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HAMILTONIAN FORMALISM FOR HYDRODYNAMIC PLASMA MODELS

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We construct canonical variables for a hydrodynamic description of a plasma. We use those to obtain simple formulae for the matrix elements for the interaction of waves in the plasma both when there is a magnetic field present and when it is not present. We find exact relations between the increments of decay instability and the kernels of the kinetic equations and we obtain criteria for the applicability of the kinetic equations for wave packets which are narrow in k-space.

INTRODUCTION

We solve in the present paper the problem of the introduction of canonical variables in hydrodynamic plasma models. This problem is of interest from two points of view.

First, the introduction of canonical variables enables us easily to calculate the matrix elements for the interaction of waves in the plasma. Although these matrix elements have been evaluated by many different methods[1-7] (mainly starting from the kinetic equations) the expressions obtained are very complicated and not very lucid and the problem to prove their symmetry is also a non-trivial one. In the present paper we derive simple closed formulae for the matrix elements which automatically possess all necessary symmetry properties.

Second, the introduction of canonical variables enables us to include the theory of waves in a plasma in the framework of a general theory of waves in non-linear media. Recently[8,9] attempts have been made to construct such a theory on the basis of a Lagrangian formalism and Hamilton's variational principle. This method has, however, not been very fruitful, since for many actual media the Lagrangian is non-local and can not be written down explicitly. The method involving a direct use of the Hamiltonian formalism has turned out to be more fuitful. If one succeeds in writing down the equations for the medium in terms of such variables they have the form of Hamilton's equations and after changing to the normal variables of the linearized problem these equations become of standard form and one can apply general methods to them. The introduction of such canonical variables is, generally speaking, a non-trivial problem. Davydov[10] introduced in 1949 canonical variables in the equations of hydrodynamics, while Schlömann[11] introduced them into the spin-wave theory (see also[12]). The greatest effort has been spent on introducing canonical variables into the equations of the general theory of relativity. The importance of this problem is determined by the fact that the introduction of canonical variables is a necessary condition for the quantization of the classical equations. Dirac[13] was the first to solve satisfactorily the problem of introducing canonical variables into the equations of the gravitational field. Very recently canonical variables have been introduced into the theory of surface waves[14] and into magnetohydrodynamics.[15]

The introduction of canonical variables enables us to establish some general relationships for the interaction of waves in non-linear media. We find in the present paper one such simple relationship—the connection between the increment of the decay instability and the kernel of the wave kinetic equation. We succeed by using this connection to establish criteria for the applicability of the random phase principle and of the kinetic equations for waves which in k-space have a narrow spectral distribution.

1. HAMILTONIAN FORMALISM IN HYDRODYNAMICS

To a large extent we shall use for the construction of the Hamiltonian formalism for a plasma the canonical variables found by Davydov[10] for the hydrodynamics of a perfect fluid. In the present section we present a procedure for the introduction of these variables into the hydrodynamic equations with a non-local density dependence of the pressure.

We consider a medium described by the set of equations

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\qquad \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}+\nabla\frac{\delta\mathscr{E}}{\delta\rho}=0. \tag{1}$$

Here $\mathscr{E}$ is the internal energy of the medium which is an arbitrary functional of the density ρ. In the particular case when $\mathscr{E}=\int\epsilon(\rho)d\mathbf{r}$, where $\epsilon(\rho)$ is the energy density, Eq. (1) describes a perfect barotropic fluid. The set (1) conserves the total energy of the system

$$H=\frac{1}{2}\int\rho v^2\,d\mathbf{r}+\mathscr{E}.$$

We make the following change of variables

$$\mathbf{v}=\frac{\lambda}{\rho}\nabla\mu+\nabla\Phi. \tag{2}$$

A direct check shows easily that the equation of continuity can be written in the form

$$\partial\rho/\partial t=\delta H/\delta\Phi. \tag{3}$$

Substituting (2) into the Euler equations we get

$$\frac{\lambda}{\rho}\nabla\left\{\frac{\partial\mu}{\partial t}+(\mathbf{v}\nabla)\mu\right\}+\nabla\mu\left\{\frac{\partial}{\partial t}\frac{\lambda}{\rho}+(\mathbf{v}\nabla)\frac{\lambda}{\rho}\right\}+\nabla\left\{\frac{\partial\Phi}{\partial t}+\frac{1}{2}v^2-\frac{\lambda}{\rho}(\mathbf{v}\nabla)\mu+\frac{\delta\mathscr{E}}{\delta\rho}\right\}=0. \tag{4}$$

Equation (4) is satisfied if we put

$$\frac{\partial\lambda}{\partial t}+\operatorname{div}\lambda\mathbf{v}=0,\quad \frac{\partial\mu}{\partial t}+(\mathbf{v}\nabla)\mu=0,$$

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}v^2-\frac{\lambda}{\rho}(\mathbf{v}\nabla)\mu+\frac{\delta\mathscr{E}}{\delta\rho}=0. \tag{5}$$

We check easily by direct calculation that Eqs. (5) can be written in the form

$$\frac{\partial\lambda}{\partial t}=\frac{\delta H}{\delta\mu}\quad \frac{\partial\mu}{\partial t}=-\frac{\delta H}{\delta\lambda}\quad \frac{\partial\Phi}{\partial t}=-\frac{\delta H}{\delta\rho} \tag{6}$$

The pairs of variables μ, λ and ρ, Φ are thus canonical variables.

