CONTRIBUTION TO THE NONLINEAR THEORY OF RELAXATION OF A "MONOENERGETIC" BEAM IN A PLASMA

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The dynamics of relaxation of an initially "monoenergetic" beam in a plasma was investigated. It is shown that the instability of the beam particles captured in the oscillation potential well plays an important role in the relaxation process. Equations are derived for the oscillation excitation and for the diffusion in the beam, with allowance for this instability. It follows from an examination of these equations that the distribution function of the beam is close to plateau-like at times $t \sim \omega_p^{-1} n_0/n_1$ (ω_p—plasma frequency, n_1, n_0—beam and plasma densities). During the succeeding relaxation stage, the characteristic time of which is $t \sim \Omega^{-1} n_0/n_1$ ($\Omega = \omega_p (n_1/n_0)^{1/3}$—the oscillation frequency of the captured particles) accelerated particles appear in the beam and a nearly Maxwellian velocity distribution is established.

1. It is known that the relaxation of an initially monoenergetic beam in a plasma due to the development of two-stream instability[1] can be considered within the framework of the quasilinear theory[2,3]. Such an analysis was carried out in[4] and it follows from it that in the one-dimensional model the relaxation terminates with formation of a plateau on the distribution function of the beam $f(v)$ at velocities $v \le v_0$ (v_0 is the initial beam velocity). The influence of the captured particles on the relaxation process is neglected in such an analysis.

Kadomtsev and Pogutse[5] called attention to the fact that the capture of the beam particles in potential wells of the waves excited by them leads to the formation in the plasma of charge-density bunches (macroparticles) with sufficiently large lifetimes. As a result of the radiation of waves by these plasmoids, the stationary state obtained in the quasilinear theory, with a plateau on the distribution function, turns out to be unstable in accordance with the results of computer experiments on the interaction of the beam with the plasma[6].

One of the possible modifications of the equations of the quasi-linear theory, consisted in[5], is to take into account in these equations the spontaneous radiation of the plasmoids, which is coherent over distances on the order of the wavelength $\lambda \sim v_0/\omega_p$ (ω_p is the plasma frequency).

We show in the present paper that when a plasmoid of particles captured in the potential well of a wave and executing phase oscillations in the well passes through a plasma, an instability develops connected with the polarization radiation of the plasmoid. When averaged over the period $T = 2\pi/\Omega$ of the phase oscillations, the plasmoid particles do not exchange energy with the wave that has captured them, and this leads to the occurrence in the plasma of nonlinear monochromatic waves of stationary amplitude or of an amplitude that oscillates in time[7-10]. Such waves, however, are unstable against the excitation of satellites in the phase-velocity interval $\sim \Omega/k$ (k—wave number), and unlike the usual two-stream instability, the excited waves move not only more slowly than the beam of the captured particles, but also faster.

The increment of the instability on the captured beam particles $\gamma_{tr} \sim f^2(v)$ is comparable with the linear growth increment $\gamma_L \sim \partial f/\partial v$ and the instability on the captured particles may exert an appreciable influence on the process of relaxation of the beam with an initially "δ-like" velocity distribution. When this instability is taken into account, the relaxation of an initially monoenergetic beam proceeds in the following manner. During the initial stage, the characteristic time of which is $t_1 \sim \omega_p^{-1}(n_0/n_1)^{1/3}$ (n_1—beam density, n_0—plasma density, $(n_1/n_0)^{1/3} \ll 1$), there is excited in the plasma a monochromatic wave with $k = \omega_p/v_0$, corresponding to the maximum of the linear increment. After a time $\sim t_1$ the electric field of the wave reaches a maximum $\sim [4\pi n_1 m v_0^2 (n_1/n_0)^{1/3}]^{1/2}$. It subsequently oscillates with the frequency of the phase oscillations of the captured particles $\Omega = \omega_p (n_1/n_0)^{1/3}$. The instability of such a solution leads to a broadening of the spectrum of the oscillations, and during the succeeding stage of relaxation in the plasma there are excited many waves with random phases. Just as in the quasilinear theory, the relaxation of the beam during this stage is described by a system of equations (32) and (35) for the distribution of the beam and for the spectral density of the oscillations. An investigation of this system of equations shows that at times $t_2 \sim \omega_p^{-1} n_0/n_1$ the distribution function of the beam is of the form

$$f(v) \sim [v + v_0/(\sqrt{e} - 1)]^{-1}, \quad v \leqslant v_0,$$

i.e., it is sufficiently close in form to a plateau. The fast particles with velocities $v > v_0$, observed in experiments on the interaction of the beam with the plasma[11] and obtained by computer calculations[6], appear in our analysis[1)] at times $t_3 \sim \Omega^{-1} n_0/n_1$. The beam distribution function at these times turns out to be close to Maxwellian:

[1)]Our analysis pertains to the relaxation of a beam in a homogeneous plasma. Allowance for the inhomogeneity of the plasma makes it possible for fast particles to appear within the framework of the quasilinear theory [12].

