INFLUENCE OF TRANSPORT CURRENT ON THE DISTRIBUTION OF INDUCTION IN A SUPERCONDUCTING CYLINDER OF THE SECOND KIND

F. F. TERNOVSKIĬ

Moscow State University

Submitted March 23, 1970; resubmitted June 26, 1970

Zh. Eksp. Teor. Fiz. 60, 1790–1803 (May, 1971)

A condition is formulated under which undamped current can circulate in a closed superconducting circuit containing a section in the mixed state. The thermodynamic potential (Gibbs free energy) (2.13), (4.5) is found; it has a minimum at a fixed current in the sample. Assuming that the vortex filaments in a long circular cylinder placed in a longitudinal magnetic field and carrying transport current are helical lines with equal pitch, the density of the free energy (3.21) is calculated and the distribution of the induction (5.1) is determined. The critical current is estimated.

1. INTRODUCTION

IT is known from experiment[1,2] that a superconducting cylinder (of the second kind) placed in an external magnetic field parallel to its axis is capable, when in the mixed state, of carrying a considerable transport current.

If the sample is extremely pure, so that its magnetization curve is almost reversible, then the critical current in the longitudinal field is higher by several orders of magnitude than in a transverse field.

Sufficiently well founded attempts to calculate the critical current in a longitudinal field were made by Bergeron[3] and Boyd.[4] Bergeron, however, used for the description of the mixed state the Goodman laminar model, which, as is well known,[5] is not sufficiently realistic, while Boyd did not take into account the bending of the vortex filaments.

Our analysis is based on Abrikosov's vortex model[6] and takes into account the bending of the vortex filaments.

To avoid complications connected with the need for considering the contact between the superconductor and the normal metal and to be able to make consistent use of the principles of equilibrium thermodynamics, we shall assume that our sample (a long circular cylinder) is part of some closed superconducting circuit, and that the magnetic field parallel to its axis is produced by a short-circuited superconducting solenoid (see Fig. 1). In the main part of this article it will be assumed that the sample is made of an ideal superconductor of the second kind and is situated in an external field close to H_{c2}—the second critical field. We shall assume that the remaining part of the superconducting circuit is in the Meissner state, so that all the vortex filaments start and end on the surface of the sample.

In order to excite a current in such a closed superconducting circuit, it is necessary to cool it below the critical temperature in the presence of a weak external field, change the sample to the mixed state (by making current flow through the aforementioned solenoid), and turn off the external field, at a sufficiently slow rate if necessary. If there exists inside the superconducting circuit a closed contour (line γ_1 of Fig. 1) that is not intersected by the vanishing vortex filaments produced or moving upon change of the external magnetic field, then the magnetic field flux through the surface covering this contour (more accurately, the corresponding fluxoid), will be conserved, and current will be produced in the circuit.

To prove this statement rigorously, it is necessary to have an expression for the electric field in superconductors. At present there is no known sufficiently general expression of this type. We shall therefore use a different analysis method. Namely, we postulate that the system of superconductors under consideration can exist, at least at small values of the current, in a relatively stable state. Inasmuch as the absolute minimum of the free energy of our system occurs at zero current, the determination of the metastable current states is a problem of finding a conditional extremum. The corresponding additional condition, as we shall show, is determined fully uniquely. It should obviously coincide with the conservation law that ensures the possibility of exciting the current in the closed circuit under consideration.

2. THERMODYNAMIC POTENTIALS FOR SUPERCONDUCTORS

In this section we shall consider a superconductor or a system of superconductors of arbitrary connectivity, placed in a thermostat and not interacting with any other external bodies (the energy flux through an infinitely remote surface is equal to zero). This, for example, might be the system described in Sec. 1.

The free energy of such a system can be represented in the form

$$\mathcal{F} = \int F_s \, dV_s + \int \frac{\mathbf{H}_e^2}{8\pi} dV_e, \tag{2.1}$$

where F_s is the density of the free energy of the superconductor, the integration in the first term is over the volumes of all the superconductors under consideration, H_e is the magnetic field outside the superconductors ($H_e = 0$ at infinity), and the integration in the second term extends over all space that is external with respect to the superconductors.

