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CONTRIBUTION TO THE THEORY OF LINEAR WAVE CONVERSION IN AN INHOMOGENEOUS PLASMA

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Linear conversion of waves in an inhomogeneous plasma described by a fourth order differential equation with inhomogeneous coefficients is considered. It is demonstrated that in the vicinity of points where the coefficient of the second derivative vanishes, a quasisurface wave may arise which propagates in the direction perpendicular to the density gradient. It is shown in the magnetohydrodynamic approximation that a sharp change in the shape and polarization of the initial magnetosonic wave also occurs when the velocity of sound considerably exceeds the Alfvén velocity in the inhomogeneity region. Furthermore, the conditions under which surface waves may be generated at a sharp boundary between two plasma media that is perpendicular to the magnetic field are elucidated. The coefficients of reflection, refraction and conversion of magnetohydrodynamic waves at the boundary are obtained

I. INTRODUCTION

An extensive literature exists on the problems of linear conversion of waves on plasma inhomogeneities. The study of wave conversion is of interest in applications to the problems of heating, radiation and confinement of a plasma in magnetic mirrors as well as in the elucidation of certain astrophysical phenomena. The conversion of plasma waves into electromagnetic waves, and vice versa, at a plasma-vacuum boundary and at a sharp boundary between two plasma media were first considered in[1,2]. In this case, the problem reduces to that of satisfying definite boundary conditions at the interface. The sharp boundary approximation is justified if the thickness a of the transitional layer over which the plasma parameters change appreciably is much smaller than the wave-length ($a \ll \lambda$).

In the opposite limiting case, when the wavelength is much smaller than the characteristic dimension of an inhomogeneity ($\lambda \ll a$), the coupling between waves of two different modes is described, as a rule[3-5], by a fourth order differential equation with inhomogeneous coefficients:

$$\alpha^2\Psi^{IV} + V_2(x)\Psi^{II} + V_1(x)\Psi = 0, \qquad (1)$$

where $\alpha/a^2 \ll 1$, $V_2/a^2 \sim 1$, and $V_1 \sim 1$.

Thus, for example, for the case when the plasma density gradient is directed along the magnetic field (the x axis) and the perturbed quantities are proportional to $\exp(ik_y y - i\omega t)$, the coupling between a slow and a fast magnetosonic wave is described in the magnetohydrodynamic approximation by Eq. (1) with the coefficients[5,6]:

$$\alpha^2 = s^4/\omega^4; \quad V_1(x) = \frac{s^2}{v_A^2(x)} - \frac{k_y^2 s^2}{\omega^2}\left(1 + \frac{s^2}{v_A^2(x)}\right),$$

$$V_2(x) = \frac{s^2}{\omega^2}\left[1 + \frac{s^2}{v_A^2(x)} - \frac{k_y^2 s^2}{\omega^2}\right]. \qquad (2)$$

Here, s is the velocity of sound in the plasma in the absence of an external magnetic field, and $v_A = \sqrt{H_0^2/4\pi\rho_0}$ is the Alfvén velocity. Equation (1) is usually solved using: the perturbation method[7,8], the method of phase integrals[6,8,9], and also the Wasow asymptotic theory and its modifications[4,5,10-12]. The question as to which method is applicable depends, generally speaking, on the behavior of the functions V_1 and V_2 in the plasma region being considered. However, the realms of applicability of these methods may partially overlap. We note also that certain problems connected with the stability of plasma and liquid oscillations also lead to Eq. (1)[13].

In the region of mild inhomogeneity where the geometrical optics approximation (WKB) is applicable, the four linearly independent solutions of Eq. (1) have, as is well known, the form

$$\Psi \sim \frac{k_{1,2}^{-1/2}}{(V_2^2/4\alpha^4 - V_1/\alpha^2)^{1/4}} \exp\left\{\pm i\int^x k_{1,2}(x)\,dx\right\}, \qquad (3)$$

where

$$k_1(x) = \left(\frac{V_2}{2\alpha^2} + \sqrt{\frac{V_2^2}{4\alpha^4} - \frac{V_1}{\alpha^2}}\right)^{1/2}, \qquad (3a)$$

$$k_2(x) = \left(\frac{V_2}{2\alpha^2} - \sqrt{\frac{V_2^2}{4\alpha^4} - \frac{V_1}{\alpha^2}}\right)^{1/2}. \qquad (3b)$$

It may be said that the same wave modes propagate in a mildly inhomogeneous plasma as in a homogeneous one. In magnetohydrodynamics, when the coefficients have the form (2), the solutions with the "wave numbers" $k_1(x)$ and $k_2(x)$ correspond to slow and fast magnetosonic waves. For the Wasow asymptote to be applicable to the analysis of Eq. (1), it is necessary for the coefficient $V_2(x)$ to vanish at some point inside the plasma. Wave conversions, which take place in the vicinity of the points where $V_2 = 0$, are, in the opinion of the authors of[4,5], complete and have been given the name "anomalous conversions" by these authors. At the same time, under the conditions of applicability of the other methods, which obtains, as a rule, is small conversions of the waves. For example, the method of phase integrals gives for magnetohydrodynamic waves an exponentially small conversion[6,8]. Therefore, elucidation of the meaning of "anomalous conversion" and a thorough analysis of the conditions of applicability of the Wasow asymptote to concrete equations are called for.