$$f(v) \sim \exp\left[-\frac{v\sqrt{v^2+v_0^2}}{v_0^2}\right].$$

2. In this section we consider the initial stage of the instability of a monoenergetic beam in a plasma, on which excitation of a narrow wave packet is possible, with increment close to the maximum value

$$\gamma_{max} = \frac{\sqrt{3}}{2^{4/3}}\omega_p\left(\frac{n_1}{n_0}\right)^{1/3}.$$

In the linear theory, the time variation of the amplitude of an individual harmonic of this packet is determined by the relation

$$E_k(t) = E_k(0)\exp\left[\gamma_{max}t\left(1 - \frac{2^{5/6}}{9}\frac{(k-k_0)^2}{k_0^2}\left(\frac{n_0}{n_1}\right)^{2/3}\right)\right]. \quad (1)$$

Here $k_0 = \omega_p/v_0$ is the wave number corresponding to the maximum of the increment. It follows from (1) that the amplitude E_k is comparable with E_{k_0} in the wave-number interval

$$\Delta k = k_0\left(\frac{n_1}{n_0}\right)^{1/3}\frac{3}{\sqrt{2^{5/6}\gamma_{max}t}} \approx k_0\left(\frac{n_1}{n_0}\right)^{1/3}\left[\ln\frac{E_{max}}{E(0)}\right]^{-1/2} \quad (2)$$

where E_{max} is the maximum value of the amplitude of the electric field, determined by relation (8) (see below); $E(0)$ is the initial value of the amplitude. We confine ourselves further to a consideration of the case when the spectrum of the natural oscillations of the plasma is sufficiently rarefied and only one harmonic is contained in the wave-number interval Δk. In this case the initial stage of the instability corresponds to excitation of a monochromatic wave. (The opposite limiting case, when many waves are excited during the initial stage of the instability is considered in[4] with the aid of quasi-nonlinear equations.)

At a low beam density ($n_1 \ll n_0$), the amplitude of the excited wave is also sufficiently small, and for particles determining the dispersion of the wave there is satisfied the condition for applicability of the linear approximation $e\varphi_0 \ll mv_0^2$ (φ_0 is the amplitude of the potential). The frequency of the wave is close to the resonant frequency of the plasma ω_p and the excitation of the higher harmonics with frequency $n\omega_p$ ($n \geq 2$) can be neglected (a simple estimate shows that the amplitude of these harmonics is $E_n \sim \gamma\omega_p^{-1}E_1 \ll E_1$).

On the basis of the foregoing, the electric field of the excited wave can be represented in the form

$$E(t,x) = E(t)\sin(kx - \omega_p t + \alpha(t)). \quad (3)$$

Integrating the linearized equations of the oscillations of the plasma particles in this field and substituting the result in the Poisson equation, we obtain in the usual manner the following equations for the amplitude and phase of the wave:

$$\frac{dE}{dt} = 2e\omega_p\int_{-\lambda/2}^{\lambda/2} d\xi \sin(k\xi + \alpha)n_b(t,\xi),$$
$$E\frac{d\alpha}{dt} = 2e\omega_p\int_{-\lambda/2}^{\lambda/2} d\xi\cos(k\xi+\alpha)n_b(t,\xi) \quad (4)$$

($\lambda = 2\pi/k$). In these equations $\xi = x - v_0t$; to derive them we used the condition $kv_0 = \omega_p$. The quantity

$$n_b(t,\xi) = \int dv f_b(t,\xi,v)$$

is the electron density in the beam. Using the fact that the distribution function is the integral of motion $f_b(t,\xi,v) = f_b^0(v(0)) = n_1\delta(v(0) - v_0)$ and the Liouville theorem of conservation of the phase volume $d\xi dv = d\xi(0)dv(0)$[2)], we write equations (4) in the following form:

$$\frac{dE}{dt} = 2e\omega_p n_1\int_{-\lambda/2}^{\lambda/2} d\xi(0)\sin[k\xi(t,\xi(0)) + \alpha],$$
$$E\frac{d\alpha}{dt} = 2e\omega_p n_1\int_{-\lambda/2}^{\lambda/2} d\xi(0)\cos[k\xi(t,\xi(0)) + \alpha]. \quad (5)$$

The trajectories of the particles are determined by integrating the equations of motion:

$$\frac{dv}{dt} = -\frac{e}{m}E(t)\sin(k\xi+\alpha), \quad v = \frac{d\xi}{dt} + v_0. \quad (6)$$

The system of equations (5)—(6) describes excitation of a monochromatic wave by an electron beam. At small amplitudes $e\varphi_0 \ll m\gamma^2/k^2$ it follows from these equations that the amplitude has an exponential growth during the instability, $E \sim \exp(\gamma t)$. Indeed, linearizing the equations of motion of the beam particles at such amplitudes, we get

$$\xi = \xi(0) + \frac{e}{2im}\left[E(t)\frac{\exp\{i(k\xi(0)+i\alpha)\}}{\mu^2} - \text{c.c.}\right], \quad (7)$$
$$\mu = i\gamma - d\alpha/dt.$$

Substituting in (5) the quantity $\xi = \xi(0) + \delta\xi$ and confining ourselves to the approximation linear in $\delta\xi$, we obtain

$$\mu^3 = \frac{1}{2}\omega_b^2\omega_p, \quad \omega_b^2 = \frac{4\pi e^2 n_1}{m} = \frac{n_1}{n_0}\omega_p^2, \quad (7')$$

which coincides with the result of the linear theory.

