GREEN'S FUNCTION OF A PHOTON IN A CONSTANT HOMOGENEOUS ELECTROMAGNETIC FIELD OF GENERAL FORM

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The covariant structure of the polarization operator of a photon, following from relativistic, gauge, and charge invariance, is obtained in general form in the presence of an external electromagnetic field with an intensity which is constant in time and space. The polarization operator is represented in diagonal form. The photon Green's function is expressed in terms of the polarization operator in general form with the aid of Dyson's equation. Maxwell's equations in an external field are solved in general form. The polarization operator is calculated in the loop approximation, when the exact Green's functions of the Dirac equation in the electromagnetic field under consideration are taken as the electron propagators.

1. INTRODUCTION

RECENTLY the problem of the propagation of light in empty space, occupied by an external electromagnetic field, has attracted attention, this problem being similar to the problem of propagation in an anisotropic medium. The case of an external electromagnetic field of special form, when the vectors of the electric and magnetic fields are constant, equal in magnitude and mutually orthogonal (the so-called "crossed field") was studied earlier by the authors[1] and by Narozhnyĭ.[2] In this connection the covariant structure of the exact polarization operator was obtained,[1] the polarization operator was calculated to second order in the radiative corrections with exact account of the external field (the loop approximation), and the photon Green's function was obtained as the solution of Dyson's equation, where the calculated polarization operator[1,2] was used as the kernel. Expressions were obtained in general form for the free electromagnetic waves as solutions of Maxwell's equations in the "medium."[1] The existence of different branches in the photon spectrum was indicated, including spurion branches.[1] An investigation of the photon spectrum in a crossed field was carried out by Ritus.[3] In the present article we carry out the same kind of program for the case of an external constant and homogeneous field of more general form, where the final form of the polarization operator and the Green's function is given in a diagonal representation.

In the special case of a constant homogeneous external field with $\mathbf{E}\cdot\mathbf{H} = 0$, the problem under consideration was studied in article[4] to the lowest-order approximations with respect to the intensity of the external field. We shall not use approximations of such a type anywhere.

2. GENERAL CONSIDERATIONS

An electromagnetic field having a field tensor $F_{\mu\nu} = -F_{\nu\mu}$ which does not depend on the coordinates is given in the most general case by the vector potential (the Lorentz gauge is used)

$$A_\mu(x) = -{}^1/{}_2 F_{\mu\nu} x_\nu. \tag{1}$$

Since the renormalized polarization operator $\Pi^R_{\mu\nu}(k, k' \mid A)$ is a gauge-invariant quantity, then it is clear that the linear dependence of the vector potential (1) on the coordinates should not lead to the appearance of an exchange of energy-momentum between the photon and the external field (such an exchange arises for the electron, whose mass operator is not gauge-invariant). The fact that $\Pi^R_{\mu\nu}(k, k' \mid A)$ actually contains a δ-function, expressing the law of conservation of the energy-momentum of the photon, may be proved formally[1] with the aid of an expansion of $\Pi^R_{\mu\nu}$ in a functional series in the external field.

Now let us investigate the tensor structure of $\Pi^R_{\mu\nu}$. There are four independent vectors at our disposal:[1)] k_μ, $F_{\mu\nu}k_\nu$, $F^2_{\mu\nu}k_\nu$ and $F^3_{\mu\nu}k_\nu$, out of which 16 tensors (including $\delta_{\mu\nu}$) can be formed with the aid of the operation of taking the direct products, thus exhausting all possible tensors of the second rank in four-dimensional space. Since the polarization operator is a symmetric tensor, out of the constructed tensors we can only use the 10 symmetric ones. The transversality condition on the polarization operator, $\Pi^R_{\mu\nu}k_\nu = 0$ (gauge invariance), contains effectively four conditions (according to the number of independent vectors) and we have left six independent transverse symmetric tensors for the construction of $\Pi^R_{\mu\nu}$. It turns out that two of these contain odd powers of the tensor $F_{\mu\nu}$. Noting that there are only four independent scalars at our disposal: k^2, $k_\mu F^2_{\mu\nu}k_\nu$, $\mathfrak{F} = (1/4)F_{\mu\nu}F_{\mu\nu}$, $\mathfrak{G} = (1/4)F_{\mu\nu}F^*_{\mu\nu}$ (here $F^*_{\mu\nu} = (1/2)i\epsilon_{\mu\nu\lambda\kappa}F_{\lambda\kappa}$), and all of these are even in $F_{\mu\nu}$, we conclude that it is impossible to construct functions which are even in $F_{\mu\nu}$ (which the polarization operator is according to Furry's theorem) by using these two tensors. Thus, the polarization operator of a photon in the external field (1) has the form