Our principal aim in the present paper is to elucidate the physical meaning of the term "anomalous conversion" of waves, using the coupling between magnetohydrodynamic waves (mhd) as an illustrative example. It is necessary to distinguish between two possible types of conversion. The first is conversion in the sense of a transition from the solution (wave) with the "wave number" $k_1(x)$ to the solution with the "wave number" $k_2(x)$ or $-k_2(x)$. Such a conversion occurs most strongly in the vicinity of points where $k_1 \approx k_2$, $k_1 \approx -k_2$ (i.e., where $V_2^2/4\alpha^2 \approx V_1$), and it may be called the conversion proper. The other kind of conversion is connected with a sharp change in $k_1(x)$ itself in the vicinity of certain points over distances much less than the dimensions of an inhomogeneity. It is precisely such type that the so-called "anomalous conversion" corresponds to. For the latter type, it is not only necessary that the second derivative coefficient V_2 vanished at some point x_0 inside the plasma but, also, other conditions imposed on the coefficients V_1 and V_2 to make the Wasow asymptote valid at distances from x_0 which are smaller than the characteristic dimension of the inhomogeneities, must be fulfilled.

It is shown in Sec. 3 by way of an illustration that "anomalous conversion" of mhd waves in the indicated sense may occur only in a sufficiently dense plasma ($v_A^2 \ll s^2$). At the same time, the Wasow asymptotes, together with the boundary conditions, describe the conversion proper. It turns out that in the case when $V_2(x)$ [Eq. (2)] vanishes at some point inside the plasma, only waves with the "wave number" $k_1(x)$—slow magnetosonic waves—can freely propagate in the plasma. The "wave number" $k_2(x)$ is then purely imaginary. In this connection, whenever a slow magnetosonic wave is incident at an oblique angle on a plasma layer with a slowly varying density, there appears in the vicinity of the plane $x = x_0$, where $V_2(x_0) = 0$, a quasisurface wave which propagates in the direction perpendicular to the inhomogeneity gradient and rapidly decays with increasing distance from this plane. This phenomenon is, in many ways, analogous to the appearance of a surface wave whenever mhd waves are incident at an oblique angle on the boundary between two plasma media.

The conversion of mhd waves at a sharp plasma boundary ($\lambda \ll a$) perpendicular to the magnetic field is studied in Sec. 4. Mhd wave conversion has been studied before with a different geometry in[14]. Conversion coefficients for certain particular cases were calculated in this paper but the question of the appearance of surface waves was not considered. In contrast to[14], the coefficients of reflection, refraction and conversion of mhd waves are determined in their general forms in Sec. 4. Furthermore, conditions for the appearance of surface waves are found. It is shown that the generated surface wave in the thin layer near the boundary gives rise to a flow of electromagnetic energy along the boundary.

2. EQUATIONS DESCRIBING THE CONVERSION OF MHD WAVES IN AN INHOMOGENEOUS PLASMA

In the framework of magnetohydrodynamics, the propagation of three wave modes—Alfvén and two modes of magnetosonic waves: slow and fast— is possible in a homogeneous and weakly inhomogeneous plasma. As is well known, magnetosonic waves do not interact with the Alfvén waves. Therefore, we shall not, in future, consider the latter. Let, for the sake of definiteness, the magnetic field H_0 be directed along the x axis and let the waves have a fixed frequency ω and a component k_y of the wave vector along the direction perpendicular to the field (the perturbations are proportional to $\exp(ik_y y - i\omega t)$). Then to the slow and fast magnetosonic waves correspond, respectively, the values $k_x = k_1$ and k_2:

$$k_1 = \pm\sqrt{\frac{1}{2}\left(\frac{\omega^2}{s^2}+\frac{\omega^2}{v_A^2}-k_y^2+\sqrt{\left[\frac{\omega^2}{s^2}+\frac{\omega^2}{v_A^2}+k_y^2\right]^2-4\frac{\omega^4}{v_A^2 s^2}}\right)}, \tag{4a}$$

$$k_2 = \pm\sqrt{\frac{1}{2}\left(\frac{\omega^2}{s^2}+\frac{\omega^2}{v_A^2}-k_y^2-\sqrt{\left[\frac{\omega^2}{s^2}+\frac{\omega^2}{v_A^2}+k_y^2\right]^2-4\frac{\omega^4}{v_A^2 s^2}}\right)}. \tag{4b}$$

