ROTON VISCOSITY AND THERMAL CONDUCTIVITY OF SUPERFLUID HELIUM

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Submitted October 15, 1970

Zh. Eksp. Teor. Fiz. 60, 1178–1184 (March, 1971)

It is shown that the kinetic equation for the calculation of the roton part of the viscosity and thermal conductivity in superfluid helium with the complete collision integral can reduce an equation in the τ approximation. An expression is derived for the roton viscosity and thermal conductivity in terms of an unknown cross section for roton-roton scattering.

THE viscosity and thermal conductivity of superfluid helium have been found theoretically in the works of Landau and Khalatnikov.[1,2] Both these quantities can be divided into phonon and the roton parts. For the calculation of the roton part, Boltzmann's equation has been solved in[1,2] in the τ approximation, where τ is the mean time between roton collisions. Such an approach makes it possible to calculate the desired quantities with an accuracy to within a constant factor of the order of unity. In the present research, it will be shown that the kinetic equation can be solved for rotons with a complete collision integral; this solution is identical with the solution of Landau and Khalatnikov. The resultant solution makes it possible to obtain an expression for the viscosity and thermal conductivity in terms of an unknown cross section of roton-roton scattering. The comparison of these quantities with their experimental values imposes some restrictions on the law of roton interaction.

1. VISCOSITY

The linearized kinetic equation for the roton distribution function, which may be written conveniently in the form $n = n_0(1 + g)$, where $n_0 = e^{-F/kT}$ is the equilibrium distribution function, and $\epsilon = \Delta + (p - p_0)^2/2\mu$ is the roton energy of the roton with momentum p, has the form

$$\frac{1}{2}\frac{n_0}{kT}\frac{\partial\varepsilon}{\partial p_i}p_k\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)=-\frac{2\pi}{\hbar}\int w(\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4)\times\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)n_0(\varepsilon_1)n_0(\varepsilon_2)\times(g_1+g_2-g_3-g_4)\frac{d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4}{(2\pi\hbar)^6}. \tag{1}$$

Here u_i is the i-th component of the normal motion, $g_1 = g(\mathbf{p}_1)$ and so on.

The roton scattering probability $w(\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4)$ is normalized so that it goes over into $\frac{1}{2}|V(\mathbf{p}_1-\mathbf{p}_3)+V(\mathbf{p}_1-\mathbf{p}_4)|^2$ in the Born approximation, where $V(\mathbf{p}) = \int e^{i\mathbf{p}\cdot\mathbf{r}}V(\mathbf{r})d\mathbf{r}$ is the Fourier transform of the interaction potential.

It is clear from the symmetry of the problem that the solution (1) can be sought in the form

$$g(\mathbf{p})=\nu_i\nu_k\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)q(p),$$

where $\nu_i = p_i/p$ is a unit vector in the $\mathbf{p}$ direction. The velocity field can be so chosen that the angular dependence of $g(\mathbf{p})$ is described by the Legendre polynomial $P_2(\cos\theta)$ (for brevity, $P_2(\theta)$ in what follows), and the following equation is obtained for $q(\mathbf{p})$:

$$\frac{1}{2kT}\frac{\partial\varepsilon}{\partial p}pP_2(\theta_1)=-\frac{2\pi}{\hbar}\int w\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\times\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)n_0(\varepsilon_2)\left[q(p_1)P_2(\theta_1)+q(p_2)P_2(\theta_2)-q(p_3)P_2(\theta_3)-q(p_4)P_2(\theta_4)\right]\frac{d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4}{(2\pi\hbar)^6}. \tag{2}$$

Integration over $d\mathbf{p}_4$ in the collision integral removes the δ function of the momenta. In the remaining integrals, it is convenient to transform to other variables:[1] in the integral over $d\mathbf{p}_2$, to spheroidal coordinates with the polar axis along $\mathbf{p}_1$, and integration over $d\mathbf{p}_3$ is replaced by $p_3p_4P^{-1}dp_3dp_4d\varphi$, where P is the absolute value of the vector $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$, and φ is the angle between the planes $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_3, \mathbf{p}_4)$. Because of the Boltzmann factor $n_0(\epsilon)$, the integration actually goes over the region in which all the momenta are close to the value p_0, as a consequence of which we can set $p_1 = p_2 = p_3 = p_4 = p_0$ in all non-special functions. This corresponds to the expansion in $(p - p_0)/p_0$, and, inasmuch as $p - p_0 \sim \sqrt{\mu kT}$ in the problem, then $\sqrt{\mu kT}/p_0 \approx 0.1$ is a small parameter.