Each solution of the set of four equations (3)–(5) generates a well-defined solution of the hydrodynamical equations. Conversely, each solution of the hydrodynamic equations generates a certain class of solutions of the set (3)–(5). To establish this correspondence it is sufficient to establish the correspondence between the velocity $\mathbf{v}$ and the set of quantities λ, μ, Φ at some time.

We take the curl of Eq. (2) and get

$$\operatorname{rot}\mathbf{v}=\left[\nabla\frac{\lambda}{\rho}\times\nabla\mu\right]. \tag{7}$$

It is well known (see, e.g.,[16]) that one can, moreover non-uniquely, find from Eq. (7) λ/ρ and μ. Moreover, taking the divergence of Eq. (5) one can determine Φ. The set of variables λ, μ, ρ, and Φ are thus completely equivalent to the usual hydrodynamic variables ρ and $\mathbf{v}$.

For an incompressible fluid, ρ = const, the quantity Φ is not an independent variable and can be eliminated using the relation div v = 0. In this case there is only one pair of independent variables λ, μ—they are known as the Clebsch variables.[16]

We consider now the equations of relativistic hydrodynamics:

$$\partial\rho/\partial t+\operatorname{div}\rho\mathbf{v}=0,$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)\mathbf{p}+m\nabla\frac{\delta\mathscr{E}}{\delta\rho}=0,\quad \mathbf{p}=\frac{m\mathbf{v}}{(1-v^2/c^2)^{1/2}}.$$

Using the directly verified relation $(\mathbf{v}\nabla)\mathbf{p} = -[\mathbf{v}\operatorname{curl}\mathbf{p}]+\nabla W$, $W=\sqrt{(m^2c^4+p^2c^2)}$ we reduce the equation of motion to the form

$$\frac{\partial\mathbf{p}}{\partial t}+\nabla W-[\mathbf{v}\operatorname{rot}\mathbf{p}]+m\nabla\frac{\delta\mathscr{E}}{\delta\rho}=0. \tag{8*}$$

Moreover, we perform the transformation

$$\frac{\mathbf{p}}{m}=\frac{\mathbf{v}}{(1-v^2/c^2)^{1/2}}=\frac{\lambda}{\rho}\nabla\mu+\nabla\Phi. \tag{9}$$

After substituting (9) into (8) we get the equations

$$\frac{\partial\lambda}{\partial t}+\operatorname{div}\lambda\mathbf{v}=0,\quad \frac{\partial\mu}{\partial t}+(\mathbf{v}\nabla)\mu=0,$$

$$\frac{\partial\Phi}{\partial t}+\frac{W}{m}-\frac{\lambda(\mathbf{v}\nabla)\mu}{\rho}+\frac{\delta\mathscr{E}}{\delta\rho}=0. \tag{10}$$

We check by direct calculation that Eqs. (10) can be written in the form (6), where

$$H=\int\frac{\rho}{m}W\,d\mathbf{r}+\mathscr{E}$$

is the total energy of the liquid.

So far we have not chosen a definite form of the functional of the internal energy of the medium. In the simplest case of a perfect barotropic fluid we have

$$\mathscr{E}=\int\varepsilon(\rho)\,d\mathbf{r},$$

where $\epsilon(\rho)$ is the internal energy density:

$$\frac{d\varepsilon(\rho)}{d\rho}=\int\frac{1}{\rho}\frac{dP}{d\rho}d\rho,$$

$P(\rho)$ is the pressure of the fluid.

One can consider the quantity $\epsilon(\rho)$ as the zeroth term of the expansion of the internal energy density in powers of $\nabla\rho$; there will then be a dispersion of the waves in the medium. If one takes the weak dispersion into account one must in that expansion retain the term proportional to $(\nabla\rho)^2$. Weakness of the dispersion assumes also that the non-linearity is small; in the general form we have for a weakly dispersive medium

$$\mathscr{E}=\int\left[\frac{1}{2}c^2\frac{\delta\rho^2}{\rho_0}+\frac{1}{3}qc^2\frac{\delta\rho^3}{\rho_0^2}+\frac{1}{2}\lambda_1(\nabla\rho)^2\right]d\mathbf{r},$$

$$\frac{\delta\mathscr{E}}{\delta\rho}=c^2\left(\frac{\delta\rho}{\rho_0}+q\frac{\delta\rho^2}{\rho_0^2}\right)-\lambda_1\Delta\rho. \tag{11}$$

2. HAMILTONIAN FORMALISM FOR A PLASMA

The simplest hydrodynamical models to describe a plasma directly refer to the type (4). We consider the hydrodynamics of electrons interacting with a potential field in a plasma without a magnetic field:

$$\partial\rho/\partial t+\operatorname{div}\rho\mathbf{v}=0,$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=\nabla\left(\frac{e}{m}\varphi-\frac{3T}{m\rho_0}\delta\rho\right),$$

$$\Delta\varphi=-4\pi\frac{e}{m}\delta\rho,\quad \delta\rho=\rho-\rho_0. \tag{12}$$