In a mildly inhomogeneous plasma the dependence on x in the wave has the form of (3), where k_1 and k_2 given by (4a) and (4b) are functions of x. If k_x is considered as a function of the density, then it turns out that each of the branches k_1, k_2 is continuous and that they do not intersect anywhere when $k_y \neq 0$ ($k_2(x)$ may be imaginary). In a rarefied plasma ($s^2 \ll v_A^2$), $k_1 \approx \omega/s$ and $k_2 \approx \sqrt{\omega^2/v_A^2 - k_y^2}$; in a very dense plasma ($s^2 \gg v_A^2$), $k_1 \approx \omega/v_A$ and $k_2 \approx \sqrt{\omega^2/s^2 - k_y^2}$.

Let us now write out the equations describing the conversion of magnetosonic waves in an inhomogeneous plasma. We shall assume that the density of the plasma ρ_0 and the magnetic field H_0 in the unperturbed state depend only on the coordinate x while the perturbed quantities are proportional to $\exp(ik_y y - i\omega t)$. Then from the equations of magnetohydrodynamics we obtain the following system:

$$\frac{d\rho}{dx} - \frac{i\omega}{s^2}(\rho_0 v_x) = 0; \qquad \frac{d(\rho_0 v_x)}{dx} - i\omega\rho + \frac{i\omega\rho_0}{H_0}H_x = 0;$$
$$\frac{dH_x}{dx} + ik_y H_y = 0;$$
$$\frac{dH_y}{dx} + i\left(\frac{\omega^2}{k_y v_A^2} - k_y\right)H_x - ik_y\frac{s^2 H_0}{v_A^2\rho_0}\rho = 0. \tag{5}$$

Here ρ, v_x, H_x, and H_y are the density, velocity and magnetic field perturbations. In a homogeneous plasma these quantities are, for the four possible values of k_x, (4a) and (4b), connected by the following relations:

$$\rho_0 v_x = \frac{s^2}{\omega}k_x\rho, \qquad H_x = \frac{(\omega^2 - k_x^2 s^2)H_0}{\omega^2\rho_0}\rho,$$
$$H_y = -\frac{k_x(\omega^2 - k_x^2 s^2)H_0}{k_y\omega^2\rho_0}\rho. \tag{6}$$

It is easy to show from the system (5) that the variables ρ, $\rho_0 v_x$, H_x, and H_y are continuous across a region with a steep gradient. This continuity furnishes the boundary conditions at the interface between two plasma media differing from each other in density and magnetic field (ρ_0, H_0 to the left of the boundary, and ρ_{01}, H_{01}—to the right). If to the left of the interface

$$\rho \sim \sum_{i=1}^{4} C_i \exp(ik_i x),$$

while to the right

$$\rho \sim \sum_{i=1}^{4} C_i' \exp(ik_i' x),$$

then it follows from the continuity of ρ and the quantities (6) that (at the interface $x = 0$):

$$\sum C_i = \sum C_i', \quad \sum C_i k_i = \sum C_i' k_i',$$

$$\frac{H_0}{\rho_0}\sum C_i(\omega^2 - k_i^2 s^2) = \frac{H_{01}}{\rho_{01}}\sum C_i'(\omega^2 - k_i'^2 s^2);$$

$$\frac{H_0}{\rho_0}\sum C_i k_i(\omega^2 - k_i^2 s^2) = \frac{H_{01}}{\rho_{01}}\sum C_i' k_i'(\omega^2 - k_i'^2 s^2). \quad (7)$$

These equations will be used in Sec. 4 to find the coefficients of reflection, refraction and conversion of m.h. waves.

We also give here the expression for the electromagnetic energy flux **S**. As is well known, the energy flux in magnetohydrodynamics has the form: $\mathbf{S} = s^2\mathbf{v}\rho + \frac{1}{4}\pi^{-1}(\mathbf{H} \times (\mathbf{v} \times \mathbf{H}_0))$. For monochromatic waves (considered as the limit of quasimonochromatic waves) in a homogeneous plasma:*

$$\mathbf{S} = \tfrac{1}{2}s^2\mathbf{v}\rho + \tfrac{1}{8}\pi^{-1}[\mathbf{H}^*[\mathbf{v}\mathbf{H}_0]] + \text{c.c.}$$

Using the formulas (6), we find that the components S_x and S_y have the form

$$S_x = \frac{k_x s^2}{2\rho_0\omega}\left[s^2 + \frac{v_A^2(\omega^2 - k_x^2 s^2)}{\omega^2 - k^2 v_A^2}\right]|\rho|^2 + \text{c.c.};$$

$$S_y = \frac{(\omega^2 - k_x^2 s^2)}{2\rho_0\omega k_y}\left[s^2 + \frac{v_A^2(\omega^2 - k_x^2 s^2)}{\omega^2}\right]|\rho|^2 + \text{c.c.} \quad (8)$$

$$(k^2 = k_x^2 + k_y^2).$$

If k_x is real, the direction of **S** coincides with the direction of the group velocity. We note further that for real k_x the quantity $k_x S_x$ is positive. Consequently, the energy flux is directed along the wave and not in the opposite direction as in certain problems connected with wave conversion[3].