It is convenient to introduce the dimensionless variables

$$p_1-p_0=t\sqrt{2\mu kT},\quad p_2-p_0=x\sqrt{2\mu kT},\quad p_3-p_0=y\sqrt{2\mu kT},\quad p_4-p_0=z\sqrt{2\mu kT}.$$

Because of the rapid convergence of the integrals, we can carry out the integration over dx, dy and dz from $-\infty$ to $+\infty$; then the kinetic equation is written down in the form

$$\frac{p_0}{\sqrt{2\mu kT}}t=-\frac{(2\mu kT)^{3/2}p_0^4}{(2\pi)^4\hbar^7kT}\int\frac{w}{P}n_0(x^2)\delta(t^2+x^2-y^2-z^2)\times\left[q(t)+q(x)P_2(\theta_{12})-q(y)P_2(\theta_{13})-q(z)P_2(\theta_{14})\right]\times\sin\theta_{12}\,d\theta_{12}\,d\varphi\,dx\,dy\,dz. \tag{3}$$

In the transition from (2) to (3), the formula for addition of Legendre polynomials is used, and integration is carried out over $d\varphi_2$, as a result of which there remain in the equation functions of the angles θ_{12} (between $\mathbf{p}_1$ and $\mathbf{p}_2$), θ_{13}, and θ_{14}. The cross section of roton scattering w can be assumed to depend only on φ, P and the total energy of the colliding rotons $E = \epsilon_1 + \epsilon_2 = 2\Delta + kT(x^2 + t^2)$ (this will be discussed in more detail below). Then the kernel of Eq. (3) becomes an even function of x, y, and z and upon substitution in the integral of any odd function q(t), the terms containing q(z), q(y) and q(z), vanish in integration over dx, dy and dz, respectively. Equation (3) takes on here the form of the kinetic equation in the τ approximation:

$$p_0 t / \sqrt{2\mu kT} = -q(t) / \tau_r, \tag{4}$$

where

$$\frac{1}{\tau_r} = \frac{p_0 \mu N_r}{2\hbar^4} \bar{w}, \quad \bar{w} = \int e^{-x^2} \frac{w}{\cos(\theta/2)} \frac{do}{4\pi} \frac{dx}{\sqrt{\pi}} \tag{4'}$$

and the number of rotons per unit volume is

$$N_r = \frac{2p_0^2 (\mu kT)^{1/2} e^{-\Delta/T}}{(2\pi)^{3/2}\hbar^3}.$$

If we describe the interaction of the rotons by the potential $V_0 \delta(\mathbf{r}_1 - \mathbf{r}_2)$, as was done in[1], then

$$1/\tau_r = 2|V_0|^2 p_0 \mu N_r / \hbar^4.$$

This is smaller by a factor of two than the similar quantity in[1]; the difference is connected with the fact that in the given work the statistical weight of the final states of the rotations was taken to be twice as large as is necessary.

The solution of Eq. (4) $q(t) = p_0 \tau_r t/\sqrt{2\mu kT}$ is an odd function of t and therefore also satisfies the initial equation (3). From the general theory of the Boltzmann equation, it is known that the homogeneous equation corresponding to (3) does not have solutions; therefore the found solution is the only one.

By substituting

$$\delta n = n_0 \nu_i \nu_k \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) q(t)$$

in the expression for the momentum flux

formula $$\pi_{ik} = \int \frac{\partial \varepsilon}{\partial p_i} p_k \delta n \frac{d\mathbf{p}}{(2\pi\hbar)^3},$$

we find the viscosity

$$\eta = 2p_0 \hbar^4 / 15\mu^2 \bar{w}, \tag{5}$$

where $\bar{w}$ is taken for t = 0.

As has already been mentioned, Eq. (3) is described with accuracy to terms $\sim \sqrt{\mu kT}/p_0$; the function q(t) is found with the same accuracy. A more detailed analysis shows that if we keep in the equation two terms of the given small parameter, then corrections appear for the even part of q(t) of the order of $\sqrt{\mu kT}/p_0$ and to the odd part of order $\mu kT/p_0^2$. In the calculation of the momentum flux, the even part of q(t) acquires the additional factor $\sqrt{\mu kT}/p_0$ in comparison with the odd part. Therefore, the corrections to the coefficient of viscosity will be of the order $\mu kT/p_0^2 \sim 1\%$.