The internal energy of the plasma is the sum of the electrostatic energy

$$\mathscr{E}_{es}=\frac{1}{8\pi}\int(\nabla\varphi)^2\,d\mathbf{r}=\frac{e^2}{2m^2}\int\frac{\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}'$$

and the gas-kinetic energy

$$\mathscr{E}_T=\frac{3}{2}\frac{T}{m\rho_0}\int\delta\rho^2\,d\mathbf{r}.$$

It is clear that

$$\frac{e}{m}\varphi=-\frac{e^2}{m^2}\int\frac{\delta\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'=-\frac{\delta\mathscr{E}_{es}}{\delta\rho}, \tag{13}$$

$$\frac{3T}{m\rho_0}\delta\rho=\frac{\delta\mathscr{E}_T}{\delta\rho}.$$

Equations (13) show that the set (12) belong to type (1).

We consider now the hydrodynamics of slow motions of a non-isothermal plasma:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\quad \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}+\frac{e}{M}\varphi=0, \tag{14}$$

$$\Delta\varphi=-\frac{4\pi e}{M}(\rho-\rho_0e^{e\varphi/T}).$$

Here M is an ion mass and T the electron temperature. The internal energy E of the plasma has the form

$$\mathscr{E}=\frac{1}{8\pi}\int(\nabla\varphi)^2d\mathbf{r}+\mathscr{E}_T,$$

*[v rot p] ≡ v × curl p.

where

$$\mathscr{E}_T=\frac{1}{M}\int T\rho_0\left\{e^{e\varphi/T}\left(\frac{e\varphi}{T}-1\right)+1\right\}d\mathbf{r}$$

is the thermal energy of the electron gas.

Evaluating the variational derivative of the total internal energy with respect to the ion density we get

$$\frac{\delta\mathscr{E}}{\delta\rho}=\int\varphi(\mathbf{r}')\left\{-\frac{1}{4\pi}\Delta\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}+\frac{e^2n_0}{T}e^{e\varphi/T}\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}\right\}d\mathbf{r}'$$

($n_0 = \rho_0/M$). On the other hand, varying the Poisson equation we find

$$-\frac{1}{4\pi}\Delta\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}+\frac{n_0e^2}{T}e^{e\varphi/T}\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}=\frac{e}{M}\delta(\mathbf{r}-\mathbf{r}').$$

Comparing these expression, we obtain

$$\frac{\delta\mathscr{E}}{\delta\rho}=\frac{e}{M}\varphi,$$

whence follows that the set (14) also belongs to type (1).

We consider now a relativistic electron plasma interacting with an arbitrary, not necessarily potential, electromagnetic field:

$$\partial\rho/\partial t+\operatorname{div}\rho\mathbf{v}=0,$$

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\nabla\right]\mathbf{p}=e\mathbf{E}+\frac{e}{c}[\mathbf{v}\mathbf{H}]-\frac{3T}{m}\nabla\frac{\delta\rho}{\rho_0},$$

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\quad \operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\frac{e\rho}{m}\mathbf{v},$$

$$\operatorname{div}\mathbf{E}=\frac{4\pi e}{m}(\rho-\rho_0).$$

We introduce the scalar and vector potentials φ and $\mathbf{A}$ and use the Coulomb gauge for $\mathbf{A}$:

$$\operatorname{div}\mathbf{A}=0.$$

The Poisson equation then results:

$$\Delta\varphi=-\frac{4\pi e}{m}(\rho-\rho_0).$$

It is well known that in the Coulomb gauge the vector potential is a canonical variable. The vector which is its canonically conjugate turns out to be

$$\mathbf{B}=\frac{1}{4\pi c}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)=-\frac{\mathbf{E}}{4\pi c}.$$

Substituting it into the Maxwell equations we get

$$\frac{\partial\mathbf{B}}{\partial t}=-\frac{1}{4\pi}\Delta\mathbf{A}-\frac{e\rho c}{m}\frac{\mathbf{p}}{W},\quad\frac{\partial\mathbf{A}}{\partial t}=4\pi c^2\left(\mathbf{B}-\frac{1}{4\pi c}\nabla\varphi\right).\tag{16}$$

We rewrite the equations of motion in the form

$$\frac{\partial\mathbf{p}}{\partial t}+\nabla W-[\mathbf{v}\operatorname{rot}\mathbf{p}]=e\mathbf{E}+\frac{e}{c}[\mathbf{v}\mathbf{H}]-\frac{3T}{m}\nabla\frac{\delta\rho}{\rho_0}$$

and change to the generalized momentum

$$\mathbf{p}=\mathbf{p}_1-(e/c)\mathbf{A}.$$

The vector $\mathbf{p}_1$ satisfies the equation

$$\partial\mathbf{p}_1/\partial t+\nabla W-[\mathbf{v}\operatorname{rot}\mathbf{p}_1]+e\nabla\varphi=0.\tag{17}$$

We introduce the variables

$$\frac{\mathbf{p}_1}{m}=\frac{\lambda}{\rho}\nabla\mu+\nabla\Phi.\tag{18}$$

Equation (17) is similar to Eq. (8) and after substituting (18) into (17) we get the equations

$$\frac{\partial\lambda}{\partial t}+\operatorname{div}(\lambda\mathbf{v})=0,\quad\frac{\partial\mu}{\partial t}+(\mathbf{v}\nabla)\mu=0,$$

$$\frac{\partial\Phi}{\partial t}+\frac{W}{m}-\frac{\lambda}{\rho}(\mathbf{v}\nabla)\mu-\frac{e}{m}\varphi+\frac{3T}{m\rho_0}\delta\rho=0.\tag{19}$$