It is easy in the quasiclassical approximation ($\lambda \ll a$) to obtain from the system (5), neglecting the differential coefficients of $\rho_0(x)$ and $H_0(x)$, Eq. (1) with the coefficients (2), where ψ is any of the quantities ρ, ρ_0, v_x, H_x, or H_y. We consider in the following section the conversion of waves in a mildly inhomogeneous plasma described by Eq. (1).

3. INVESTIGATION INTO THE LINEAR CONVERSION OF WAVES IN A MILDLY INHOMOGENEOUS PLASMA

Let us suppose, for the sake of definiteness, that the plasma density monotonously increases in a certain layer from some value $\rho_{0\min}$ to $\rho_{0\max}$ with $\rho_0(-\infty) = \rho_{0\min}$ and $\rho_0(+\infty) = \rho_{0\max}$. The conversion of mhd waves is considered below. However, the method used in the investigation is applicable also to the study of the conversion of other types of waves which are described by equations of the kind (1).

As was indicated in the introduction, we shall be interested in the case, when the coefficient $V_2(x)$ of the second derivative of this equation vanishes somewhere inside the plasma. As is easily seen, for this to happen it is necessary and sufficient that

$$\frac{\omega^2}{s^2} + \left(\frac{\omega^2}{v_A^2}\right)_{min} < k_y^2 < \frac{\omega^2}{s^2} + \left(\frac{\omega^2}{v_A^2}\right)_{max}. \quad (9)$$

In that case $k_y^2 s^2 > \omega^2$ and, consequently, the function $V_1(x)$ (formula (2)) is negative everywhere in the plasma. Then the "wave number" k_1, (4a), will be real while the "wave number" k_2, (4b), will be purely imaginary. To the values $k_x = \pm k_2$ correspond decreasing or increasing in the direction of the x-axis solutions. Although we use in this paper the asymptote of the solutions of Eq. (1) which does not coincide everywhere with the asymptote obtained by the WKB method, it is convenient for the interpretation of the results to use the concept of "wave numbers" $k_1(x)$, $k_2(x)$ given by (3a) and (3b).

Let us write Eqs. (1) and (2) in dimensionless variables:

$$\gamma^2\Psi^{IV} + \gamma\left[1 + \beta\frac{v_A^2(0)}{v_A^2(\xi)} - \delta\right]\Psi^{II} + \left[\beta\frac{v_A^2(0)}{v_A^2(\xi)} - \delta\left(1 + \beta\frac{v_A^2(0)}{v_A^2(\xi)}\right)\right]\Psi = 0. \quad (10)$$

Here, $\xi = x/a$, where a is the characteristic dimension of an inhomogeneity; $\gamma = s^2/\omega^2 a^2 \ll 1$; $\beta = s^2/v_A^2(0)$; $\delta = k_y^2 s/\omega^2 > 1$. Let us suppose that the coefficient of the second derivative vanishes at the point $x = 0$ and let us in the neighborhood of this point substitute a linear function for this coefficient and a constant for the coefficient of ψ[1)]. Then Eq. (10) takes the form

$$\gamma^2\Psi^{IV} + \gamma V_2'\xi\Psi^{II} + V_1\Psi = 0. \quad (11)$$

Here,

$$V_1 = -[\delta + (1-\delta)^2], \quad V_2' = -\beta\frac{d\ln v_A^2}{d\xi}\bigg|_{\xi=0}$$

The coefficient of the second derivative $\gamma V_2'$ is of the order of $\gamma\beta$. In order to make this coefficient equal to unity, let us make a change of the independent variable $\zeta = (\gamma\beta)^{1/3}\xi$, after which Eq. (11) becomes

$$\frac{1}{\lambda^2}\Psi^{IV} + \zeta\Psi^{II} + b\Psi = 0, \quad (12)$$

where $\lambda^2 = V_2'/\gamma^2$ and $b = V_1/V_2'(\gamma\beta)^{4/3}$. The four linearly independent solutions of Eq. (12) can be exactly expressed in terms of a Laplace contour integral for four definite contours of integration in the complex region. Wasow[10] has found by the method of steepest descent asymptotic representations of the solutions of Eq. (12) for $\lambda \to \infty$ and sufficiently large ζ, if $b = 1$. The asymptote has been extended in[11] to the case when b is an arbitrary but bounded (as $\lambda \to \infty$) function of λ. The Wasow asymptotes are used in this work to obtain approximate solutions for sufficiently large but finite values of λ. It is easy to show by a direct application of the method of steepest descent that for the Wasow

*$[\mathbf{H}^*[\mathbf{v}\mathbf{H}_0]] \equiv \mathbf{H}^* \times [\mathbf{v} \times \mathbf{H}_0]$.