The state of the superconductors will be described in this section with the aid of quasimicroscopic parameters (i.e., quantities averaged over the volume with a characteristic dimension much larger than atomic but smaller than the coherence length). The quantity F_S depends on the magnetic field intensity h, on the vector potential **A** (h = curl **A**), and on the wave function (or pairing potential)

$$\psi = |\psi| \exp[2\pi i\theta] = f \exp[2\pi i\theta].$$

From the gauge-invariance requirements it follows that the vector potential can enter in F_S only in the form curl **A** or $\mathbf{A} - \varphi_0\nabla\theta$, where $\varphi_0 = hc/2e$ is the magnetic-flux quantum. Therefore

$$\delta F_s = \frac{1}{4\pi}\mathbf{h}\,\mathrm{rot}\,\delta\mathbf{A} - \frac{1}{c}\mathbf{j}(\delta\mathbf{A} - \varphi_0\nabla\delta\theta) + \ldots \quad (2.2)$$

where we have omitted terms corresponding to the variations of the other parameters. The physical meaning of the vector **j** will become clear after the equilibrium conditions are determined (see (2.7)).

If we use for F_S the Ginzburg-Landau expression[7]

$$F_s = \alpha f^2 + \beta\frac{f^4}{2} + \frac{1}{2m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi\right|^2 + \frac{h^2}{8\pi} + \mathrm{const},$$

then, as can readily be verified, Eq. (2.2) is satisfied, with

$$\mathbf{j} = \frac{4e^2f^2}{mc}(\varphi_0\nabla\theta - \mathbf{A}). \quad (2.3)$$

In the derivation of the Ginzburg-Landau equations the independent parameters are usually assumed to be ψ and ψ^*. This results in a loss of certain unique effects due to the fact that the function θ may not be single-valued. We shall assume that the independent parameters are $f = |\psi|$ and θ. Variation with respect to f leads to the equation

$$\alpha f + \beta f^3 + \frac{2e^2 f}{mc^2}(\varphi_0\nabla\theta - \mathbf{A})^2 - \frac{\hbar^2}{2m}\nabla^2 f = 0, \quad (2.4)$$

which differs only in the notation from the corresponding Ginzburg-Landau equation. We shall therefore omit in the variation of the free energy the terms proportional to δf or $\nabla\delta f$, and use expression (2.2), the applicability of which is not connected with the assumption that the quantity $|\psi|^2$ is small.

We recall first certain known facts concerning the phase of the pairing potential. The function ψ should be single-valued. Therefore the change of the quantity θ on going around any region that violates the single connectivity of the superconductor should be equal to an integer.

Let us assume that the superconductor in question can be made singly-connected with the aid of a finite number of cuts (S_1 and S_2 in Fig. 1). In the resultant region it is possible to separate a single-valued branch of the function θ. This function will have in the general case different values on the two sides of the aforementioned cuts. The difference of these values for the cut S_α will be denoted by $[\theta]_\alpha$. We have

$$[\theta]_\alpha = \int_1^2 \nabla\theta\, dl; \quad (2.5)$$

here the points 1 and 2 are on opposite sides of the cut S_α, and the contour γ_α joining these points should be chosen inside the region where the function θ is uniquely defined. Therefore, in particular, it must not cross any of the cuts. The quantities $\varphi_0[\theta]$ are customarily called fluxoids.

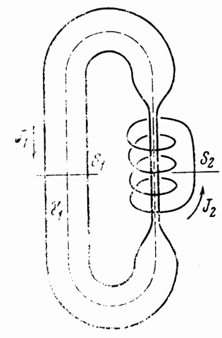

FIG. 1

As shown by Abrikosov,[6] when going around an axial line (such lines will be denoted by Γ_k) of a vortex filament, the phase should change by 2π, i.e., θ should change by unity. Therefore in those cases when the superconductor is in the mixed state, it is necessary to draw besides the cuts S_α also cuts S_k joining the lines Γ_k with the surface of the sample.