$$\Pi_{\mu\nu}{}^R(k, k' \mid A) = (2\pi)^4 \delta(k + k')\Pi_{\mu\nu}{}^R(k) \tag{2}$$

[1)]The summation over repeated Greek indices is to be understood everywhere in this article. The n^{th} power of a tensor means the successive contraction of n tensors with respect to inner indices.

$$=(2\pi)^4\delta(k+k')\sum_{i=1}^{4}\pi_i(k^2,kF^2k,\mathfrak{F},\mathfrak{G})\psi^{(i)}_{\mu\nu}(k,F),$$

where $\psi^{(i)}_{\mu\nu}$ denotes the set of symmetric, transverse, linearly independent matrices which are even functions of F. For example,

$$\psi^{(1)}_{\mu\nu}=k^2\delta_{\mu\nu}-k_\mu k_\nu,\quad \psi^{(2)}_{\mu\nu}=-(F_{\mu\rho}k_\rho)(F_{\nu\lambda}k_\lambda),$$
$$\psi^{(3)}_{\mu\nu}=-k^2(\delta_{\mu\rho}-k_\mu k_\rho/k^2)F_{\rho\varkappa}{}^2(\delta_{\varkappa\nu}-k_\varkappa k_\nu/k^2),\tag{3}$$
$$\psi^{(4)}_{\mu\nu}=(F_{\mu\rho}k_\rho)(F_{\nu\varkappa}{}^3k_\varkappa)+(F_{\mu\rho}{}^3k_\rho)(F_{\nu\varkappa}k_\varkappa).$$

Instead of (2) it is more convenient to have the polarization operator in the physically more meaningful diagonal form. In order to change to diagonal form, let us consider Π^R in the following special representation:

$$\Pi_{\mu\nu}{}^R(k)=\sum_{m,\,n=1}^{4}d_\mu^{\,m_{\rm com}}\Pi^{mn}d_\nu^{\,n_{\rm com}},\qquad \Pi^{mn}=d_\mu{}^m\Pi_{\mu\nu}{}^R(k)d_\nu{}^n,\tag{4}$$

where d^m_μ are the four (complex) eigenvectors of the antisymmetric tensor $F_{\mu\nu}$ ($F_{\mu\nu}d^m_\nu=f_md^m_\mu$) expressed in terms of its eigen matrices

$$A_{\mu\nu}{}^m=\frac{-\bar f_m{}^2\delta_{\mu\nu}+f_mF_{\mu\nu}+F_{\mu\nu}{}^2-i\bar f_mF_{\mu\nu}{}^*}{2(f_m{}^2-\bar f_m)^2},\qquad m=1,2,3,4,$$
$$A_{\mu\rho}{}^mF_{\rho\nu}=F_{\mu\rho}A_{\rho\nu}{}^m=f_mA_{\mu\nu}{}^m,\tag{5}$$
$$A_{\mu\rho}{}^mA_{\rho\nu}{}^n=\delta_{mn}A_{\mu\nu}{}^m,\qquad \operatorname{Sp}A^m=1$$

with the aid of the relations

$$d_\mu{}^m=A_{\mu\nu}{}^mk_\nu(kA^mk)^{-1/2},\qquad d_\mu{}^n\,d_\mu{}^m=\delta_{m,\,n_{\rm com}}.\tag{6}$$