1) Asymptotes of the solutions of Eq. (1) for arbitrary forms of the functions $V_1(x)$, $V_2(x)$, provided that $V_2(0) = 0$, have been obtained in [11]. However, the use of the more exact forms of the coefficients would have made the computations highly complicated without changing significantly the approximate representation of the solutions of Eq. (10) for $|x| < a$. For $|x| > a$ the solutions should approximate to the form (3) which was obtained by the WKB method.

asymptotes to be suitable for the purpose indicated, it is sufficient to impose on b a less stringent limitation, namely, that b may be unbounded for $\lambda \to \infty$ (as in (12)), but must satisfy the inequality:

$$|b| \ll \lambda^2 |\zeta|^2. \tag{13}$$

The seven solutions of Eq. (12) obtained in[10] satisfy the relations:

$$A_1 + A_2 + A_3 = V,\ U_3 - U_2 = A_1,\ U_1 - U_3 = A_2. \tag{14}$$

We use below the following asymptotic representations:

$$A_1 \sim \pi^{1/2}\lambda^{-3/2}\zeta^{-5/4}\exp[{}^2/_3 i\lambda\zeta^{3/2} + {}^1/_4 i\pi], \qquad -{}^2/_3\pi < \arg\zeta < {}^4/_3\pi, \tag{15}$$

$$A_2 \sim -\pi^{1/2}\lambda^{-3/2}\zeta^{-5/4}\exp[-{}^2/_3 i\lambda\zeta^{3/2} - {}^1/_4 i\pi], \qquad -{}^4/_3\pi < \arg\zeta < {}^2/_3\pi, \tag{16}$$

$$A_3 \sim \pi^{1/2}\lambda^{-3/2}\zeta^{-5/4}\exp[-{}^2/_3 i\lambda\zeta^{3/2} - {}^1/_4 i\pi], \quad 0 < \arg\zeta < 2\pi, \tag{17}$$

$$U_2 \sim \pi i\zeta^{1/2} b^{-1/2} H_1^{\,1}(2\zeta^{1/2} b^{1/2}), \quad 0 < \arg\zeta < {}^4/_3\pi, \tag{18}$$

$$U_3 \sim \pi i\zeta^{1/2} b^{-1/2} H_1^{\,1}(2\zeta^{1/2} b^{1/2}), \qquad -{}^2/_3\pi < \arg\zeta < {}^2/_3\pi, \tag{19}$$

$$V \sim 2\pi i\zeta^{1/2} b^{-1/2} J_1(2\zeta^{1/2} b^{1/2}), \quad \text{for all } \zeta. \tag{20}$$

Here, J_1 and H_1^1 are the Bessel function and the Hankel function of the first kind, and $\arg b = \pi$. The formulas (15)–(19) give the asymptotes of the solutions A_k, U_k for those regions of ζ where they have the simplest form. Conversion of the waves is described thanks to the fact that in other regions of the complex ζ plane, these solutions have different asymptotic representations (the Stokes effect).

Let us now proceed directly to the study of wave conversion. It turns out that the problem must be solved slightly differently in the two cases when the wave is incident in the direction of increasing plasma density and when it is in the opposite direction. Let us first investigate the case when the wave is incident in the direction of decreasing density, i.e., in the direction of negative values of x. The boundary conditions are: the absence of solutions increasing to infinity and solutions in the form of waves impinging from the left on an inhomogeneity region (from $-\infty$). This case, to the best of our knowledge, has never before been considered. For $\zeta > 0$ the solution A_2 having (16) as its asymptote corresponds to a wave traveling to the left. But it turns out that if we analytically continue this solution around the point $\zeta = 0$ to the upper or lower half of the complex plane, ζ, then it exponentially increases to the left of ζ. In order to satisfy the boundary conditions, we must allow the existence of a wave which is exponentially damped in the direction of incidence of the initial wave, i.e., we must assume that to the right (for $\zeta > 0$) the solution may be represented in the form:

$$\begin{aligned} A_2 + U_3 \sim &-\pi^{1/2}\left|\frac{V_2'}{\gamma^2}\right|^{-3/4}\zeta^{-5/4}\exp\left[-\frac{2}{3}i\left(\frac{V_2'}{\gamma^2}\right)^{1/2}\zeta^{3/2} - \frac{i\pi}{4}\right] \\ &+ \frac{\pi^{1/2}}{\gamma\beta}\left|\frac{V_2'}{V_1}\right|\left|\zeta\frac{V_1}{V_2'}\right|^{1/4}\exp\left[-2(\gamma\beta)^{-2/3}\left|\zeta\frac{V_1}{V_2'}\right|^{1/2} + \frac{i\pi}{2}\right]. \end{aligned} \tag{21}$$