Let us now calculate the variation of the free energy. From (2.1) and (2.2) we get

$$\delta\mathcal{F} = \int\left[\frac{1}{4\pi}\mathbf{h}\,\mathrm{rot}\,\delta\mathbf{A} - \frac{1}{c}\mathbf{j}(\delta\mathbf{A} - \varphi_0\nabla\delta\theta)\right]dV_s + \frac{1}{4\pi}\int \mathbf{H}_e\,\mathrm{rot}\,\delta\mathbf{A}\,dV_e.$$

Integrating by parts and recognizing that the tangential components of the vector potential should be continuous on the surfaces of the superconductors, we obtain

$$\delta\mathcal{F} = \frac{1}{4\pi}\int\left[\left(\mathrm{rot}\,\mathbf{h} - \frac{4\pi}{c}\mathbf{j}\right)\delta\mathbf{A} + \frac{4\pi\varphi_0}{c}\mathbf{j}\nabla\delta\theta\right]dV_s + \frac{1}{4\pi}\int \mathrm{rot}\,\mathbf{H}_e\delta\mathbf{A}\,dV_e + \frac{1}{4\pi}\int[\mathbf{H}_e - \mathbf{h}, \delta\mathbf{A}]_n\,d\sigma.$$

The integration in the last term is over the surfaces of the superconductors, and we discard the integral over the infinitely remote surface.

We consider the integral

$$\int \mathrm{div}(\mathbf{j}\theta)\,dV_s = \sum_\alpha [\theta]_\alpha J_\alpha + \sum_k \int \mathbf{j}\,d\sigma_k + \int \theta j_n\,d\sigma,$$

where J_α is the flux of the vector **j** through the cut S_α, and we have taken into account the fact that on going around the vortex filament $[\theta]_k = 1$. In calculating the analogous integral with the function $\theta' = \theta + \delta\theta$ it is necessary to bear in mind that variation may result in

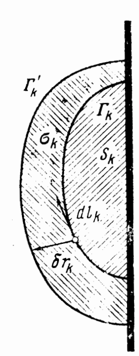

FIG. 2. Illustrating the variation of the function θ. The heavy line represents the surface of the superconductor. Γ_k and Γ_k' are the axial lines of the vortex filament numbered k before and after the variation.

a change in the form and position of the vortex filaments, so that the functions θ' correspond to new cuts S'_k. We shall assume that each such cut consists of a surface S_k and a surface σ_k covering the lines Γ_k, Γ'_k and the segments joining the ends of these lines along the surface of the sample (see Fig. 2). Further, let dl_k be an element of length of the line Γ_k, and dr_k be the displacement of this element upon variation. Then $d\sigma_k = \delta r_k \times dl_k$ and we obtain*

$$\int \operatorname{div}(\mathbf{j}\delta\theta)\,dV_s = \sum_\alpha J_\alpha \delta[\theta]_\alpha + \sum_k \int_{\Gamma_k} \mathbf{j}[\delta\mathbf{r}_k, d\mathbf{l}_k] + \int \delta\theta j_n\, d\sigma.$$

Using this result, we can represent the variation of the free energy in the form

$$\delta\mathcal{F} = \int\left[\frac{1}{4\pi}\left(\operatorname{rot}\mathbf{h} - \frac{4\pi}{c}\mathbf{j}\right)\delta\mathbf{A} - \frac{\varphi_0}{c}\delta\theta \operatorname{div}\mathbf{j}\right]dV_s + \int \delta\mathbf{A}\operatorname{rot}\mathbf{H}_e\,dV_e + \frac{1}{4\pi}\int\left([\mathbf{H}_e - \mathbf{h}, \delta\mathbf{A}]_n + \frac{4\pi\varphi_0}{c} j_n\delta\theta\right)d\sigma + \frac{\varphi_0}{c}\sum_\alpha J_\alpha\delta[\theta]_\alpha - \frac{\varphi_0}{c}\sum_k\int_{\Gamma_k}[d\mathbf{l}_k, \mathbf{j}]\delta\mathbf{r}_k. \tag{2.6}$$

The variations $\delta\mathbf{A}$, δr_k, and $\delta\theta$ are independent of one another. Therefore in any equilibrium state we have

$$\operatorname{rot}\mathbf{h} = 4\pi\mathbf{j}/c, \tag{2.7}$$

$$d\mathbf{f}_k = \varphi_0 c^{-1}[d\mathbf{l}_k, \mathbf{j}] = 0, \quad k = 1, 2, \ldots \tag{2.8}$$

inside the superconductors,

$$\operatorname{rot}\mathbf{H}_e = 0 \tag{2.9}$$

on the outside, and

$$\mathbf{H}_{et} = \mathbf{h}_t, \quad j_n = 0 \tag{2.10}$$

on the boundary.