We easily check directly that we can write Eqs. (19) in the form (6) where the Hamiltonian H has the form

$$H=\int\frac{\rho}{m}W\,d\mathbf{r}+\frac{3}{2}\frac{T}{m\rho_0}\int\delta\rho^2\,d\mathbf{r}+\frac{1}{8\pi}\int(\operatorname{rot}\mathbf{A})^2\,d\mathbf{r}$$
$$+2\pi c^2\int\mathbf{B}^2\,d\mathbf{r}-c\int(\mathbf{B}\nabla)\varphi\,d\mathbf{r}+\frac{1}{4\pi}\int\varphi\Delta\varphi\,d\mathbf{r}.\tag{20}$$

When the subsidiary condition

$$\operatorname{div}\mathbf{B}=-\frac{1}{4\pi c}\Delta\varphi$$

is satisfied, which is identically the same as the Poisson equation, the Hamiltonian (20) is the same as the total energy of the plasma. For Eqs. (16) we have $\partial\mathbf{B}/\partial t = -\delta H/\delta\mathbf{A}$; $\partial\mathbf{A}/\partial t = \delta H/\delta\mathbf{B}$; it is clear from Eq. (17) that if $\operatorname{curl}\mathbf{p}_1 = \operatorname{curl}(\mathbf{p} + (e/c)\mathbf{A}) \equiv 0$ at some time, this will be true at all later times. From this follows that there exists a particular type of plasma motion for which $\lambda\equiv\mu\equiv 0$ which is the analogue of the potential motion of an ordinary liquid.

3. TRANSITION TO NORMAL VARIABLES

Let the medium be described by a pair of canonical variables—the generalized coordinate $q(\mathbf{r}, t)$ and the generalized momentum $p(\mathbf{r}, t)$. The equations of the medium have the form

$$\partial q/\partial t=\delta H/\delta p,\quad\partial p/\partial t=-\delta H/\delta q.\tag{21}$$

The Hamiltonian H (which usually is the same as the energy of the medium) can be expanded in powers of p and q. For a uniform medium which is invariant under a simultaneous change in the sign of the coordinates and of the time we can write the first term of the expansion in the form

$$H_0=\int\{A_kp_kp_k^*+iB_k(p_kq_k^*-p_k^*q_k)+C_kq_kq_k^*\}\,d\mathbf{k};$$
$$A_k=A_{-k}=A_k^*,\quad B_k=B_{-k}=B_k^*,\quad C_k=C_{-k}=C_k^*.$$

Waves with a dispersion relation

$$\omega_k^2=A_kC_k+B_k^2,\quad\omega_{-k}^2=\omega_k^2.$$

can propagate in the medium. The canonical transformation

$$p_k=\frac{1}{2}\left(\frac{C_k}{\omega_k}\right)^{1/2}(a_k+a_{-k}^*),$$

$$q_k=\frac{i}{2}\left(\frac{\omega_k}{C_k}\right)^{1/2}\left[\left(1+\frac{B_k}{\omega_k}\right)a_k-\left(1-\frac{B_k}{\omega}\right)a_{-k}^*\right],$$

reduces Eqs. (1) to the single equation

$$\partial a_k/\partial t+i\delta H/\delta a_k^*=0,$$

and the Hamiltonian H_0 to the form

$$H_0^{(2)}=\int\omega_ka_ka_k^*\,d\mathbf{k}.$$

The rest of the Hamiltonian is a power series in the a_k and a_k^*. Three-wave processes are described by the following term of this series:

$$H^{(3)}=\int(V_{kk_1k_2}a_k^*a_{k_1}a_{k_2}+V_{kk_1k_2}^*a_ka_{k_1}^*a_{k_2}^*)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)dk\,dk_1\,dk_2,$$
$$V_{kk_1k_2}=V_{kk_2k_1},\quad V_{-k,-k_1,-k_2}=V_{kk_1k_2}^*.\tag{22}$$

The quantity $V_{kk_1k_2}$ contains all information about three-wave processes in the medium.

In the case when the medium is described by N pairs of canonical variables several (but not more than N) types of wave can occur in it.

We change to the normal variables for the hydrodynamical plasma models. We first of all consider a plasma without a magnetic field. In that case we can restrict ourselves to "vortex-less" motions and put $\lambda \equiv \mu \equiv 0$. We then have for the quadratic Hamiltonian

$$H^{(2)} = \frac{1}{2}\int \rho_0 v^2\, d\mathbf{r} + \frac{1}{8\pi}\int (\operatorname{rot}\mathbf{A})^2\, d\mathbf{r} + 2\pi c^2 \int \mathbf{B}^2\, d\mathbf{r} + \frac{1}{4\pi}\int \varphi\Delta\varphi\, d\mathbf{r} - c^2\int (\mathbf{B}\nabla)\varphi\, d\mathbf{r} + \frac{3}{2}\frac{T}{m\rho_0}\int \delta\rho^2\, d\mathbf{r},$$

$$\mathbf{v} = \nabla\Phi - \frac{e}{mc}\mathbf{A}. \tag{23}$$

There are in the plasma Langmuir waves with amplitude a_k and a dispersion law $\omega_k = \omega_p(1 + \frac{3}{2}(kr_D)^2)$ and electromagnetic waves of different circular polarizations b_k^+ and b_k^- with a dispersion law $\Omega_k = \sqrt{(\omega_p^2 + c^2k^2)}$.