2. THERMAL CONDUCTIVITY

For the calculation of the thermal conductivity, terms proportional to ∇T should be kept in the left side of the kinetic equation. We get

$$-\frac{n_0}{kT}\frac{\nabla T}{T}\left(\mathbf{p}\frac{ST}{\rho_n} - \varepsilon\frac{\partial \varepsilon}{\partial \mathbf{p}}\right) = -I(\delta n) \tag{6}$$

(S is the entropy of a unit volume of helium, ρ_n the normal density, I the collision integral). We seek a solution in the form $n = n_0(1 + g)$, where $g = q(p)T^{-1}(\nu\nabla T)$, ν a unit vector in the $\mathbf{p}$ direction. On the left side of Eq. (6), there remains only the principal term in the small parameter $\sqrt{\mu kT}/p_0$, and the transformation of the collision integral as in Sec. 1. As a result, we have

$$-\varepsilon\frac{\partial \varepsilon}{\partial p} = \frac{(2\mu kT)^{3/2} p_0^4}{(2\pi)^4 \hbar^7} \int \frac{w}{P} \delta(x^2 + t^2 - y^2 - z^2) n_0(x^2) \tag{7}$$
$$\times [q(t) + q(x)\cos\theta_{12} - q(y)\cos\theta_{13} - q(z)\cos\theta_{14}]\, dx\, dy\, dz\, do.$$

By reasoning the same way as in the case of viscosity, we come to the conclusion that

$$q(t) = -\frac{4\pi^{3/2}\hbar^7 e^{\Delta/kT}}{p_0^2 \mu^2 kT \bar{w}} t(\Delta + kTt^2) \tag{8}$$

is a solution of Eq. (7). However, the homogeneous equation corresponding to (7) now has a solution. It is easy to prove that q = const causes the collision integral to vanish. In order to make the solution completely defined, we make use of the condition of the absence of mass flow

$$\int \mathbf{p}\, \delta n\, d\mathbf{p} = 0 \tag{9}$$

and note that δn, which is proportional to the left side of (6), satisfies this condition. Therefore, $q \sim (\epsilon\partial\epsilon/\partial p - p_0 ST/\rho_n)$: however, the ratio of the second term to the first is of the order of $\sqrt{\mu kT}/p_0$ for $(p - p_0) \sim \sqrt{\mu kT}$ and one can neglect it; then (8) is the solution of Eq. (7) with the necessary accuracy.

Computing the energy flow

$$\mathbf{Q} = \int \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} n_0 q \frac{(\mathbf{p}, \nabla T)}{pT} \frac{d\mathbf{p}}{(2\pi\hbar)^3} \tag{10}$$

with the function q from (8), we find the expression for the thermal conductivity:

$$\varkappa = \frac{2\hbar^4 \Delta^2}{3\mu^2 p_0 T \bar{w}}\left[1 + \frac{3T}{\Delta} + \frac{15}{4}\left(\frac{T}{\Delta}\right)^2\right]. \tag{11}$$

The accuracy of the resultant expression, as in the case of the viscosity, is of the order of 1% and T/Δ in the roton region $\sim 1/5$; therefore, terms of order (T/Δ) and $(T/\Delta)^2$ are kept in (11).

The unknown cross section enters into the expression for the viscosity and thermal conductivity averaged with the same weight. This allows us to write down a relation not containing unknown quantities:

$$\frac{\varkappa}{\eta} = \frac{5\Delta^2}{p_0 T}\left[1 + \frac{3T}{\Delta} + \frac{15}{4}\left(\frac{T}{\Delta}\right)^2\right]. \tag{12}$$

This relation agrees well with experiment.

Figure 1 shows the theoretical temperature dependence of the ratio of the damping of second sound α_2 to the square of the frequency ω^2 in the roton region $T \gtrsim 1.6°K$, in comparison with experiment.[4] The principal contribution to α_2 is made by the thermal conductivity. The values of the thermal conductivity were computed from data on the viscosity[5] with the help of Eq. (12). The experimental points lie on or close to the computed curve, which confirms the relation (12).