Equation (2.7) shows that the vector j in formulas (2.2) and (2.3) represents the current density. The quantity df_k is obviously the force acting on the vortex-filament element dl_k.

There are many published variants of the derivation of this formula. The most general of these is apparently that of Galaĭko.[8] Expression (2.8) can be reduced to the same form as in [8] by separating the contribution of the currents circulating around the line Γ_k.

Expression (2.6), furthermore, shows that the additional conditions ensuring the existence of metastable current states is

$$[\theta]_\alpha = N_\alpha = \text{const}, \quad \alpha = 1, 2, \ldots \tag{2.11}$$

where N_α are integers (see Sec. 1).

If Eqs. (2.3), (2.4), and (2.7)–(2.10) (which will henceforth be called the local-equilibrium conditions) have solutions for a given set of numbers N_α, then the free energy has a relative minimum. At finite temperatures, transitions can occur between the states corresponding to different values of the numbers N_α. The probability of such transitions can be calculated for some simple cases.[9-11] If irreversible processes occur in the system, then Eq. (2.6) should be replaced by the inequality

$$\delta\mathcal{F} \leqslant \sum_\alpha \frac{\varphi_0}{c} J_\alpha \delta[\theta]_\alpha, \tag{2.12}$$

which determines the direction and the very possibility of realization of the aforementioned transitions.

If there are vortex filaments inside the superconductor, then there is no unique correspondence between the currents and the fluxoids. Let us assume, however, that at specified values of the currents J_α there exist values of the numbers N_α such that the inequality (2.12) is not satisfied at least for small values of these numbers. Such a state is metastable, as before, but it has the highest stability in comparison with the other nearby states. We shall call this a macroscopic equilibrium state. At small deviations from macroscopic equilibrium, by definition, we have

$$\delta G = \delta\mathcal{F} - \sum_\alpha \frac{\varphi_0}{c} J_\alpha \delta[\theta]_\alpha \geqslant 0.$$

Thus, for fixed currents (and if the local-equilibrium conditions are satisfied), the potential

$$G = \mathcal{F} - \sum_\alpha \frac{\varphi_0}{c} J_\alpha [\theta]_\alpha, \tag{2.13}$$

which we shall call, following the tradition, the Gibbs potential, has a relative minimum.

Simplifying the situation somewhat, it can be stated that in the determination of the local-equilibrium conditions one considers variations such that the vortex filaments shift from the equilibrium positions through distances that are small compared with the distances between the filaments, whereas when the conditions of the macroscopic equilibrium are determined, one considers variations such that the displacements of the vortex filaments are small only in comparison with the characteristic dimension of the sample. The displaced vortex filaments, however, remain in equilibrium with one another. Therefore, before we proceed to derive the macroscopic equilibrium conditions, we must express the Gibbs potential in terms of macroscopic parameters.

3. LOCAL EQUILIBRIUM

We consider a sample in the form of an infinite round (radius R) cylinder. The sample is placed in a magnetic field parallel to its axis (which coincides with the z axis of the cylindrical coordinate system ρ, φ, z which will henceforth be used) and carries a transport current J_1. We assume that the values of the magnetic-field intensity components at $\rho = R$ satisfy the inequalities

$$H_{c2} - H_{ez} \ll H_{c2}, \quad H_{e\varphi} \ll H_{ez}. \tag{3.1}$$

Under the conditions in question, the vortex filaments should be helical lines having a common axis coinciding with the cylinder axis. Each helical line is characterized by its own radius a and by a pitch $z_0 = 2\pi\omega^{-1}$. Its equation is

$$\rho = a, \quad \varphi = \omega z + \varphi(0).$$

If ω depends on a or $\varphi(0)$, the mutual placement of the centers of the vortex filaments in different planes z = const will be essentially different, so that it is impossible to ensure satisfaction of the conditions (2.8) simultaneously in the entire volume of the sample. Therefore in each equilibrium state the pitch of the vortex filaments z_0 can depend only on z. If the cylinder is long

*$[\delta r_k, dl_k] = \delta r_k \times dl_k$.

enough ($l \gg z_0$), ω should be simply constant in the greater part of the cylinder, as will henceforth be assumed. Under these conditions the quantities h_ρ, h_φ, h_z and A_ρ, A_φ, and A_z are functions of only ρ and $\varphi - \omega z$, and the function ψ can be represented in the form

$$\psi = \Phi(\rho, \varphi - \omega z) \exp(ikz), \tag{3.2}$$

where k is a new unknown parameter. To fix the "origin" for the quantity k, we shall assume that $A_z = 0$ when $\rho = 0$. In addition, we can choose $A_\rho = 0$ for all ρ.