The Hamiltonian (23) is diagonalized by the substitution

$$\Phi_k = -\frac{i}{k}\left(\frac{\omega_k}{2\rho_0}\right)^{1/2}(a_k - a_{-k}^*),$$

$$\delta\rho_k = k\left(\frac{\rho_0}{2\omega_k}\right)^{1/2}(a_k + a_{-k}^*),$$

$$\mathbf{A}_k = c\left(\frac{2\pi}{\Omega_k}\right)^{1/2}[\mathbf{S}_k(b_k^+ + b_{-k}^{+*}) + \mathbf{S}_k^*(b_k^- + b_{-k}^{-*})],$$

$$\mathbf{B}_k = -\frac{i}{c}\left(\frac{\Omega_k}{8\pi}\right)^{1/2}[\mathbf{S}_k(b_k^+ - b_{-k}^{+*}) + \mathbf{S}_k^*(b_k^- - b_{-k}^{-*})].$$

After diagonalization we get

$$H^{(2)} = \int \omega_k a_k a_k^*\, d\mathbf{k} + \int \Omega_k(b_k^+ b_k^{+*} + b_k^- b_k^{-*})\, d\mathbf{k}.$$

The cubic part of the Hamiltonian takes the form

$$H^{(3)} = \frac{1}{2}\int \delta\rho\, v^2 d\mathbf{r} = \frac{1}{2}\int \delta\rho\left(\nabla\Phi - \frac{e}{mc}\mathbf{A}\right)^2 d\mathbf{r}$$

$$= \frac{1}{2(2\pi)^{3/2}}\int k\left(\frac{\rho_0}{2\omega_k}\right)^{1/2}(a_k + a_{-k}^*)(\mathbf{QR})\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2,$$

$$\mathbf{Q} = \frac{\mathbf{k}_1}{k_1}\left(\frac{\omega_{k_1}}{2\rho_0}\right)^{1/2}(a_{k_1} - a_{-k_1}^*) - \frac{e}{m}\left(\frac{2\pi}{\Omega_{k_1}}\right)^{1/2}\mathbf{S}_{k_1}(b_{k_1}^+ + b_{-k_1}^{+*}) - \frac{e}{m}\left(\frac{2\pi}{\Omega_{k_1}}\right)^{1/2}\mathbf{S}_{k_1}^*(b_{k_1}^- + b_{-k_1}^{-*}),$$

$$\mathbf{R} = \frac{\mathbf{k}_2}{k_2}\left(\frac{\omega_{k_2}}{2\rho_0}\right)^{1/2}(a_{k_2} - a_{-k_2}^*) - \frac{e}{m}\left(\frac{2\pi}{\Omega_{k_2}}\right)^{1/2}\mathbf{S}_{k_2}(b_{k_2}^+ + b_{-k_2}^{+*}) - \frac{e}{m}\left(\frac{2\pi}{\Omega_{k_2}}\right)^{1/2}\mathbf{S}_{k_2}^*(b_{k_2}^- + b_{-k_2}^{-*}). \tag{24}$$

In these equations $\mathbf{S}_k$ is a unit polarization vector, satisfying the conditions

$$\mathbf{k}\mathbf{S}_k = 0, \quad \mathbf{S}_k\mathbf{S}_k = 0, \quad \mathbf{S}_k\mathbf{S}_k^* = 1, \quad \mathbf{S}_{-k} = \mathbf{S}_k^*.$$

Expanding Eq. (24) we obtain all possible terms of the Hamiltonian describing the interaction of waves in the plasma.

We note that in the non-relativistic case the expression $H^{(2)} + H^{(3)}$ is the total Hamiltonian of the system while in the relativistic case the Hamiltonian is an infinite power series in the canonical variables.

If there is a constant magnetic field in the plasma the vector potential corresponding to it is a linear function of the coordinates and one can not directly diagonalize the quadratic Hamiltonian. It is therefore necessary to carry out a canonical transformation to new variables eliminating the constant part of the vector potential. We choose it in the form

$$\mathbf{A}_0 = \tfrac{1}{2}(-\mathbf{i}y + \mathbf{j}x)H_0,$$

and we perform the canonical transformation in two stages. In the first stage we "symmetrize" the variables:

$$\lambda = \frac{\rho^{1/2}}{\sqrt{2}}(\lambda' + \mu'), \quad \mu = \frac{1}{(2\rho)^{1/2}}(\mu' - \lambda'), \quad \Phi = \Phi' + \frac{\lambda'^2 - \mu'^2}{4\rho}.$$