In the course of time the amplitudes of the potential

$$\varphi_0 \sim \frac{m}{e}\frac{|\mu|^2}{k^2} \sim \frac{m}{e}\frac{\gamma_{max}^2}{k^2}, \quad (8)$$

reach values such that the electron beam is captured in the potential well of the wave. The growth of the amplitude then stops, since the interaction of the captured particles with the wave leads to oscillations of its amplitude with a frequency $\Omega = k\sqrt{e\varphi_0/m} \approx \omega_p(n_1/n_0)^{1/3}$ (see[13,14]).

An investigation of these oscillations was carried out by integrating the nonlinear system of equations (5) and (6) with a computer (see[10]). In the dimensionless variables

$$\tau = \left(\frac{n_1}{n_0}\right)^{1/3}\omega_p t, \quad \zeta = \frac{k\xi}{2\pi}, \quad \nu = \frac{v - v_0}{v_0(n_1/n_0)^{1/3}},$$
$$\mathscr{E} = \frac{E}{[4\pi n_1 m v_0^2(n_1/n_0)^{1/3}]^{1/2}}$$

the investigated system of equations is written in the following form:

$$\frac{d\nu}{d\tau} = -\mathscr{E}\sin(2\pi\zeta+\alpha), \quad \frac{d\zeta}{d\tau} = \frac{1}{2\pi}\nu, \quad (9)$$

$$\frac{d\mathscr{E}}{d\tau} = \int_{-1/2}^{1/2} d\zeta_0\sin[2\pi\zeta(\tau,\zeta_0)+\alpha], \quad \mathscr{E}\frac{d\alpha}{d\tau} = \int_{-1/2}^{1/2} d\zeta_0\cos[2\pi\zeta(\tau,\zeta_0)+\alpha]. \quad (10)$$

The system (9)—(10) was integrated by the Runge-Kutta method for 100 particles, the initial coordinates

2) Here $\xi(0)$ and $v(0)$ are the initial values of the coordinate and of the velocity of the particle on a trajectory passing at the instant of time t through the point ξ, v of phase space.

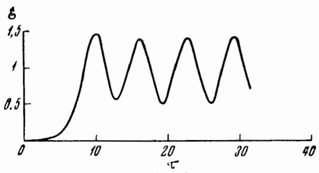

FIG. 1

of which were in the range $-\frac{1}{2} < \zeta_0 < \frac{1}{2}$ with intervals of $\Delta\zeta_0 = 1/100$. Figure 1 shows the $\mathscr{E}(\tau)$ plot obtained in this manner. When $\mathscr{E} \ll 1$, in accord with the linear theory, $\mathscr{E} \backsim e^{0.683\tau}$ (for comparison we note that the increment obtained from (7′) is $\gamma = \sqrt{3\cdot 2^{-4/3}}\,\omega_p (n_1/n_0)^{1/3} = 0.686\,\omega_p \times (n_1/n_0)^{1/3}$). When $\mathscr{E} \sim 1$, the amplitude of the wave becomes an oscillating function of τ. The amplitude of these oscillations does not decrease in the course of time. As is well known, the damping of the oscillations is due to the phase "mixing" of the captured particles, resulting from the dependence of the period of the particle oscillations on their energy. In a monoenergetic beam, the particles of which execute synchronous oscillations in the potential well of the wave excited by them, there is no phase "mixing."

3. Let us investigate now the stability of the system consisting of the plasma and of the beam of particles captured by the wave. The analysis in this section is carried out in the reference frame of the wave. In this reference frame, the equilibrium state of the beam is characterized by a velocity $u_b(\xi)$ and a density $n_b(\xi)$. The time dependence of u_b and n_b, due to the periodic changes of the amplitude of the wave, can be neglected for simplicity in the present analysis.

The perturbations of the equilibrium values of the density and the velocity of the beam are determined from the following system of equations:

$$\frac{\partial u_b'}{\partial t} + \frac{\partial}{\partial \xi} u_b u_b' = \frac{e}{m}\frac{\partial \varphi'}{\partial \xi}, \tag{11}$$

$$\frac{\partial n_b'}{\partial t} + \frac{\partial}{\partial \xi} u_b n_b' + \frac{\partial}{\partial \xi} n_b u_b' = 0, \tag{12}$$

where $\varphi'(t, \xi)$ is the potential of the perturbation wave. Putting $\varphi' \backsim e^{-i\omega' t}$ (ω' is the frequency of the perturbation wave in the reference frame of the main wave), we obtain from (11) and (12)

$$n_b' = -\frac{e}{m u_b(\xi)}\int_0^{\xi} d\xi' \exp\left[-i\omega'\int_0^{\xi'}\frac{d\xi^*}{u_b}\right] \tag{13}$$

$$\times \frac{\partial}{\partial \xi'}\left\{\frac{n_b}{u_b}\int_0^{\xi'} d\xi'' \frac{\partial \varphi'}{\partial \xi''}\exp\left[-i\omega'\int_0^{\xi''}\frac{d\xi^*}{u_b}\right]\right\}$$

$$\approx \begin{cases} -\dfrac{e n_b(\xi)}{m u_b^2(\xi)}\varphi' & \text{for } \omega' \ll k u_b \\ \dfrac{e}{m}\dfrac{\partial}{\partial \xi}\left(\dfrac{n_b(\xi)}{\omega'^2}\dfrac{\partial \varphi'}{\partial \xi}\right) & \text{for } \omega' \gg k u_b \end{cases}$$