The eigenvalues are expressed in terms of the field invariants:

$$f_{1,2}=\pm i(\mathfrak{F}+\sqrt{\mathfrak{F}^2+\mathfrak{G}^2})^{1/2},\qquad f_{3,4}=\pm(-\mathfrak{F}+\sqrt{\mathfrak{F}^2+\mathfrak{G}^2})^{1/2}\ldots\tag{7}$$

The bar over f_m in (5) denotes the following interchange of the number m: $1\longleftrightarrow 4$, $2\longleftrightarrow 3$. The subscript "com" associated with the indices in expressions (4) and (6) means an interchange of the indices according to the rule $1\longleftrightarrow 2$, $3\longleftrightarrow 4$, for example, $1_{\rm com}=2$. A more detailed utilization of the extremely convenient computational apparatus associated with the quantities (5)–(7) is described in the articles by the authors.[5,6] The square root in (6) is understood in such a manner that $\sqrt{(-\alpha)^2}=-\alpha$ for $\alpha>0$ (the cut is drawn to the right), and in (7) it is to be understood such that $if_m\bar f_m=\mathfrak{G}$.

Using any appropriate set of the matrices in (2) (for example, (3)), one can show that the 16 quantities Π^{mn} introduced by us in (4) satisfy the relations

$$\Pi^{mn}=\Pi^{nm},\quad \Pi^{11}=\Pi^{22},\quad \Pi^{33}=\Pi^{44},\quad \Pi^{14}=\Pi^{23},\quad \Pi^{13}=\Pi^{24},$$
$$-(\Pi^{13}+\Pi^{14})=(\Pi^{12}+\Pi^{11})\frac{d^{(1)}k}{d^{(3)}k}=(\Pi^{34}+\Pi^{33})\frac{d^{(3)}k}{d^{(1)}k},\tag{8}$$

so that four independent quantities remain among the elements Π^{mn}, which we choose in the following way:

$$\Lambda_1=\Pi^{11}+\Pi^{12},\quad \Lambda_2=\Pi^{11}-\Pi^{12},\quad \Lambda_3=\Pi^{13}-\Pi^{23},\quad \Lambda_4=\Pi^{33}-\Pi^{34}.\tag{9}$$

The quantities Λ_i are functions of the four scalars $\mathfrak{F}$, $\mathfrak{G}$, $(d^{(1)}k)$, $(d^{(3)}k)^2$ (the relations $(d^{(1)}k)^2+(d^{(3)}k)^2=k^2/2$ and $(d^{(3)}k)^2-(d^{(1)}k)^2=(kF^2k+\mathfrak{F}k^2)\times(2\sqrt{\mathfrak{F}^2+\mathfrak{G}^2})^{-1}$ exist). They describe the polarization operator to the same extent as the functions Π_i which appear in the more traditional way of writing expression (2); however the diagonal form of $\Pi^R_{\mu\nu}$ is easier to express in terms of Λ_i. In addition, the quantities Λ_i are much easier to find if one or the other specific form of $\Pi^R_{\mu\nu}$ is given (see the following section). Therefore in what follows we shall as a rule express all quantities in terms of Λ_i.

By using the relations (8) one can verify that the polarization operator (4) has four mutually orthogonal eigenvectors:

$$\Pi_{\mu\nu}{}^Rb_\nu^{(i)}=\varkappa_ib_\mu^{(i)},$$
$$b_\mu^{(1)}=(d^{(3)}k)(d_\mu^{(1)}+d_\mu^{(2)})-(d^{(1)}k)(d_\mu^{(3)}+d_\mu^{(4)}),\tag{10}$$
$$b_\mu^{(2,3)}=-2\Lambda_3(d_\mu^{(1)}-d_\mu^{(2)})+[(\Lambda_2-\Lambda_4)\pm\sqrt{(\Lambda_2-\Lambda_4)^2+4\Lambda_3{}^2}](d_\mu^{(3)}-d_\mu^{(4)}),$$
$$b_\mu^{(4)}=k_\mu$$