From the equations div h = 0 and h = curl **A** we get

$$h_\varphi = \omega\rho h_z - \frac{\partial a}{\partial \rho}, \quad h_\rho = \frac{1}{\rho}\frac{\partial a}{\partial \varphi}, \tag{3.3}$$

where

$$a = A_z + \omega\rho A_\varphi. \tag{3.4}$$

Using (2.3) and (2.7) we can readily establish that the induction force lines are parallel to the vortex filaments if

$$\omega k d^2 \ll 1, \tag{3.5}$$

where d is the distance between the neighboring vortex filaments. In Sec. 4 it will be shown that this inequality, as well as the inequality

$$\omega R \ll 1, \tag{3.6}$$

is well satisfied at $H_{ez} \approx H_{c2}$ and for all values of the transport current smaller than the critical value.

Along the vortex filament $d\varphi = \omega\, dz$. The following equalities therefore hold

$$B_\varphi = \omega\rho B_z, \quad B_\rho = 0, \tag{3.7}$$

where $B = \langle h \rangle$ is the induction vector. The symbol $\langle \ldots \rangle$ denotes microscopic averaging. Comparing (3.7) with (3.3), we see that in the main approximation $\langle a \rangle = 0$.

Changing over to the approximate solution of the Ginzburg-Landau equations, we put

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 + \ldots, \qquad A_{0\varphi} = \frac{C\rho}{2}\left(1 - \frac{s\omega^2\rho^2}{2}\right), \quad A_{0z} = -\omega\rho A_{0\varphi}, \tag{3.8}$$

where $C \approx H_{c2}$ and $s \sim 1$ are constants, the exact values of which we shall determine below. Such a choice of the vector $\mathbf{A}_0$ makes it possible to take into account the bending of the vortex lines even in the main approximation. We note also that the expansion in (3.8) is with respect to the two small parameters $|\psi|^2$ and $\omega\rho$.

As is well known,[6] it suffices to determine the function ψ in the main approximation, when it satisfies a certain linear equation (see below). It is convenient to represent it in the form

$$\psi = \sum_n c_n \psi_n = \sum_n c_n R_n(\rho) \exp[ikz + in(\varphi - \omega z)].$$

At a given choice of the sequence $\{\psi_n\}$, the function ψ will be the best approximation if the coefficients c_n are chosen such that the free energy has the least possible value. Substituting in (2.1) for F_s the Ginzburg-Landau expression and $\psi = \sum_n c_n\psi_n$, we get

$$\frac{\partial \mathcal{F}}{\partial c_n} = 2\int \psi^* \left[\left(\frac{i\nabla}{\varkappa} + \mathbf{A}\right)^2 \psi_n - \psi_n + |\psi|^2\psi_n\right] dV_s = 0. \tag{3.9}$$

Here and henceforth we shall use a system of units characteristic of the Ginzburg-Landau theory, retaining the same symbols as before for all the quantities. In such a system of units (see, for example, [7])

$$F_s = -|\psi|^2 + \frac{1}{2}|\psi|^4 + \left|\left(-\frac{i\nabla}{\varkappa} - \mathbf{A}\right)\psi\right|^2 + h^2,$$

where κ is the parameter of the Ginzburg-Landau theory, and Eqs. (2.3) and (2.7) take the form

$$\mathbf{j} = -\frac{i}{2\varkappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \mathbf{A}|\psi|^2, \tag{3.10}$$

$$\operatorname{rot} \mathbf{h} = \mathbf{j}. \tag{3.11}$$

We choose now the functions ψ_n such as to cancel out under the integral sign in (3.9) all the "large" terms, i.e., we stipulate that

$$\left(\frac{i\nabla}{\varkappa} + \mathbf{A}_0\right)^2 \psi_n - \psi_n = \lambda_n \psi_n, \tag{3.12}$$