In the second stage we eliminate the constant component of the potential:

$$\lambda' = \lambda'' + \omega_H^{1/2}\rho^{1/2}y, \quad \mu' = \mu'' - \omega_H^{1/2}\rho^{1/2}x,$$

$$\Phi' = \Phi'' - \frac{\omega_H^{1/2}}{2\rho^{1/2}}(x\lambda - y\mu), \quad \tilde{\mathbf{A}} = \mathbf{A} - \mathbf{A}_0, \quad \omega_H = \frac{eH_0}{mc}.$$

In the new variables (which in the following we shall write without primes, like the original ones) we get

$$\frac{\mathbf{p}}{m} = \frac{\omega_H^{1/2}}{\rho^{1/2}}(-\mathbf{i}\lambda - \mathbf{j}\mu) + \nabla\Phi - \frac{e}{mc}\tilde{\mathbf{A}} + \frac{1}{2\rho}(\lambda\nabla\mu - \mu\nabla\lambda).$$

The equations for the new variables have the form

$$\frac{\partial\lambda}{\partial t} + \omega_H^{1/2}\rho^{1/2}v_y + \frac{1}{2}(\mathbf{v}\nabla)\lambda + \frac{1}{2}\operatorname{div}(\mathbf{v}\lambda) = 0,$$

$$\frac{\partial\mu}{\partial t} - \omega_H^{1/2}\rho^{1/2}v_x + \frac{1}{2}(\mathbf{v}\nabla)\mu + \frac{1}{2}\operatorname{div}(\mathbf{v}\mu) = 0,$$

$$\frac{\partial\Phi}{\partial t} + \frac{W}{m} - \left(\mathbf{v}, \frac{\omega_H^{1/2}}{2\rho^{1/2}}(-\mathbf{i}\lambda - \mathbf{j}\mu) + \frac{1}{2\rho}(\lambda\nabla\mu - \mu\nabla\lambda)\right) = 0. \tag{25}$$

Expanding the velocity in powers of the canonical coordinates we get

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \ldots,$$

$$\mathbf{v}_0 = \frac{\omega_H^{1/2}}{\rho_0^{1/2}}(-\mathbf{i}\lambda - \mathbf{j}\mu) + \nabla\Phi - \frac{e}{mc}\tilde{\mathbf{A}},$$

$$\mathbf{v}_1 = -\frac{1}{2}\frac{\omega_H^{1/2}}{\rho_0^{3/2}}\delta\rho(-\mathbf{i}\lambda - \mathbf{j}\mu) + \frac{1}{2\rho_0}(\lambda\nabla\mu - \mu\nabla\lambda).$$

The Hamiltonian is a power series in the canonical variables:

$$H = H^{(2)} + H^{(3)} + \ldots,$$

$$H^{(2)} = \frac{1}{2}\rho_0\int v^2 d\mathbf{r} + \frac{3}{2}\frac{T}{m\rho_0}\int \delta\rho^2\, d\mathbf{r} + \frac{1}{8\pi}\int (\operatorname{rot}\mathbf{A})^2 d\mathbf{r} + 2\pi c^2\int B^2 d\mathbf{r} - c\int (\mathbf{B}\nabla)\varphi\, d\mathbf{r} + \frac{1}{4\pi}\int \varphi\Delta\varphi\, d\mathbf{r},$$

$$H^{(3)} = \frac{1}{2}\int \delta\rho\, v_0^2\, d\mathbf{r} + \rho_0\int \mathbf{v}_0\mathbf{v}_1\, d\mathbf{r}.$$

We consider the vector

$$\mathbf{d} = -\mathbf{i}\lambda - \mathbf{j}\mu.$$

By direct calculation we check the relation

$$\lambda\nabla\mu - \mu\nabla\lambda = [\mathbf{d}\operatorname{rot}[\mathbf{d}\mathbf{n}]] + (\mathbf{d}\nabla)[\mathbf{d}\mathbf{n}], \tag{26}$$

where $\mathbf{n}$ is a unit vector in the direction of H_0. Using this relation we get after some simple calculations

$$H^{(3)} = \frac{1}{(2\pi)^{3/2}}\left\{\frac{1}{2}\int \delta\rho_k(\mathbf{v}_{0k_1}\mathbf{v}_{0k_2})\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2 - \frac{1}{2}\frac{\omega_H^{1/2}}{\rho_0^{1/2}}\int \delta\rho_k(\mathbf{v}_{0k_1}\mathbf{d}_{k_2})\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2 + \frac{i}{2}\int (\mathbf{k}_2\mathbf{v}_k)(\mathbf{n}[\mathbf{d}_{k_1}\mathbf{d}_{k_2}])\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2.\right.$$

We number the types of oscillations in the plasma by an index s; the corresponding dispersion law is ω_k^s and the wave amplitude a_k^s. It is convenient also to use negative values of s, putting

$$a_k^{-s} = a_{-k}^{*s}, \quad \omega_k^{-s} = -\omega_k^s.$$

We have for the Fourier components of any real quantity ξ_k

$$\xi_k = \sum_s \xi_k^s a_k^s, \quad \xi_k^{-s} = \xi_{-k}^{*s}.$$

We introduce the velocity vector

$$\mathbf{v}_k = \sum_s \mathbf{v}_k^s a_k^s.$$

The components of the vector $\mathbf{v}_k^s$ are determined, apart from the normalization, from the linearization equations. The normalization is determined from the condition that the quadratic part of the Hamiltonian must be of the form

$$H = \sum_{s>0} \int \omega_k^s a_k^s a_k^{*s} d\mathbf{k}.$$

After determining the vectors $\mathbf{v}_k^s$ there is no difficulty in finding the total expression for the Hamiltonians of the wave interaction.