3. ROTON-ROTON SCATTERING CROSS SECTION

The unknown function w, in terms of which η and κ are expressed, is connected with the vertex part for

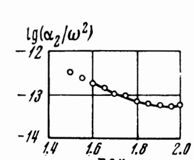

FIG. 1

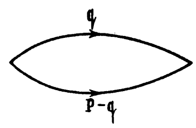

FIG. 2

roton-roton scattering:

$$w(\mathbf{p}_1\mathbf{p}_2;\ \mathbf{p}_3\mathbf{p}_4) = {}^1/_2 |\Gamma(p_1p_2;\ p_3p_4) + \Gamma(p_1p_2;\ p_4p_3)|^2. \qquad (13)$$

Here p_i is the 4 vector $(\epsilon_i, \mathbf{p})$ i = 1, 2, 3, 4 and in the arguments of Γ, $\epsilon_i = \epsilon(\mathbf{p}_i)$ everywhere. As Pitaevskiĭ has shown[6], Γ has a singularity for $E = \epsilon_1 + \epsilon_2 = 2\Delta$ and goes to zero at this point. For this reason, it is not possible to assume that all $|\mathbf{p}_i| = p_0$ in the arguments of the function w, and the energies of all rotons are exactly equal to Δ.

We shall now make clear the quantities in which $\overline{w}$ (4′) is expressed. The singularity at the vertex arises because of the logarithmic divergence of the roton loop for $E \to 2\Delta$ (see Fig. 2). The behavior of Γ near the singularity is determined from the condition

$$\Gamma(p_1p_2;\ p_3p_4) = \Gamma^{(1)}(p_1p_2;\ p_3p_4) + \frac{i}{2\pi}\int \Gamma^{(1)}(p_1p_2;\ q, P-q)$$
$$\times G(q)G(P-q)\Gamma(q, P-q;\ p_3p_4)\frac{d^4q}{(2\pi\hbar)^3}, \qquad (14)$$

where $G(q) = [\epsilon - \epsilon(\mathbf{q}) + i\delta]^{-1}$ is the Green's function of the roton, and $\Gamma^{(1)}$ the set of diagrams for the vertex not containing the special loop.

The integral over $|\mathbf{q}|$ in (14) diverges at the upper limit. In order to avoid this divergence, we split the range of integration into two: near $|\mathbf{q}| = p_0$ and near $|\mathbf{P} - \mathbf{q}| = p_0$ (the integration, and over them is denoted by A) far from them (correspondingly, B).[7] Then Eq. (14) can be rewritten in the form

$$\Gamma = \Gamma^\xi + \Gamma^\xi A\Gamma, \qquad \Gamma^\xi = \Gamma^{(1)} + \Gamma^{(1)}B\Gamma^\xi,$$

or in explicit form, after integration over the fourth component q,

$$\Gamma(p_1p_2;\ p_3p_4)$$
$$= \Gamma^\xi(p_1p_2;\ p_3p_4) + \int\limits_A \frac{\Gamma^\xi(p_1p_2;\ q, P-q)\Gamma(q, P-q;\ p_3p_4)}{E - \varepsilon(\mathbf{q}) - \varepsilon(\mathbf{P}-\mathbf{q}) + i\delta}\frac{d\mathbf{q}}{(2\pi\hbar)^3}. \qquad (15)$$

In the integral over dq, it is convenient to transform to the coordinates $|\mathbf{q}|$, $|\mathbf{P} - \mathbf{q}|$, φ (see Sec. 1) and expand Γ in a Fourier series in φ:

$$\Gamma = \sum_{m=-\infty}^{+\infty} \Gamma_m e^{im\varphi}. \qquad (16)$$

Then equations are obtained for each m separately; their solutions near the singularity depend only on the total energy E and the total momentum P of the colliding rotons, i.e.,

$$\Gamma_m(E, P) = \frac{2\pi\hbar^3 P}{\mu p_0^2}\left(\frac{2\pi\hbar^3 P}{\mu p_0^2\Gamma_m^\xi} + \ln\frac{\xi_m}{E - 2\Delta} + i\pi\right)^{-1} \qquad (17)$$

Here $E > 2\Delta$.

The parameter ξ_m divides the regions A and B. For weak interaction Γ^ξ transforms into the corresponding Born amplitude. The imaginary part of the denominator of (17) is chosen from the condition of realness of the vertex for $E < 2\Delta$; this corresponds to neglect of the possibility of conversion of two rotons into a single roton and an energetic phonon, or into two energetic phonons. The contributions of these processes to the imaginary part of the denominator of (17) contain the factors $(\Delta/\mu c^2)\sqrt{2\mu\Delta/p_0^2} \approx 0.5$ and $(\Delta/\mu c^2)(\Delta/cp_0)^2 \approx 0.1$, respectively; these can be neglected in comparison with π.