where $\lambda_n \ll 1$, and the components of the vector $\mathbf{A}_0$ are given by formulas (3.8). A limited solution of (3.12) exists when

$$\lambda_n = \frac{C}{\varkappa} - 1 + \frac{k^2 + 2k\omega + \omega^2 n(s-2)}{\varkappa^2} + \frac{\omega^2 n^2 (s-1)}{\varkappa^2}$$

and is equal to

$$\psi_n = \rho^n \exp\left[-\frac{\gamma_n\rho^2}{2} + in(\varphi - \omega z) + ikz\right],$$

where

$$\gamma_n = {}^1/_2\varkappa C + \omega k + {}^1/_2\omega^2 n(s-2).$$

The function $|\psi_n|$ has a sharp maximum at $\rho = (n/\gamma_n)^{1/2}$. Consequently, the significant values are $n \sim \gamma_n R^2 \gg 1$ (in the usual units $\gamma_n \sim \xi^{-2}$). Therefore, if $s \neq 1$, then λ_n is inadmissably large.

We shall show in Sec. 4 that the value s = 1 is singled out also in another respect. When s = 1 we have

$$\lambda_n = \frac{C - H_{c2}^*}{\varkappa} - \frac{n\omega^2}{\varkappa^2}, \quad H_{c2}^* = \varkappa - \frac{k^2 + 2\omega k}{\varkappa}, \tag{3.13}$$

$$\gamma_n = \frac{\varkappa C}{2} + \omega k - \frac{n\omega^2}{2} \equiv \gamma - \frac{n\omega^2}{2},$$

and the function ψ satisfies the equation

$$\rho\frac{\partial\psi}{\partial\rho} + \gamma\rho^2\psi + i\left(1 + \frac{\omega^2\rho^2}{2}\right)\frac{\partial\psi}{\partial\varphi} = 0. \tag{3.14}$$

Using this equation and the equations (3.11) and (3.10), in which we can substitute $\mathbf{A} = \mathbf{A}_0$, we obtain

$$h_z = H_0 - C\omega^2\rho^2 - \frac{|\psi|^2}{2\varkappa}\left(1 - \frac{\omega^2\rho^2}{2}\right) \tag{3.15}$$

and

$$2\omega H_0 = \frac{k + \omega}{\varkappa}\langle|\psi|^2\rangle|_{\rho=0}, \tag{3.16}$$

where H_0 is the integration constant. Equation (3.16) determines the connection between the parameters ω and k. In order to reconcile (3.8) with (3.15), it suffices to choose

$$C = H_0 - \langle|\psi|^2\rangle / 2\varkappa, \tag{3.17}$$

and

$$A_{1\varphi} = \frac{1}{2\varkappa\rho}\int_0^\rho (\langle|\psi|^2\rangle - |\psi|^2)\rho\, d\rho. \tag{3.18}$$

The corrections to A_{0Z} need not be determined in the present approximation, since A_{0Z} itself is of the first order of smallness.

The condition for the determination of the normalization of the function ψ is obtained from Eq. (3.9), which can be reduced with the aid of (3.8), (3.12), (3.13), (3.17), and (3.18) to the form

$$\int \psi^* \left[\frac{H_0 - H_{c2}^* - n\omega^2\varkappa^{-1}}{\varkappa} + \left(1 - \frac{1}{2\varkappa^2}\right)|\psi|^2\right]\psi_n\, dV_s = 0.$$

Multiplying this equation by c_n and summing over the values $n \approx \gamma\rho^2 \approx \kappa H_0\rho^2/2$, we obtain

$$\left(1 - \frac{1}{2\varkappa^2}\right)\langle|\psi|^4\rangle + \frac{1}{\varkappa}\left[H_0\left(1 - \frac{\omega^2\rho^2}{2}\right) - H_{c2}^*\right]\langle|\psi|^2\rangle = 0,$$

whence

$$\langle|\psi|^2\rangle = \frac{H_{c2}^* - H(\rho)}{\varkappa\beta(1 - 1/2\varkappa^2)}, \qquad (3.19)$$

where

$$\beta = \langle|\psi|^4\rangle / (\langle|\psi|^2\rangle)^2, \quad H(\rho) = H_0(1 - {}^1\!/_2\omega^2\rho^2).$$

If the vortex filaments are straight and form a quadratic lattice, then $\beta = 1.18$.[6] For a triangular lattice $\beta = 1.16$.[12] In the case considered here, $|\psi|^2$ is not strictly periodic in the plane z = const, but the distance $\rho = \omega^{-1}$ over which the quantity $\langle|\psi|^2\rangle$ changes appreciably is incomparably larger than the lattice period. Therefore there are no grounds for assuming that β differs significantly from its usual values.