We note that

$$\delta\rho_k = \rho_0 (\mathbf{k}\mathbf{v}_k^s) / \omega_k^s. \tag{28}$$

Moreover, we get from Eqs. (25) after they are linearized

$$v_x^s = \frac{i\omega_k^s}{\omega_k^{1/2}\rho_0^{1/2}} \lambda^s, \quad v_y^s = \frac{i\omega_k^s}{\omega_k^{1/2}\rho_0^{1/2}} \mu^s,$$
$$\mathbf{d}_k^s = i\frac{\omega_H^{1/2}\rho_0^{1/2}}{\omega_k^s}[\mathbf{v}_k^s \mathbf{n}]. \tag{29}$$

Substituting (28) and (29) into (27) we find

$$H^{(3)} = \sum_{ss_1s_2} \int U_{kk_1k_2}^{ss_1s_2} a_k^s a_{k_1}^{s_1} a_{k_2}^{s_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2, \tag{30}$$

where

$$U_{kk_1k_2}^{ss_1s_2} = {}^1/_3 (Q_{kk_1k_2}^{ss_1s_2} + Q_{k_1kk_2}^{s_1ss_2} + Q_{k_2k_1k}^{s_2s_1s}),$$
$$Q_{kk_1k_2}^{ss_1s_2} = \frac{\rho_0}{2}\frac{1}{(2\pi)^{3/2}}\Big(\frac{\mathbf{k}\mathbf{v}_k{}^s}{\omega_k{}^s}(\mathbf{v}_{k_1}^{s_1}\mathbf{v}_{k_2}^{s_2}) + \frac{i\omega_H}{2\omega_k^s\omega_{k_1}^{s_1}\omega_{k_2}^{s_2}}(\mathbf{n}\,[\mathbf{v}_{k_1}^{s_1}\mathbf{v}_{k_2}^{s_2}])$$
$$\times \{(\mathbf{k}\mathbf{v}_k^s)(\omega_{k_2}^{s_2} - \omega_{k_1}^{s_1}) - (\mathbf{k}_2 - \mathbf{k}_1, \mathbf{v}_k^s)\,\omega_k^s\}\Big). \tag{31}$$

Changing to a summation over positive s only we find

$$H^{(3)} = \sum_{ss_1s_2} \int \{[U_{kk_1k_2}^{ss_1s_2} a_k^s a_{k_1}^{s_1} a_{k_2}^{s_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) + \text{c.c.}]$$
$$+ [V_{kk_1k_2}^{ss_1s_2} a_k^{*s} a_{k_1}^{s_1} a_{k_2}^{s_2} + \text{к.c.}]\, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\}\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2,$$
$$V_{kk_1k_2}^{ss_1s_2} = 3U_{-k,\,k_1,\,k_2}^{-s,\,s_1,\,s_2}. \tag{32}$$

Equations (30) to (32) give explicit expressions for the Hamiltonians for the interaction of different types of waves in the plasma.

All we have said so far refers to an electron plasma. If we want to take the motion of the ions into account it is necessary to introduce instead of their density and velocity a new set of canonical variables ρ, Φ, μ, λ. Apart from the electron non-linearity there occurs in the interaction Hamiltonian a term of similar structure caused by the ion non-linearity.

4. SOME GENERAL RELATIONS

The construction of a Hamiltonian formalism automatically solves the problem of obtaining kinetic equations describing three-wave interactions and allows us to advance further and develop a regular technique for studying non-linear wave processes. As a first step along that path we establish exact relations between the kernel of the kinetic equation and the increment of a decay instability.

We consider a medium with one type of waves which is described by the Hamiltonian (22). The equations for the medium have the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\int \{V_{kk_1k_2} a_{k_1} a_{k_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$$
$$+ 2V_{k_1kk_2} a_{k_1} a_{k_2}^* \delta(\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2)\}\, d\mathbf{k}_1\, d\mathbf{k}_2. \tag{33}$$

Equations (33) have the approximate solution $a_k = A\delta(\mathbf{k} - \mathbf{k}_0)\exp[-i\omega(\mathbf{k}_0)t]$ which is a monochromatic wave. The linearization of Eq. (33) on the basis of this solution leads to an instability with an increment

$$\gamma_k = [\gamma_{max}^2 - {}^1/_4(\omega_{k_0} - \omega_k - \omega_{k_0-k})^2]^{1/2},$$
$$\gamma_{max} = 2|V_{k_0,\,k,\,k_0-k}|\,|A|. \tag{34}$$

This is a first-order decay instability;[17] it occurs if the equation

$$\omega_{k_0} = \omega_k + \omega_{k_0-k} \tag{35}$$

has non-trivial solutions.