For simplicity we shall henceforth confine ourselves to the case $\omega' \ll k u_b$. Then, using (13) and the expansion

$$\frac{n_b(\xi)}{u_b^2(\xi)} = \frac{n_{0b}}{u_{0b}^2}\sum_n \Gamma_n \exp(i n k_0 \xi)$$

(k_0 is the wave number of the fundamental wave, n_{0b} and u_{0b} are the mean values of $n_b(\xi)$ and $u_b(\xi)$), we arrive at the following equation for the potential of the perturbation wave

$$\hat{\varepsilon}_p \frac{\partial^2 \varphi'}{\partial \xi^2} = 4\pi e n_b' = -\frac{\omega_b^2}{u_{0b}^2}\sum_n \Gamma_n \exp(i n k_0 \xi)\,\varphi', \tag{14}$$

where $\omega_b^2 = 4\pi e^2 n_{0b}/m$, and $\hat{\epsilon}_p$ is the usual operator of the dielectric constant of the plasma in the reference frame of the main wave:

$$\hat{\varepsilon}_p \exp[i(k\xi - \omega' t)] = \left(1 - \frac{\omega_p^2}{(\omega' + k v_{ph})^2}\right)\exp[i(k\xi - \omega' t)].$$

We seek a solution of (14) by expansion in the small parameter $\omega_b^2/k^2 u_{0b}^2$. The solution is of the form

$$\varphi'(t, \xi) = \sum_n \varphi^{(n)}(t)\exp\{i[(k + n k_0)\xi - \omega' t]\}. \tag{15}$$

From (14) in the highest order in the small parameter there follows a dispersion equation relating ω' with k:

$$1 - \frac{\omega_p^2}{(\omega' + k v_{ph})^2} - \frac{\omega_b^2}{k^2 u_{0b}^2}\Gamma_0 = 0. \tag{16}$$

After determining from (14) the expansion coefficients of $\varphi^{(n)}$ at $n \neq 0$, we find that in our case $k \approx k_0 \approx \omega_p/v_{ph}$ the largest coefficient is the one corresponding to $n = -2$, for which $\epsilon_p(k + nk_0, \omega') \approx 0$. For $\varphi^{(-2)}$ we get from (14)

$$\varphi^{(-2)} = \frac{\omega_b^2}{(k - 2k_0)^2 u_{0b}^2}\Gamma_{-2}\varphi^{(0)}\left[\varepsilon_p(\omega', k - 2k_0) + \delta\frac{\partial \varepsilon_p}{\partial \omega'}(\omega', k - 2k_0) - \frac{\omega_b^2}{(k - 2k_0)^2 u_{0b}^2}\right]^{-1}, \tag{17}$$

where we have substituted $d\varphi^{(0)}/dt = -i\delta\varphi^{(0)}$. Substituting in (14) φ' from (15) and gathering terms proportional to $e^{ik\xi}$, we obtain in the usual manner an equation for the determination of δ:

$$\delta = -\frac{1}{4}\left(\frac{\omega_b^2}{u_{0b}^2}\right)^2 \frac{\omega_p^2}{k^2(2k_0 - k)^2}\frac{|\Gamma_2|^2}{\Delta(k, k_0) + \delta}$$

$$\Delta(k\ k_0) = (k - 2k_0)v_{ph} + \omega_p + \omega'(\omega_b^2/2\omega_p)(v_{ph}^2/u_{0b}^2). \tag{18}$$

From this equation we have for the growth increment $\gamma = \operatorname{Im}\delta$

$$\gamma(k, k_0) = \frac{1}{2}\left[\left(\frac{\omega_b^2}{u_{0b}^2}\right)^2 \frac{\omega_p^2|\Gamma_2|^2}{k^2(2k_0 - k)^2} - \Delta^2\right]^{1/2} \quad \text{for } \Delta \leqslant \frac{\omega_b^2}{u_{0b}^2}\frac{\omega_p|\Gamma_2|}{k(2k_0 - k)} \tag{18'}$$

$$\gamma(k, k_0) = 0 \quad \text{for} \quad \Delta > \frac{\omega_b^2}{u_{0b}^2}\frac{\omega_p|\Gamma_2|}{k(2k_0 - k)}. \tag{18''}$$