This asymptote is not valid to the left of $\zeta = 0$ when $\arg\zeta = \pi$. To obtain the required asymptote, we use the identity: $A_2 + U_3 = V + U_2 - A_3$, which is a consequence of (14). The function $(V + U_2 - A_3)$ and, consequently, the solution for $\arg\zeta = \pi$ may be represented in the form

$$\begin{aligned} (V + U_2) - A_3 \sim &\frac{\pi^{1/2}}{\gamma\beta}\left|\frac{V_2'}{V_1}\right|\left|\zeta\frac{V_1}{V_2'}\right|^{1/4}\exp\left[2i(\gamma\beta)^{-2/3}\left|\zeta\frac{V_1}{V_2'}\right|^{1/2} + i\frac{\pi}{2}\right. \\ &\left. + i\frac{\pi}{4}\right] - \pi^{1/2}\left|\frac{V_2'}{\gamma^2}\right|^{-3/4}\zeta^{-5/4}\exp\left[-\frac{2}{3}\left(\frac{V_2'}{\gamma^2}\right)^{1/2}|\zeta|^{3/2} - i\frac{\pi}{4}\right]. \end{aligned} \tag{22}$$

If we take into account the fact that the perturbation $\psi \sim \exp(ik_y y - i\omega t)$, then it becomes clear that the first terms in the formulas (21), (22) correspond to a wave traveling at an oblique angle to the direction of the density gradient while the second terms correspond to a wave traveling in the plane x = 0 along the y axis and decaying with increasing distance from either side of this plane. Thus, the traveling wave is partly transformed into a quasisurface wave to which corresponds the imaginary "wave number" k_2, (3b). By the definition given in the introduction, this is the conversion proper.

In the case, when the wave is incident in the direction of increasing density (in the positive direction of the x axis), we obtain by using, as in[5], the second identity of (14):

$$\begin{aligned} &\frac{\pi^{1/2}}{\gamma\beta}\left|\frac{V_2'}{V_1}\right|\left|\zeta\frac{V_1}{V_2'}\right|^{1/4}\exp\left[-2i(\gamma\beta)^{-2/3}\left|\zeta\frac{V_1}{V_2'}\right|^{1/2} - i\frac{\pi}{2} - i\frac{\pi}{4}\right] \\ &\to -\pi^{1/2}\left|\frac{V_2'}{\gamma^2}\right|^{-3/4} \\ &\times\zeta^{-5/4}\exp\left[\frac{2i}{3}\left(\frac{V_2'}{\gamma^2}\right)^{1/2}\zeta^{3/2} + i\frac{\pi}{4}\right] - \frac{\pi^{1/2}}{\gamma\beta}\left|\frac{V_2'}{V_1}\right|\left|\zeta\frac{V_1}{V_2'}\right|^{1/4} \\ &\times\exp\left[-2(\gamma\beta)^{-2/3}\left|\zeta\frac{V_1}{V_2'}\right| - i\frac{\pi}{2}\right], \end{aligned} \tag{23}$$

where the expression on the left hand side of (23) gives the solution to the left of the point x = 0 ($\arg\zeta = \pi$) and the right hand side gives the solution to the right ($\arg\zeta = 0$). For $\zeta > 0$ the second term of the solution describes the appearance of a quasisurface wave in a thin layer immediately to the right of the plane x = 0. The width of the layer in which this "quasisurface wave" propagates is given by

$$\Delta x \sim \gamma\beta\left|\frac{V_2'}{V_1}\right| a \sim \frac{\gamma\beta}{1 + \beta + \beta^2}a.$$

Clearly, in both the cases considered the location of the layer inside the plasma in which the quasisurface wave propagates (the plane x = 0, where $V_2(0) = 0$) depends on the angle of incidence and, in the general case, on the density profile.