with the eigenvalues

$$\varkappa_1=\frac{\Lambda_1k^2}{2(d^{(3)}k)^2}\qquad \varkappa_{2,3}=\frac{-\Lambda_2-\Lambda_4\pm\sqrt{(\Lambda_2-\Lambda_4)^2+4\Lambda_3{}^2}}{2},\qquad \varkappa_4=0\tag{11}$$

and its diagonal representation is given by

$$\Pi'^R_{\mu\nu}(k,k'|A)=(2\pi)^4\delta(k+k')\Pi_{\mu\nu}{}^R(k)=(2\pi)^4\delta(k+k')\sum_{i=1}^{3}\varkappa_i\frac{b_\mu{}^{(i)}b_\nu{}^{(i)}}{(b^{(i)})^2}\tag{12}$$

The renormalized photon Green's function $D_{\mu\nu}$ in an external electromagnetic field satisfies the generalized Dyson equation[7]

$$(k^2\delta_{\mu\rho}-k_\mu k_\rho)D_{\rho\nu}(k,k')-\frac{1}{(2\pi)^4}\int\Pi_{\mu\rho}{}^R(k,k_1|A)D_{\rho\nu}(-k_1,k')d^4k_1$$
$$=(2\pi)^4\delta(k+k')\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right),\tag{13}$$

solving this equation in the case of the field (1) by substituting (12) into it, we obtain (to within the longitudinal part which remains arbitrary)

$$D_{\mu\nu}=(2\pi)^4\delta(k+k')\sum_{i=1}^{3}\frac{1}{k^2-\varkappa_i}\frac{b_\mu^{(i)}b_\nu^{(i)}}{(b^{(i)})^2}.\tag{14}$$

The vector potential of free electromagnetic waves, propagating in space occupied by the external field (1), satisfies the equation which is obtained from (13) by discarding its right-hand part (Maxwell's equation in the "medium"). It has the following form (to within the physically unimportant longitudinal part)

$$A_\mu(k)=\sum_{i=1}^{3}b_\mu^{(i)}\delta(k^2-\varkappa_i)f_i(k),\tag{15}$$

where the $f_i(k)$ are arbitrary functions of the momentum. Expressions (14) and (15) show that for each 4-polarization $b^{(i)}_\mu$ there is a separate dispersion law which is obtained by solving the equation

$$k^2=\varkappa_i(k^2,kF^2k,\mathfrak{F},\mathfrak{G}).\tag{16}$$

In the special case $\mathfrak{G}=0$ (a reference system exists in which only an external magnetic field exists ($\mathfrak{F}>0$) or only an external electric field exists ($\mathfrak{F}<0$)) the relation $F^3_{\mu\nu}=-2\mathfrak{F}F_{\mu\nu}$ arises, and therefore for the construction of the matrices $\psi^{(i)}_{\mu\nu}$ in (2) we have one less vector at our disposal. Therefore the matrix $\psi^{(4)}_{\mu\nu}$ in (3) becomes linearly dependent on $\psi^{(2)}_{\mu\nu}$ and does not participate anymore in the expansion (2). In this special case, as one can show, there is one more relation which supplements the relations (8): $\Lambda_3=0$. Under this condition the eigenvectors $b^{(2,3)}_\mu$ given by (10) become (to within normalization) independent of the supporting dynamical information of the invariant functions Λ_i, similar to the way this occurs in the general case for the vector $b_\mu^{(1)}$. Instead of (10), for $\mathfrak{G} = 0$ we have

$$b_\mu^{(1)} = (F_{\mu\rho}^2 k_\rho) k^2 - k_\mu (kF^2k), \quad b_\mu^{(2)} = (F_{\mu\rho}^* k_\rho), \quad b_\mu^{(3)} = F_{\mu\rho} k_\rho, \quad b_\mu^{(4)} = k_\mu. \tag{17}$$