The obtained relations determine the increment of the instability on the captured particles of the beam. The investigated instability arises at $\omega' \lesssim k u_{0b} \approx \Omega$ ($\Omega \approx \omega_p (n_1/n_0)^{1/3}$—the oscillation frequency of the captured particles). When $\omega' > k u_{0b}$, as follows from (13), the right-hand side of (18) is proportional to $(\omega')^{-4}$, and with increasing ω' the increment decreases like $(\omega')^{-2}$. The instability leads to excitation of waves that move both more slowly and more rapidly than the main wave, i.e., with $\omega' < 0$ and with $\omega' > 0$. It also follows from (18) that instability takes place in the wave-number interval $\Delta k \approx \Omega/v_{ph}$. During the initial stage of the beam relaxation, when the entire beam is captured in the potential well, i.e., $n_{0b} \sim n_1$, the maximum increment γ, determined from (18), is of the order of Ω. Subsequently, as a result of the "smearing" of the beam we have during the instability $n_{0b} \ll n_1$ and an increment $\gamma \ll \Omega$. During this relaxation stage many waves are excited in the plasma and the growth increment of the wave φ_k is determined by averaging the relation (18′) over the interval of the wave numbers $\Delta k_0 \sim \Omega/v_{ph}$. The result takes the form

$$\bar{\gamma}_k \approx \frac{v_{ph}}{\Omega}\int \gamma(k\ k_0)\,dk_0 = \frac{v_{ph}}{4\Omega}\left(\frac{\omega_b^2}{u_{0b}^2}\right)^2 \frac{\omega_p^2|\Gamma_2|^2}{k^2(2k_0-k)^2} \times \int \frac{\gamma}{\Delta^2/4+\gamma^2}\,dk_0 = \frac{\pi}{4}|\Gamma_2|^2 \frac{\omega_b^4\omega_p^2}{k^2(2k_0-k)^2 u_{0b}^4\Omega}. \tag{19}$$

4. From the analysis presented in the preceding sections it follows that during the initial stage of relaxation of the monoenergetic beam in the plasma, the beam is captured in the potential well of the wave excited by it and executes in the well oscillations with frequency $\Omega \approx \omega_p(n_1/n_0)^{1/2}$. The velocity distribution of the beam particles at x = const and t = const remains "δ-like"; during the course of time, the maximum of this distribution oscillates in the range $\Delta v \approx \pm\Omega/k$. The instability of such a state of the plasma-plus-beam system leads to excitation in the wave spectrum of satellites shifted relative to the main wave by an amount $\sim\Omega/k$. At sufficiently large amplitudes of the satellites, part of the beam is captured by them. Further broadening of the oscillation spectrum causes the distribution function of the beam to acquire a complicated and tangled form, being a superposition of distribution functions of a large number of "captured" beams.

Accordingly, we seek the distribution function F(t, x, v), which is the solution of the equation

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial F}{\partial v} = 0, \tag{20}$$

in the form $F = F_0 + \delta F$, where the background distribution function $F_0(t, x, v)$ is represented in the form

$$F_0 = f(t, v)G. \tag{21}$$

f(t, v) is a "smooth" function of v, describing slow diffusion of the beam in velocity; the singularities of the distribution F_0 are contained in G. The "smoothing" of these singularities is attained by averaging F_0 over the velocity interval $\Delta v \sim \Omega/k$, i.e.,

$$\frac{k}{\Omega}\int_{\Delta v} F_0\,dv = f, \quad \frac{k}{\Omega}\int_{\Delta v} G\,dv = 1. \tag{22}$$

The singularities of the distribution function F_0 are connected with the capture by the waves of particles from the initially monoenergetic beam, and have the form $\delta(v - v_b(t, x))$; $v_b(t, x)$ is the velocity of the bunch of captured particles. The distribution function δF describes small perturbations of the "background" by the trial waves and satisfies the condition $\delta F \ll F_0$. For f(t, v) we have from (20) the equation

$$\frac{\partial f}{\partial t} = -\frac{e}{m}\left\langle \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial v}\delta F\right\rangle. \tag{23}$$

In the right-hand side of this equation, besides averaging over distances on the order of the wavelength $2\pi/k$, we average over the velocity intervals $\Delta v \sim \Omega/k$.

The perturbation of the background distribution function δF, using (21) and (22), can be represented in the form

$$\delta F = F_1 + F_2, \tag{24}$$

where the distribution function F_1, satisfying the equation

$$\frac{\partial F_1}{\partial t} + v\frac{\partial F_1}{\partial x} + \frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial v}G = 0, \tag{25}$$

leads in (23) to a quasilinear collision integral. The distribution function F_2 satisfies the equation

$$\frac{\partial F_2}{\partial t} + v\frac{\partial F_2}{\partial x} + \frac{e}{m}\frac{\partial\varphi_0}{\partial x}\frac{\partial F_2}{\partial v} + \frac{e}{m}\frac{\partial\varphi}{\partial x}f\frac{\partial G}{\partial v} = 0 \tag{26}$$