It also follows from the formulas (21)–(23) that a transition from the short-wavelength solutions, A_2, to the long-wavelength ones, $(V + U_2)$, occurs in the neighborhood of the point $\zeta = 0$. This transition was given the name "anomalous conversion" in[4,5]. Clearly, an abrupt change in the shape of the wave results in a sharp change in its polarization (the ratio of H_x to H_y)[2)]. However, the vicinity referred to above should be really small—at least smaller than the characteristic dimension of an inhomogeneity. Otherwise, the change in the shape of the wave cannot be considered as sharp, i.e., the formulas (21)–(23) should be valid for $|x| < a$. It was noted above that the Wasow asymp-

[2)]Contrary to what the authors of [4,5] think, "anomalous conversion" is connected not with the transition from the "wave number" k_1, (3a), to the "wave number" k_2, (3b), but with the sharp change in the real quantity $k_1(x)$. To the left of the point x = 0, where $V_2 < 0$, $k_1 \approx \sqrt{V_1/V_2}$ while to the right where $V_1 > 0$, $k_1 \approx \sqrt{V_2/\alpha^2}$.

totes are suitable for the approximate description of the solutions only when the inequality (13), which is equivalent to the following inequality:

$$|x| \gg \frac{|V_1|^{1/3}}{V_2'\beta} a. \tag{24}$$

is fulfilled. The inequality (24) is compatible with the requirement that $|x| < a$ only if $\beta \gg 1$. If $\beta \lesssim 1$, then for $|x| < a$ the traveling wave has a shape which differs from the shape indicated above and "anomalous conversion" does not occur. However, a damped solution exists for small $|x|$ which do not satisfy the inequality (24). This solution must be expressed directly in terms of a contour integral. Thus, the inference of the existence of a quasisurface wave is valid for dense as well as rarefied plasma.

To conclude this section, we note that the analysis presented above is not valid for all angles of incidence. In very dense $(s^2 \gg v_A^2)$ and very tenuous $(s^2 \ll v_A^2)$ plasmas it is valid for angles of incidence lying close to $\pi/4$. Indeed, supposing that the wave is incident from a region of almost homogeneous plasma and taking into account the condition (9), we find that for $\beta \ll 1$, when $k_1 \approx \omega/s$

$$\sqrt{1+\left(\frac{s^2}{v_A^2}\right)_{min}} < \operatorname{tg}\varphi < \sqrt{1+\left(\frac{s^2}{v_A^2}\right)_{max}},$$

while for $\beta \gg 1$, when $k_1 \approx \omega/v_A$,

$$\sqrt{1+\left(\frac{v_A^2}{s^2}\right)_{min}} < \operatorname{tg}\varphi < \sqrt{1+\left(\frac{v_A^2}{s^2}\right)_{max}} \quad \left(\operatorname{tg}\varphi = \frac{k_y}{k_1}\right).$$

If, however, $\beta \sim 1$ and the wave is incident on a plasma layer in which $\rho_{0min} \ll \rho_{0max}$; then a quasisurface wave of the kind already described arises almost for all angles of incidence exceeding $\pi/4$. "Anomalous conversion" of a magnetosonic wave is possible only in a plasma of very high density, $\beta \gg 1$, when the angles of incidence are close to $\pi/4$.

4. LINEAR CONVERSION OF MHD WAVES FROM A SHARP BOUNDARY

The appearance of a wave of the surface type, which exists in some thin plasma layer, may also be expected in a highly inhomogeneous region. We consider here, in its general form, the conversion of magnetosonic waves at the boundary between two plasma media when the external magnetic field in both media is perpendicular to the boundary. The method for the solution of this problem was indicated in Sec. 2.

Suppose that the boundary is located at $x = 0$ and that the wave, in which the density perturbation $\rho \sim \exp(ik_y y + ik_1 x - i\omega t)$, is incident on the boundary from the left. Assuming that, in the general case, both modes of transmitted and reflected magnetosonic waves may exist and, putting the amplitude of the incident wave equal to unity, we find that for $x = 0$,

$$\exp[ik_1x] + R\exp[-ik_1x] + T_1\exp[-ik_2x] = D\exp[ik_1'x] + T_2\exp[ik_2'x].$$

Here, k_1 is the projection on the x axis of the wave vector of the incident--slow or fast—magnetosonic wave; k_2—that of the other wave mode in the first medium; and k_1', k_2'—the respective projections on the x axis of the wave vectors of the first and second wave modes in the second medium. It follows from the radiation principle that the signs of the quantities k_1, k_2, k_1', k_2' should be chosen so as to make them positive and real or positive and imaginary. Writing out the rest of the equations of the system (7), we obtain for the coefficients of reflection R, refraction D and conversion T_1 and T_2, an inhomogeneous system of four equations. Solving this system, we find in the general case:

$$R = [(A+B)(A-C)(k_2+k_2')(k_1-k_1') + BC(k_1'+k_2)(k_1-k_2')]/F,$$
$$D = 2A(A-C)k_1(k_2+k_2')/F,$$
$$T_1 = B(A-C)k_1(k_1'-k_2')/F,$$
$$T_2 = ABk_1(k_1'+k_2)/F, \tag{25}$$

where

$$A = k_1^2 - k_2^2, \quad B = \frac{\rho_0 H_{01}}{\rho_{01} H_0}\left(k_1'^2 - \frac{\omega^2}{s^2}\right) - k_1^2 + \frac{\omega^2}{s^2}$$
$$C = \frac{\rho_0 H_{01}}{\rho_{01} H_0}\left(k_2'^2 - \frac{\omega^2}{s^2}\right) - k_2^2 + \frac{\omega^2}{s^2},$$
$$F = (A+B)(A-C)(k_2+k_2')(k_1+k_1') + BC(k_1'+k_2)(k_1+k_2').$$