We note now that it follows from (3.7) and (3.15) that

$$B = \langle h\rangle = (1 + \omega^2\rho^2)^{1/2}\langle h_z\rangle = H(\rho) - \frac{\langle|\psi|^2\rangle}{2\varkappa}. \qquad (3.20)$$

Expressions (3.20) and (3.19) differ from the corresponding results of Abrikosov[6] only in that the quantity $H = H_0(1 - \omega^2\rho^2/2)$ depends on the coordinates, and in the fact that H_{c2} is replaced in this case by the smaller quantity (owing to the presence of the current) $H_{c2}^* = \kappa - (k^2 + 2\omega k)\kappa^{-1}$.

Returning again to the usual units, we obtain

$$F_s = \frac{\langle h^2\rangle}{8\pi} - \frac{H_c^2}{8\pi}\langle|\psi|^4\rangle = \frac{B^2}{8\pi} - \frac{(H_{c2}^* - B)^2}{8\pi[1 + \beta(2\varkappa^2 - 1)]}. \qquad (3.21)$$

From (3.21), (3.20), and (3.19) it follows that

$$H = \frac{1}{4\pi}\left(\frac{\partial F_s}{\partial B}\right)_{\omega,k} = B - \frac{H_{c2}^* - B}{(2\varkappa^2 - 1)\beta + 1}. \qquad (3.22)$$

Finally, we note that with the aid of (3.19) we can represent (3.16) in the form

$$\omega = (k + \omega)\frac{H_{c2}^* - H_0}{H_0\beta(2\varkappa^2 - 1)}. \qquad (3.23)$$

Only two out of the three parameters ω, k, and H_0 introduced by us are independent. We could determine them if we knew the boundary conditions for the tangential components of the vectors H or B. Since, however, some part (which we shall show to be appreciable) of the transport current can flow along the surface of the sample, the determination of these boundary conditions is a separate nontrivial problem, constituting one of the problems of macroscopic equilibrium.

4. MACROSCOPIC EQUILIBRIUM

The Gibbs potential (2.13) for the system shown in Fig. 1 is

$$G = \mathscr{F} - \frac{\varphi_0 J_1[\theta]_1}{c} - \frac{\varphi_0 J_2[\theta]_2}{c}. \qquad (4.1)$$

According to (2.3) and (2.5)

$$\varphi_0[\theta]_1 = \oint_{\gamma_1}\left(\mathbf{A} + \frac{mc}{4e^2f^2}\mathbf{j}\right)d\mathbf{l}. \qquad (4.2)$$

Since the contour γ_1 must not intersect any of the cuts joining the axial lines of the vortex filaments with the surface, the line γ_1 in the interior of the sample must coincide with its axis. In the remaining part of the circuit, which is in the Meissner state (B = 0), we choose a line γ_1 far from the surface, such as to make $\mathbf{j} \approx 0$ along this line. As a result we obtain

$$\varphi_0[\theta]_1 = \Phi_1 + \int_0^l \frac{mc}{4e^2f^2}j_z\, dz,$$

where Φ_1 is the flux of the magnetic field through the surface covering the contour γ_1, and in the second term the integration proceeds along the cylinder axis.

We shall henceforth neglect systematically all the effects due to the fact that the sample has a finite length. In this approximation we can assume that at fixed currents J_1 and J_2 the distribution of the field outside the sample does not change when the quantities ω, k, or B are varied. We can therefore put

$$\varphi_0[\theta]_1 = \int_0^l dz\int_0^R B_\varphi\, d\rho + \int_0^l \frac{mc}{4e^2f^2}j_z\, dz + \text{const}. \qquad (4.3)$$

In the same approximation

$$\varphi_0[\theta]_2 = n\int B_z\, dV_s + \text{const}, \qquad (4.4)$$

where n is the number of turns per unit length in the solenoid producing the longitudinal field $H_{ez} = 4\pi n J_2 c^{-1}$.

