$.

Expressing $\cos\Phi$ on the right-hand side of the first equation in Eq. (28) in terms of $\mathscr{E}$ given by Eq. (29), we obtain the equation for the field amplitude according to which $\mathscr{E}$ varies periodically between 0 and $\mathscr{E}_m$, where $\mathscr{E}_m$ satisfies the condition

$$w_m{}^3 + 2\frac{v_0{}^2}{c^2}\gamma_0{}^2\frac{\omega_0}{\Omega} w_m{}^2 - \frac{v_0{}^2}{c^2}\gamma_0{}^2\frac{\omega_0{}^2}{\Omega^2} w_m - 2\gamma_0{}^3\frac{\omega_0}{\Omega} = 0. \tag{30}$$

The solution of Eq. (30) for $\gamma_0(\omega_0/\Omega)^{2/3} \lesssim 1$ can be written in the form

$$w_m \approx \gamma_0\left(\frac{\omega_0}{\Omega}\right)^{1/3}\left[2^{1/3} - \frac{2}{3}\gamma_0\left(\frac{\omega_0}{\Omega}\right)^{2/3}\right]. \tag{31}$$

According to this result, the maximum radiation energy density is

$$\frac{\mathscr{E}_m{}^2}{4\pi} = n m v_0{}^2\gamma_0{}^2\left(2\frac{\omega_0}{\Omega}\right)^{2/3} \tag{32}$$

and, when $\gamma_0(\omega_0/\Omega)^{2/3} \le 1$, it is comparable with the beam energy density.[5)] The characteristic time for the process turns out to be

$$T \sim \frac{1}{\mu_p} \sim \gamma_0 \Omega^{-1/3}\omega_0{}^{-2/3}. \tag{33}$$

It follows that the presence of the resonator enhances the beam radiation, and hence the growth rate becomes a cubic function. The maximum field amplitude is then proportional to the square of the growth rate, and this agrees with (19).

Substituting Eq. (33) in Eq. (20), and retaining only one term in the sum, we obtain

$$\sigma_{min} \sim \frac{v_T{}^2\omega_H}{c\mu_p{}^2}\frac{q}{\gamma_0{}^3}. \tag{34}$$

In conclusion, let us consider two numerical expressions. For a proton storage ring with $\gamma_0 = 2$, $R = 300$ cm, $T_\perp = 25 \times 10^{-6}\, Mc^2\gamma_0$, and $\sigma = 1$ cm[17] we find that: $\omega_H = 1.7 \times 10^8\ \text{sec}^{-1}$, $\omega_p = 1.7 \times 10^3 n_p^{1/2}$, $v_T^2 = 2 \times 10^{16}\ \text{cm}^2/\text{sec}^2$. According to Eq. (20), we then have $\sigma_{min} = 3 \times 10^7/n_p$ cm. Therefore, an appreciable focusing of the proton ring ($\sigma_{min} \lesssim 0.1$ cm) can occur for $n_p \gtrsim 3 \times 10^8\ \text{cm}^{-3}$, and the beam contraction time is $\tau \lesssim 10^{-6}$ sec.

For the electron storage ring at the Physico-technical Institute of the Ukrainian Academy of Sciences[18] for which $\gamma_0 = 200$, $R = 50$ cm, $T_\perp = 4 \times 10^{-4}\, mc\gamma_0$, and $\sigma = 1$ cm, we find that: $\omega_H = 12 \times 10^{10}\ \text{sec}^{-1}$, $\omega_e = 5 \times 10^4 n_e^{1/2}$, $v_T^2 = 8 \times 10^{19}\ \text{cm}^2/\text{sec}^2$. Hence, it follows that: $\sigma_{min} = 4 \times 10^{10}/n_e$ cm, i.e., effective focusing occurs for $n_e \gtrsim 4 \times 10^{10}\ \text{cm}^{-3}$, and the corresponding time is $\tau \lesssim 10^{-6}$ sec.

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