and describes the perturbations of the "background" distribution function, which lead to the instability on captured beam particles, which was considered in the preceding section. Accordingly, we have retained in the equation for F_2 the nonlinear term $\sim(\partial\varphi_0/\partial x)\times(\partial F_2/\partial v)$. Here $\varphi_0(t, x)$ is the potential of the main wave, with which the capture is connected, and $\varphi(t, x)$ is the potential of the trial waves. Confining ourselves, as before, to the case $\omega' < ku_b$ (ω' is the frequency of the perturbation in the reference frame of the main wave and u_b is the velocity of the oscillations of the particles in the potential well), we obtain in terms of the variables x and $\epsilon = (1/2)m(v - v_{ph})^2 - e\varphi_0$ the following solution for F_2:

$$F_2 \approx -e\varphi f\partial G/\partial\varepsilon. \tag{27}$$

From this we get a formula for the density perturbation

$$n_2 = \int_{\Delta v} F_2\,dv \sim -\frac{e\varphi}{mu_b^2}f\frac{\Omega}{k}, \quad \Delta v \sim \frac{\Omega}{k}, \tag{28}$$

which coincides with the corresponding formula (13) (the density of the particles captured in the potential well, $n_b = f\Omega/k$). In its derivation we have integrated in (28) by parts and taken account of the fact that $G \sim \delta(u - u_b)$, $u = v - v_\Omega$.

The potential of the electric field of the oscillations is represented, as usual, in the form

$$\varphi(t, x) = \sum_k \varphi_k \exp[i(kx - \omega_k t)],$$

where, just as in[5], φ_k is connected with the perturbation of the density of the captured particles n_{2k} by an equation similar to (14):

$$k^2\varepsilon_p(\omega = kv)\varphi_k = -4\pi e n_{2k}. \tag{29}$$

In this equation we have neglected small corrections to the dielectric constant of the plasma $\infty \int F_1 dv$. From (29) we have

$$\varphi_k = 4\pi e n_{2k}\left\{P\left[\frac{\omega_p^2}{v^2} - k^2\right]^{-1} + \pi i\delta\left(\frac{\omega_p^2}{v^2} - k^2\right)\right\}. \tag{30}$$

Using (30) and (28), we can easily calculate the contribution made to the collision integral by the distribution function F_2:

$$-\frac{e}{m}\left\langle \frac{\partial\varphi}{\partial x}\frac{\partial F_2}{\partial v}\right\rangle = -\frac{e}{m}\frac{\partial}{\partial v}\left\langle \sum_k ik\varphi_k \exp[i(kx - \omega_k t)] \times \sum_{k'} \frac{k'}{\Omega'} n_{2k'} \exp[i(k'x - \omega_{k'}t)]\right\rangle$$
$$= \frac{4\pi^2 e^2}{m}\frac{\partial}{\partial v}\left[\sum_k \frac{k^2|n_{2k}|^2}{\Omega}\delta\left(k^2 - \frac{\omega_p^2}{v^2}\right)\right] \approx \frac{2\pi e^4}{m^3}\frac{\partial}{\partial v}\left[f^2(v)\frac{k^3}{\Omega^3}|\varphi_k|^2\right]. \tag{31}$$

Here $k = \omega_p/v$ and $\Omega \approx ku_{0b}$ is the frequency of the oscillations of the captured particles. Substituting in the right-hand side of (24) the value of δF from (25), using (31), and adding the quasilinear collision integral (the contribution from the distribution function F_1), we arrive ultimately at the following equation for $f(t, v)$:

$$\frac{\partial f}{\partial t}=\frac{e^2}{m^2}\frac{\partial}{\partial v}\left[\frac{|E_k|^2}{v}\left(\frac{\partial f}{\partial v}+\frac{2\pi e^2}{m\Omega^3}\omega_p f^2\right)\right] \quad (k=\omega_p/v) \tag{32}$$

We can obtain analogously an equation for the spectral noise density $|E_k|^2(t)$. In its derivation, besides the usual instability with increment $\backsim \partial f/\partial v$, it is necessary to take into account also the excitation of the oscillations by the captured beam particles. The corresponding change in the oscillation energy is determined from the equation

$$\frac{\partial^{(2)}}{\partial t}\int_{\Delta k}|E_k|^2\,dk=-2\pi e v\int dk(ik\varphi_k n_{2k}^*+\text{c.c.}) \tag{33}$$
$$(k\approx\omega_p/v,\ \Delta k\sim\Omega/v).$$

Substituting in this equation $\int|E_k|^2dk\approx|E_k|^2\Delta k$ and using (28) and (30), we obtain

$$\frac{\partial^{(2)}|E_k|^2}{\partial t}=\frac{8\pi^3e^2v^2}{\Omega}|n_{2k}|^2\approx\frac{8\pi^3e^4f^2(v)\omega_p^2}{m^2\Omega^3}|\varphi_k|^2. \tag{34}$$

The growth increment determined by this formula coincides, accurate to a factor on the order of unity, with the increment (19). Taking into account also the excitation of oscillations with increment $\sim\partial f/\partial v$ (the contribution from the distribution function F_1), we obtain the following equation for $|E_k|^2(t)$ $(k=\omega_p/v)$:

$$\frac{\partial|E_k|^2}{\partial t}=\frac{4\pi^2e^2}{mk^2}\omega_p\left(\frac{\partial f}{\partial v}+\frac{2\pi e^2}{m\Omega^3}\omega_p f^2\right)|E_k|^2. \tag{35}$$

It is easily seen that from (32) and (35) we get conservation of the total energy

$$W=\frac{m}{2}\int v^2f\,dv+\frac{1}{4\pi}\sum_k|E_k|^2$$

and of the total momentum

$$P=m\int vf\,dv+\frac{1}{4\pi}\sum_k\frac{k}{\omega_p}|E_k|^2$$

of the system of waves and particles.