One easily obtains from Eqs. (33) a kinetic equation for the quantity n_k determined by the relation $\langle a_k a_{k_1}^* \rangle = \delta(\mathbf{k} - \mathbf{k}_1) n_k$ (see[2]). We have

$$\frac{\partial n_k}{\partial t} = 2\pi \int \{|V_{kk_1k_2}|^2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega_k - \omega_{k_1} - \omega_{k_2})$$
$$\times (n_{k_1}n_{k_2} - n_k n_{k_1} - n_k n_{k_2}) + 2|V_{k_1kk_2}|^2 \delta(\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2)$$
$$\times \delta(\omega_k - \omega_{k_1} + \omega_{k_2})(n_{k_1}n_{k_2} + n_k n_{k_1} - n_k n_{k_2})\, d\mathbf{k}_1\, d\mathbf{k}_2. \tag{36}$$

Equations (34)–(36) show a direct connection between the increment of the decay instability and the kernel of the kinetic equation; these relations enable us to estimate the limits of the applicability of the kinetic equation.

Let a wave packet, which is narrow in k-space, with maximum width Δk propagate in the medium. We then have for a characteristic time

$$\frac{1}{\tau} \sim \frac{|V|^2}{\omega_k'}\frac{k}{\Delta k}\int n_k dk.$$

As $\Delta k \to 0$ this time tends to zero. On the other hand, the time of the non-linear interaction must in any case be larger than the inverse of the increment of the monochromatic wave. For the monochromatic wave we can formally introduce $n_k = |A|^2\delta(\mathbf{k} - \mathbf{k}_0)$. In that case

$$\gamma_{max} = 2\Big(|V|^2 \int n_k dk\Big)^{1/2}.$$

From the requirement $1/\tau \ll \gamma_{max}$ we obtain a limitation for the width of the packet which is necessary in order that the kinetic equation can be applied:

$$\frac{\Delta k}{k} \geqslant \frac{1}{\omega_k}\Big(|V| \int n_k dk\Big)^{1/2}. \tag{37}$$

If there are no solutions of Eq. (35) second-order processes are the main ones. In that case one can eliminate the cubic terms in the Hamiltonian by a specially chosen canonical transformation (see[18]). The interaction between the waves is now described by the Hamiltonian

$$H^{(4)} = \frac{1}{2} \int T_{kk_1k_2k_3} a_k^* a_{k_1}^* a_{k_2} a_{k_3} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3$$

and the equation

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i \int T_{kk_1k_2k_3} a_{k_1}^* a_{k_2} a_{k_3} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3. \quad (38)$$

Equation (38) has an exact solution:

$$a_k = A \exp(-i\tilde{\omega}_{k_0} t)\, \delta(\mathbf{k} - \mathbf{k}_0), \qquad \tilde{\omega}_{k_0} = \omega_{k_0} + T_{k_0k_0k_0k_0} |A|^2, \quad (39)$$

corresponding to a monochromatic wave with a shifted frequency. The linearization of Eq. (38) on the basis of the solution (39) leads to an instability with an increment[19]

$$\gamma = [\gamma_{max}^2 + {}^1/_4 (2\tilde{\omega}_{k_0} - \tilde{\omega}_{k_0-\varkappa} - \tilde{\omega}_{k_0+\varkappa})^2]^{1/2},$$

$$\gamma_{max} = |T_{k_0,\, k_0,\, k_0+\varkappa,\, k_0-\varkappa}|\, |A|^2,$$

$$\tilde{\omega}_{k_0-\varkappa} = \omega_{k_0-\varkappa} + 2T_{k_0,\, k_0-\varkappa,\, k_0,\, k_0-\varkappa} |A|^2,$$

$$\tilde{\omega}_{k_0+\varkappa} = \omega_{k_0+\varkappa} + 2T_{k_0,\, k_0+\varkappa,\, k_0,\, k_0+\varkappa} |A|^2.$$

Equation (38) corresponds to the kinetic equation

$$\frac{\partial n_k}{\partial t} = 2\pi \int |T_{kk_1k_2k_3}|^2 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times (n_{k_1} n_{k_2} n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3})\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3. \quad (40)$$

As before, there is a direct connection between the increment of the instability and the kernel of the kinetic equation.

Reasoning as before we find the condition for the applicability of the kinetic Eq. (40). We note that we have for a wave packet which is narrow in k-space

$$\omega_k = \omega_{k_0} + (\Delta\mathbf{k}, \mathbf{v}) + \frac{1}{2}\left(\Delta k_\alpha \Delta k_\beta \frac{\partial^2 \omega}{\partial k_\alpha \partial k_\beta}\right),$$

whence we find, if we use the relation $\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$,

$$\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3} \approx \frac{(\Delta k)^2}{k^2} \omega_k.$$

For the time of the non-linear interaction we have the estimate

$$\frac{1}{\tau} \sim \frac{|T|^2}{\omega_k} \left(\int n_k\, d\mathbf{k}\right)^2 \left(\frac{k}{\Delta k}\right)^2.$$

For the increment of the decay instability we have

$$\gamma_{max} = 2|T| \int n_k\, d\mathbf{k}.$$

From the condition $\gamma_{max} \gg 1/\tau$ we find

$$\frac{\Delta k}{k} \gg \left(\frac{|T|}{\omega_k} \int n_k\, d\mathbf{k}\right)^{1/2}. \quad (41)$$

If the packets are so narrow in k-space that conditions (37) and (41) are not satisfied, the kinetic equations for n_k are inapplicable and we must study them in the framework of a more exact theory.

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