In this connection the transition from the representation (2), (3) to the diagonal representation (12) is accomplished by a linear transformation, in accordance with

$$\frac{b_\mu^{(1)} b_\nu^{(2)}}{(b^{(1)})^2} = \frac{kF^2k\psi_{\mu\nu}^{(3)} - 2\mathfrak{F}k^2\psi_{\mu\nu}^{(2)}}{(kF^2k + 2\mathfrak{F}k^2)kF^2k}, \quad \frac{b_\mu^{(2)} b_\nu^{(2)}}{(b^{(2)})^2} = \frac{\psi_{\mu\nu}^{(1)}}{k^2} - \frac{\psi_{\mu\nu}^{(3)} + \psi_{\mu\nu}^{(2)}}{kF^2k + 2\mathfrak{F}k^2},$$

$$\frac{b_\mu^{(3)} b_\nu^{(3)}}{(b^{(3)})^2} = \frac{\psi_{\mu\nu}^{(2)}}{kF^2k}; \quad \varkappa_1 = \Pi_1 k^2 + \Pi_3 (kF^2k + 2\mathfrak{F}k^2),$$
$$\varkappa_2 = \Pi_1 k^2, \quad \varkappa_3 = \Pi_1 k^2 + \Pi_2 kF^2k + \Pi_3 2\mathfrak{F}k^2. \tag{18}$$

In the other special case, when $\mathfrak{F} = 0$, such a type of "kinematization" of the basis is not observed.

In the limit $\mathfrak{F} = 0$, $\mathfrak{G} = 0$ (the crossed field case) the eigenvectors are again given by expressions (17), where the order of the limiting transitions perhaps affects the quantities Λ_i but does not have any effect on the eigenvectors $b_\mu^{(i)}$ and on the eigenvalues κ_i given by (11). Thus, if first $\mathfrak{F} \to 0$ and then $\mathfrak{G} \to 0$, then $\Lambda_3 \neq 0$, and the "kinematization" of the vectors (10) guarantees that in this case $(\Lambda_2 - \Lambda_4) \to 0$. The diagonal form of Eqs. (12), (14), and (15) can be imparted to the results of the work on the crossed field[1,2] with the aid of the relations (18); in order to do this it is sufficient to set $\mathfrak{F} = 0$ in the latter equations.

In concluding this section we emphasize that the results contained in it are not related to any kind of approximations, but follow exclusively from the general properties of relativistic, gauge, and charge invariance of the theory. In the general case $\mathfrak{F} \neq 0$, $\mathfrak{G} \neq 0$ the polarization operator, the photon Green's function, and the free electromagnetic waves are expressed in terms of four functions Λ_i of the invariants appearing in the problem. We now go on to the calculation of these functions in a specific approximation.

3. THE LOOP APPROXIMATION

The renormalized polarization operator which enters into Eq. (13) is given by

$$\Pi_{\mu\nu}^R(k, k'|A) = \Pi_{\mu\nu}(k, k'|A) - \Pi_{\mu\nu}(k, k'|0) + (2\pi)^4\delta(k + k')(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\mathcal{T}(k^2), \tag{19}$$

where the unrenormalized polarization operator $\Pi_{\mu\nu}$ is given by the expression

$$\Pi_{\mu\nu}(k, k'|A) = -ie^2 \int e^{-i(kx+k'y)} \operatorname{Sp} \gamma_\mu G(x, x')\Gamma_\nu(x', x'', y)G(x'', x)\,dx\,dy\,dx'\,dx''. \tag{20}$$

and the third term in (19) is the renormalized polarization operator in the absence of the external field.

Let us consider the approximation $\Gamma_\nu = \gamma_\nu\delta(x' - x'')\delta(x' - y)$ and as $G(x, x')$ we take Schwinger's[8] exact solution of the Dirac equation for the Green's function in the presence of the field (1)

$$(\hat{\partial}_x + m - ie\hat{A}(x))G(x, x') = \delta(x - x'). \tag{21}$$

The corresponding calculations are described in detail by the authors in the preprint[6] and lead to the following answer:

$$\Pi_{\mu\nu}^R(k) = \frac{e^2}{2\pi^2}\int_0^\infty\int_0^\infty d\nu_1\,d\nu_2 \Bigg[\Bigg\{\left(\frac{\cos(eF^*\nu_2)\operatorname{sh}(eF\nu_1)}{\operatorname{sh}(eF(\nu_1+\nu_2))}\bigg|_{\mu\lambda} k_\lambda\right)$$
$$\times\left(\frac{\cos(eF^*\nu_1)\operatorname{sh}(eF\nu_2)}{\operatorname{sh}(eF(\nu_1+\nu_2))}\bigg|_{\nu\varkappa} k_\varkappa\right) - \left(\frac{\sin(eF^*\nu_2)\operatorname{sh}(eF\nu_1)}{\operatorname{sh}(eF(\nu_1+\nu_2))}\bigg|_{\mu\lambda} k_\lambda\right)$$
$$\times\left(\frac{\sin(eF^*\nu_1)\operatorname{sh}(eF\nu_2)}{\operatorname{sh}(eF(\nu_1+\nu_2))}\bigg|_{\nu\varkappa} k_\varkappa\right)$$
$$-\left(\frac{\cos(eF^*\nu_2)\operatorname{sh}(eF\nu_1)}{\operatorname{sh}(eF(\nu_1+\nu_2))}\right)\bigg|_{\mu\nu}\left(k_\lambda\frac{\cos(eF^*\nu_1)\operatorname{sh}(eF\nu_2)}{\operatorname{sh}(eF(\nu_1+\nu_2))}\bigg|_{\lambda\varkappa} k_\varkappa\right)\Bigg\}$$
$$\times\exp\left\{-i\left(k_\lambda\frac{\operatorname{sh}(eF\nu_1)\operatorname{sh}(eF\nu_2)}{eF\operatorname{sh}(eF(\nu_1+\nu_2))}\bigg|_{\lambda\varkappa} k_\varkappa\right)\right\}\cdot$$
$$\times\frac{e^2 f_1 f_3}{\operatorname{sh}(ef_1(\nu_1+\nu_2))\operatorname{sh}(ef_3(\nu_1+\nu_2))} - \frac{\nu_1\nu_2}{(\nu_1+\nu_2)^4}(k_\mu k_\nu$$
$$-k^2\delta_{\mu\nu})\exp\left\{-i\frac{\nu_1\nu_2}{\nu_1+\nu_2}k^2\right\}\Bigg]e^{-i(\nu_1+\nu_2)m^2} + (\delta_{\mu\nu}k^2 - k_\mu k_\nu)\mathcal{T}^{(2)}(k^2). \tag{22}$$

Here by the tensor function $\Phi_{\mu\nu}(F)$ one understands its expansion in a series, where $(F^nF^{*m})_{\mu\nu} \equiv F_{\mu\mu_2}F_{\mu_2\mu_3}\ldots F_{\mu_n\nu_1}F^*_{\nu_1\nu_2}\ldots F^*_{\nu_m\nu}$. The fact that the tensor powers $F^n_{\mu\nu}$, beginning with $n = 4$, can be linearly expressed in terms of powers with $n < 4$, makes it possible to specify functions of the tensor in closed form with the aid of the equation

$$\Phi_{\mu\nu}(F^*, F) = \sum_{j=1}^4 \Phi(-if_j, f_j)A_{\mu\nu}^j, \tag{23}$$

where the $A^j_{\mu\nu}$ are the matrices defined in (5). However, the form (22) is more convenient in connection with the necessity to use the polarization operator as an intermediate expression.