The analysis in this section pertains to the case of sufficiently small initial velocity scatter in the beam, $\Delta v_0\ll\Omega/k$, and a rarefied oscillation spectrum $\Delta v_{ph}\sim\Omega/k$ (Δv_{ph} is a change in the phase velocity of the neighboring harmonics of the oscillations spectrum), when the most significant effects are those connected with the captured particles. In the case of a nearly-continuous oscillation spectrum, $\Delta v_{ph}\ll\Omega/k$, the quasilinear approximation is valid.

5. In this section we investigate the relaxation of the electron beam in the plasma with the aid of the system (32) and (33). During the initial stage of this process, the characteristic time of which is $t_2\sim\omega_p^{-1}n_0/n_1$, the diffusion in the beam occurs predominantly in the region $v<v_0$. Just as in[15], the diffusion is described by a wave with a sufficiently steep front, propagating into the region of smaller velocities. The spectral density of the oscillations excited thereby is determined from the energy integral of Eqs. (32) and (35):

$$|E_k|^2=\frac{4\pi^2mv^3}{\omega_p}\int_{v_{min}}^{v}(f-f_0)\,dv, \tag{36}$$

where $f_0=f(t=0)$.

The frequency of the oscillations of the captured particles Ω can be estimated as follows (see[5]):

$$\Omega^4\approx\frac{e^2}{\pi m^2}\int k^2|E_k|^2\,dk\approx\frac{e^2}{\pi m^2}|E_k|^2\frac{\Omega\omega_p^2}{v^3}. \tag{37}$$

Substituting $|E_k|^2$ from (36), we obtain

$$\Omega^3\approx\frac{4\pi e^2}{m}\omega_p\int f\,dv\approx\omega_p^3\frac{n_1}{n_0}. \tag{37'}$$

Diffusion leads to establishment of a stationary distribution in the beam, for which

$$\frac{1}{f^2}\frac{df}{dv}=-\frac{1}{2n_1},$$

i.e.,[3)]

$$f(v)=2n_1/(v+C_0). \tag{38}$$

The constant C_0 is determined from the condition for the conservation of the number of particles in the beam:

$$\int_0^{v_0}\frac{dv}{v+C_0}=\frac{1}{2},$$

i.e.,

$$C_0=v_0/(\sqrt{e}-1). \tag{39}$$

The limits v^0_{min} and v^0_{max} of the distribution (38) are determined by the relations

$$f(v^0_{min})=f_p(v^0_{min}),\quad v^0_{max}\approx v_0, \tag{40}$$

where $f_p(v)$ is the distribution function of the plasma particles. The time of establishment of the distribution (38) is

$$t_2\approx\frac{1}{\omega_p}\frac{n_0}{n_1}\ln\Lambda,$$

where Λ is the ratio of the energy of the oscillations excited by the beam to the energy of the thermal fluctuations of the field in the beam,

$$\ln\Lambda\approx\ln\left[n_1\lambda_D^3\frac{mv_0^2}{T}\right]\approx L_g,$$

λ_D is the Debye radius in the beam and L_g is the Coulomb logarithm.

The solution obtained is quite close to that obtained in[4] in the analysis of relaxation of the monoenergetic beam within the framework of the quasilinear theory. The distribution (38), however, is unstable and varies with a characteristic time

$$t_3\approx\frac{1}{\Omega}\frac{n_0}{n_1}\ln\Lambda.$$

In considering this stage of relaxation it is necessary to take into account in (32) and (35) the fact that the phase velocities of the oscillations excited by the captured particles, ω/k, differ from the particle velocity v by an amount $\sim\Omega/k$. Taking this circumstance into account, it is necessary to replace $f^2(v)$ in the increment of instability on the captured particles by $\langle f^2\rangle$, and $|E_k|^2$ in the coefficient of the diffusion connected with this instability should be replaced by $\langle|E_k|^2\rangle$ (the averaging is carried out over the velocity interval $\sim\Omega/k$). In the present paper we confine ourselves to a qualitative investigation of the system of equations obtained in this manner for f and $|E_k|^2$. It

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[3)]It follows from (38) that formula (37′) for Ω is valid apart from a logarithmic factor. A more rigorous analysis, which takes into account the $\Omega(v)$ dependence, would in this case be an exaggeration of the accuracy.