If even one of the quantities k_2 or k_2' is imaginary, then a surface wave appears which travels along the boundary and decays over distances of the order of $1/|k_2|$ from the boundary. It is easy to find from the dispersion equation for magnetosonic waves that the wave number k_2 can be imaginary only in the case of the fast wave (the branch (4b)) and, then, if and only if $k_y^2 > \omega^2/(v_A^2 + s^2)$. (In particular, the case in which $k_y^2 > \omega^2/s$ was considered in Sec. 3.) Consequently, the conversion into a surface wave at a boundary can occur only if the incident wave pertains to the slow magnetosonic wave mode (i.e., k_1 is given by formula (4a). It is easy to deduce from this that whenever a slow magnetosonic wave is incident on the boundary between two plasma media at an angle exceeding the critical angle:

$$\varphi_{cr} = \min\left\{\arcsin\left(\frac{3}{2} + \frac{s^2}{v_{A0}^2} + \frac{v_{A0}^2}{s^2}\right)^{-1/2}, \arcsin\left(\frac{3}{2} + \frac{s^2}{v_{A1}^2} + \frac{v_{A1}^2}{s^2}\right)^{-1/2}\right\}.$$

a surface wave arises at the boundary. As the angle of incidence of the slow magnetosonic wave approaches the critical angle, the angle at which the resulting fast magnetosonic wave propagates in one of the two media approaches $\pi/2$ and, subsequently, formally becomes imaginary which corresponds to the appearance of a surface wave. For a very dense $(\beta \gg 1)$ and very rarefied $(\beta \ll 1)$ plasma the conversion into a surface wave occurs practically at all angles of incidence, except the very small ones. (For normal incidence, the original system (5) is not applicable, but it is clear that in this case the waves are not coupled to each other.)

For a rarefied plasma $(\beta \ll 1)$ when the discontinuity in the density at the boundary is small, $\kappa = (\rho_{01} - \rho_0)/\rho_0 \ll 1$, and the angles of incidence considerably exceed the critical angle, we find

$$A \approx \frac{\omega^2}{s^2}(1+\beta+\delta), \quad B \approx \frac{\omega^2}{s^2}\frac{\varkappa}{2}\beta^2, \quad C \approx \frac{\omega^2}{s^2}\varkappa(1+\delta),$$

where $\delta = k_y^2 s^2/\omega^2$. We obtain from this: $|R| \sim \beta\kappa$, $|1 - D| \sim \beta\kappa$, $T_1 \sim \beta^2\kappa$, $T_2 \sim \beta^2\kappa$. Thus, the conversion coefficients for a rarefied plasma are very small even in comparison with the reflection coefficient. The conversion coefficients T_1, T_2 assume their largest values at incidence angles close to the critical angle, when $k_2 \approx 0$ and $k_2' \approx 0$.

Sometimes the coefficients of reflection, refraction and conversion are determined from relations involving not the amplitudes of the waves but the normal components of the energy flux density. It is easy to find these relations in the general case, using (25) and the formula (8) for S_x. For a slow magnetosonic wave in a rarefied ($\beta \ll 1$) as well as in a dense ($\beta \gg 1$) plasma, $S_x \gg S_y$, i.e., the energy flux is in a direction almost perpendicular to the boundary. The coefficient of conversion into a surface wave is, by the new definition, equal to zero (since $S_x = 0$ when k_x is imaginary). The vanishing, according to the new definition, of the conversion coefficient does not mean that energy is not expended in the generation of the surface wave. However, the transient process is not considered here. In the steady-state wave propagation picture, only the energy flux S_y, (8), which is different from zero and is such that $k_y S_y > 0$, is associated with the surface wave. In other words, energy is transported by this wave along the boundary in the direction of propagation of the surface wave. Since $S_y \sim \exp(-2|k_2||x|)$, then the energy flux exists in a thin surface layer near the boundary.

Thus, under certain conditions, when a slow magnetosonic wave is incident on a highly inhomogeneous plasma layer ($a \ll \lambda$) as well as on a mildly inhomogeneous layer ($a \gg \lambda$), the propagation of a surface type of wave is possible. In the case of a mildly inhomogeneous plasma, the location of the layer in which the quasisurface wave propagates depends upon the angle of incidence of the initial wave and, in general, on the density profile.

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