Taking (13), (16) into account, as well as the definition of $\overline{w}$, we get

$$\overline{w} = \sum_{m=-\infty}^{+\infty}\int |\Gamma_{2m}|^2 e^{-x^2}\frac{\sin\theta\, d\theta}{\cos(\theta/2)}\frac{dx}{\sqrt{\pi}} = 2\sum_m\int |\Gamma_{2m}|^2 e^{-x^2}\frac{dP}{p_0}\frac{dx}{\sqrt{\pi}}. \qquad (18)$$

Because of the identical character of the rotons, only even (2m) harmonics enter into (18), and from the symmetry property $\Gamma(p_1p_2;\ p_3p_4) = \Gamma(p_3p_4;\ p_1p_2)$, the contributions of the 2m and −2m harmonics are equal.

Integration over dx in (18) is easily carried out by noting that the important region is $x \sim 1$. In the subsequent integration over t, the region $t \sim 1$ is important; therefore, one can neglect $\ln(x^2 + t^2)$ in the denominator of (17) after the substitution $E - 2\Delta = kT(x^2 + t^2)$ in comparison with large $\ln(\xi_n/kT)$, or in comparison with π. Finally, we have

$$\overline{w} = 2\sum_{m=-\infty}^{+\infty}\int_0^{2p_0} |\Gamma_{2m}'|^2\frac{dP}{p_0}, \qquad (19)$$

where

$$|\Gamma_{2m}'|^2 = \left(\frac{2\pi\hbar^3 P}{\mu p_0^2}\right)^2\left[\left(\frac{2\pi\hbar^3 P}{\mu p_0^2\Gamma_{2m}^\xi} + \ln\frac{\xi_m}{kT}\right)^2 + \pi^2\right]^{-1}. \qquad (20)$$

As $T \to 0$, the quantity $\ln(\xi_m/kT) \to \infty$ and from this arises an additional temperature dependence of the roton parts κ and η of the form $(\ln T)^2$; however, the contribution of the rotons to the transport coefficient becomes comparable with the phonon only for $T \gtrsim 1.4$°K, and the logarithm is insufficiently large in this region.

For the harmonic Γ_{2m} with very strong or very weak coupling Γ_{2m}^ξ, one can complete the integration over dP in (19). For $2\hbar^3P/\mu p_0^2 \ll \Gamma_{2m}^\xi(P)$ we have

$$\int_0^{2p_0} |\Gamma_{2m}'|\frac{dP}{p_0} = \frac{32\pi^2\hbar^6}{3\mu^2 p_0^2}\left[\left(\ln\frac{\Delta}{T}\right)^2 + \pi^2\right]^{-1}, \qquad (21)$$

in the opposite case $2\hbar^3P/\mu p_0^2 \ll \Gamma_{2m}^\xi(P)$,

$$\int_0^{2p_0} |\Gamma_{2m}'|^2\frac{dP}{p_0} = \int_0^{2p_0} |\Gamma_{2m}^\xi(P)|^2\frac{dP}{p_0}, \qquad (22)$$

however, the contribution of these harmonics is small in comparison with (21).

Using the experimental value of the viscosity in the roton region, which amounts to $\eta = 1.3 \times 10^{-5}$ poise,[5] one can determine the minimum number of terms in the sum (19). If we assume that each harmonic makes the maximum possible contribution $32\hbar^6/3\mu^2p_0^2$, then the sum (19) should contain no more than four terms.

In connection with the possibility of formation of a bound state of two rotons,[6] one should note that, as is seen from (17), several such states can exist, corresponding to different m. However, it is not possible to draw definite conclusions as to the presence of these states in helium on the basis of data on the viscosity.

The author thanks L. P. Pitaevskiĭ for comments on the singularity of the vertex part of the rotons and for interest in the work, A. F. Andreev, I. E. Dzyaloshinskiĭ and I. M. Khalatnikov for useful discussions.

[1] L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 19, 637, 709 (1949).
[2] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 23, 21 (1952).
[3] I. M. Khalatnikov, Vvedenie v teoriyu sverkhtekuchesti (Introduction to the Theory of Superfluidity) (Nauka Press, 1965).
[4] W. B. Hanson and J. R. Pellam, Phys. Rev. 95, 321 (1954).
[5] K. N. Zinov'eva, Zh. Eksp. Teor. Fiz. 31, 31 (1956) [Sov. Phys.-JETP 4, 36 (1957)].
[6] L. P. Pitaevskiĭ, ZhETF Pis. Red. 12, 118 (1970) [JETP Lett. 12, 82 (1970)].
[7] A. B. Migdal, Teoriya konechnykh Fermi-sistem i svoĭstva atomnykh yader (Theory of Finite Fermi Systems and the Properties of Atomic Nuclei) (Nauka Press, 1965).

Translated by R. T. Beyer
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