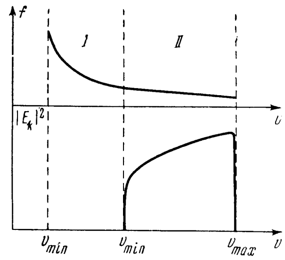

FIG. 2

follows from the equation for $|E_k|^2$ that the captured particles lead to excitation of oscillations with phase velocities ω/k larger than v^0_{max}. The oscillations are excited in the phase-velocity interval $\sim\Omega/k$ with a characteristic increment

$$\gamma_0 \approx \frac{4\pi^3 e^4 \omega_p^2}{m^2 k^2 \Omega^3} f^2(v_{max}) \approx \omega_p \frac{n_1}{n_0} \frac{v^2_{max} f^2(v_{max})}{n_1^2}. \tag{41}$$

These oscillations lead to diffusion of the particles in the indicated velocity interval, and consequently to a further displacement of the upper limit of the spectrum of the oscillations and of the beam distribution function $v_{max}(t)$. The increase of v_{max} with time is determined from the equation

$$\frac{dv_{max}}{dt} \approx \frac{\gamma_0 \Omega}{k} \frac{1}{\ln\Lambda}, \tag{42}$$

i.e., it is quite slow, with a characteristic time on the order of t_3. As a result there is time for a stationary distribution to become established when $v < v_{max}$. The lower limit of the oscillation spectrum also shifts to the right with increasing time, for otherwise an unstable discontinuity on the distribution function would appear at $v = v^0_{min}$. The instability of the discontinuity is connected with the fact that the diffusion coefficient of the particles at $v = v^0_{min}$, which is proportional to $\langle |E_k|^2 \rangle$, differs from zero.

Thus, at times $t \sim t_3$ it is possible to separate on the distribution function two regions (from v_{min} to v^0_{min} and from v_{min} to v_{max}, see Fig. 2): when $v_{min}(t) < v < v_{max}(t)$ we have the distribution (38):

$$f(t,v) = \frac{2n_1}{v + C(t)}, \tag{43}$$

When $v < v_{min}(t)$ there is established the distribution

$$f^\infty(v) = \left.\frac{2n_1}{v + C(v_{min})}\right|_{v_{min}=v}. \tag{44}$$

We determine the function $C(t)$ from the condition that the number of particles be conserved as the beam becomes diffused

$$\int\limits_{v_{min}}^{v_{max}} \delta f\, dv + f(v_{max})\,\delta v_{max} = 0. \tag{45}$$

Substituting δf from (43) and integrating with respect to v, we obtain

$$\frac{dC}{dv_{max}} = \frac{C + v_{min}}{v_{max} - v_{min}}. \tag{46}$$

The connection between v_{max} and v_{min} will be determined with the aid of the law of energy conservation for the particles and the waves:

$$\left.\frac{|E_k|^2}{4\pi}\right|_{v=v_{max}} \frac{\omega_p}{v^2_{max}} \delta v_{max} - \left.\frac{|E_k|^2}{4\pi}\right|_{v=v_{min}} \frac{\omega_p}{v^2_{min}} \delta v_{min} \tag{47}$$

$$+ \int\limits_{v_{min}}^{v_{max}} \frac{mv^2}{2} \delta f\, dv + \frac{mv^2_{max}}{2} f(v_{max})\,\delta v_{max} = 0.$$

The spectral energy density of the oscillations during the stage in question can be estimated from (37), by assuming that when $v_{min} < v < v_{max}$ the frequency is $\Omega \approx \Omega_0 = \omega_p (n_1/n_0)^{1/3}$ (from a comparison of the two terms in the increment it follows that when $\Omega > \Omega_0$ the damping of oscillations with increment $\infty\, \partial f/\partial v$ becomes predominant, as a result of which Ω decreases; when $\Omega < \Omega_0$ there is buildup of oscillations and Ω increases).

Substituting in (47)

$$|E_k|^2 = 4\pi n_0 m v^3 \Omega_0^3 / \omega_p^4$$

and neglecting the terms $\sim v/C \ll 1$ in this equation, we obtain

$$v^2_{min} = v^2_{max} - v_0^2. \tag{48}$$

Using this relation, we can readily integrate Eq. (46) under the condition $v_{min} \ll C$. The result is

$$C(v_{max}) = \frac{C_0}{[v_{max}/v_0 + \sqrt{v^2_{max}/v_0^2 - 1}]^{1/2}} \times \exp\left[\frac{v^2_{max} - v_0^2 + v_{max}\sqrt{v^2_{max} - v_0^2}}{v_0^2}\right], \tag{49}$$

$$C_0 = C(v_{max} = v_0) = \frac{v_0}{\sqrt{e} - 1}.$$

Using formulas (48) and (49), we find from (44) that the stationary distribution function $f^\infty(v)$ to which the beam relaxes is close to Maxwellian when $v \gtrsim v_0$:

$$f^\infty(v) = 2(\sqrt{e} - 1) n_1 \left[\frac{v + \sqrt{v_0^2 + v^2}}{v_0^3}\right]^{1/2} \exp\left[-\frac{v(v + \sqrt{v_0^2 + v^2})}{v_0^2}\right]. \tag{50}$$

In conclusion we note that the presented analysis of the relaxation of an electron beam in a plasma is in qualitative agreement with the results of computer experiments[6].

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