In order to obtain a specific expression for the polarization operator (22), the four functions Λ_i can easily be calculated (according to formulas (4) and (9) by using expressions (5), (6), and (23)):

$$\Lambda_i = \frac{e^2}{\pi^2}\int_0^\infty\int_0^\infty\Bigg[\lambda_i(\nu_1,\nu_2)\frac{e^2 f_1 f_3}{\operatorname{sh}(ef_1(\nu_1+\nu_2))\operatorname{sh}(ef_3(\nu_1+\nu_2))}\cdot$$
$$\times\exp\left\{-2i\sum_{j=1,3}(d^{(j)}k)^2\frac{\operatorname{sh}(ef_j\nu_1)\operatorname{sh}(ef_j\nu_2)}{ef_j\operatorname{sh}(ef_j(\nu_1+\nu_2))}\right\}$$
$$-\lambda_i^{(0)}\frac{\nu_1\nu_2}{(\nu_1+\nu_2)^4}\exp\left\{-i\frac{\nu_1\nu_2}{\nu_1+\nu_2}k^2\right\}\Bigg]e^{-i(\nu_1+\nu_2)m^2}\,d\nu_1 d\nu_2 - 2\lambda_i^{(0)}\mathcal{T}^{(2)}(k^2). \tag{24}$$

Here

$$\lambda_1 = -(d^{(3)}k)^2\frac{\operatorname{ch}(ef_3\nu_2)\operatorname{sh}(ef_1\nu_1)}{\operatorname{sh}(ef_1(\nu_1+\nu_2))}\frac{\operatorname{ch}(ef_1\nu_1)\operatorname{sh}(ef_3\nu_2)}{\operatorname{sh}(ef_3(\nu_1+\nu_2))},$$
$$\lambda_2 = (d^{(1)}k)^2\frac{\operatorname{ch}(ef_3(\nu_1+\nu_2))\operatorname{sh}(ef_1\nu_1)\operatorname{sh}(ef_1\nu_2)}{\operatorname{sh}^2(ef_1(\nu_1+\nu_2))} - \lambda_1,$$
$$\lambda_3 = (d^{(1)}k)(d^{(3)}k)\frac{[\operatorname{sh}(ef_3\nu_2)\operatorname{sh}(ef_1\nu_1)]^2}{\operatorname{sh}(ef_1(\nu_1+\nu_2))\operatorname{sh}(ef_3(\nu_1+\nu_2))}, \tag{25}$$
$$\lambda_4 = (d^{(3)}k)^2\frac{\operatorname{ch}(ef_1(\nu_1+\nu_2))\operatorname{sh}(ef_3\nu_1)\operatorname{sh}(ef_3\nu_2)}{\operatorname{sh}^2(ef_3(\nu_1+\nu_2))} - \frac{(d^{(1)}k)^2}{(d^{(3)}k)^2}\lambda_1,$$
$$\lambda_1^{(0)} = -(d^{(3)}k)^2, \quad \lambda_2^{(0)} = {}^1\!/_2k^2, \quad \lambda_3^{(0)} = 0, \quad \lambda_4^{(0)} = {}^1\!/_2k^2.$$

Expressions for Π_i in the representation given by (2) and (3) are written down in the preprint[6]. For $\mathfrak{F} \to 0$ and $\mathfrak{G} \to 0$ the corresponding expressions of articles[1,2] are obtained from Eqs. (24) and (25) with (18) taken into consideration.

One can make the following remark about the nature of the photon spectrum which is obtained upon using (24) and (25) in expressions (14), (15), (10), and (11). The two equations with $i = 2, 3$ in (16) possess solutions which pass through the vertex of the light cone $k_\mu = 0$, i.e., two branches of the spectrum exist corresponding to long wavelength massless excitations—that is, photons (double refraction). As far as the equation with $i = 1$ in (16) is concerned, formally it has its own solution on the entire light cone, $k^2 = 0$. In actual fact, however, this solution does not correspond to any kind of real electromagnetic waves since upon substituting the condition $k^2 = 0$ the corresponding 4-vector polarization (10) becomes purely longitudinal: $b_\mu^{(1)} \sim k_\mu$.

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Note Added in Proof (January 15, 1971). For a field of the magnetic type ($\mathfrak{G} = 0$, $\mathfrak{F} > 0$) and for directions of the momentum of the virtual photon such that $kF^2k = 0$, a cyclotron resonance of the vacuum polarization exists: upon using expressions (24) and (25) the polarization operator tends to infinity at the thresholds for the creation of electron-positron pairs in discrete states.

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