Substituting expressions (4.3) and (4.4) in (4.1) and leaving out an inessential constant, we get

$$G = \int\left(F_s - \frac{H_{ez}B_z}{4\pi} - \frac{RH_{e\varphi}B_\varphi}{4\pi\rho} - \frac{\hbar kcH_{e\varphi}}{4\pi eR}\right)dV_s, \qquad (4.5)$$

where we have also used the following expression, which follows from (2.3) and (3.2):

$$\frac{mc}{4e^2|\psi|^2}j_z = \frac{\hbar kc}{2e}.$$

Macroscopic equilibrium corresponds to a relative minimum of the Gibbs potential. The additional conditions for the variation are the relations (3.7) and (3.23). It is convenient to choose as the independent parameters the quantities $B(\rho)$ and k. Expressing B_φ and B_Z in terms of B and taking (3.22) into account, we get from (4.5)

$$\frac{\delta G}{\delta B(\rho)} = \frac{1}{4\pi}\left[H(\rho) - \frac{H_{ez} + \omega RH_{e\varphi}}{\sqrt{1 + \omega^2\rho^2}}\right] = 0.$$

At small values of $\omega\rho$ we obtain

$$H(\rho) = \frac{H_{ez} + \omega RH_{e\varphi}}{\sqrt{1 + \omega^2\rho^2}} \approx H_0\left(1 - \frac{\omega^2\rho^2}{2}\right). \qquad (4.6)$$

This result shows that the value s = 1 chosen by us in formulas (3.8) and (3.13) agrees with the requirements of the macroscopic equilibrium. We see also that in (3.19)

$$H_0 = H_{ez} + \omega R H_{e\varphi}. \tag{4.7}$$

Expressing B in terms of H and denoting, as usual, by

$$g_s(H, k) = F_s - \frac{BH}{4\pi}$$

the density of the Gibbs potential, we can represent (4.5) in the form

$$G = \int \left[g_s\left(\frac{H_0}{\sqrt{1+\omega^2\rho^2}}, k\right) - \frac{\hbar k c H_{e\varphi}}{4\pi e R}\right] dV_s. \tag{4.8}$$

According to (3.21) and (3.22) we have

$$g_s(H, k) = -\frac{H^2}{8\pi} - \frac{(H_{c2}^* - H)^2}{8\pi\sigma}, \tag{4.9}$$

where $\sigma = (2\kappa^2 - 1)\beta$.

We shall carry out the subsequent transformations under an assumption, which will be confirmed later on, that the important values are $\omega R \ll 1$. In addition, it follows from (3.23) that $\omega \ll k$ always. Taking these inequalities into account and introducing the notation

$$\alpha = \frac{\varkappa H_{c2} R}{\lambda \sigma H_{ez}}, \quad \Delta = 1 - \frac{H_{ez}}{H_{c2}}, \quad p = \frac{4H_{c2}^2}{\sigma\alpha^2\Delta H_{ez}^2},$$
$$\omega R = \alpha\Delta^{3/2} u, \quad 2H_{e\varphi}/H_{ez} = \alpha\Delta^{3/2} v, \tag{4.10}$$
$$q^2 = \frac{\alpha^2\Delta^2 H_{ez}}{2H_{c2}}, \quad \frac{k\lambda}{\varkappa} = \tau\sqrt{\Delta}.$$

(λ is the depth of penetration of the weak magnetic field), we can represent (4.8) in the form

$$G = \frac{H_{ez}^2}{8\pi}(-1 + \alpha^2\Delta^3 \tilde{g}) V_s, \tag{4.11}$$

where V_s is the volume of the sample and

$$\tilde{g} = \frac{u^2}{2} - uv - \frac{p}{4}\left(\frac{u}{\tau}\right)^2 - \frac{pv\tau}{2}. \tag{4.12}$$

Using (4.7) and changing over to the new notation, we get from (3.23)

$$u = \frac{\tau(1-\tau^2)}{1 + q^2 v\tau}. \tag{4.13}$$

Since $H_{c2}^* - H_{ez} = H_{c2}\Delta(1 - \tau^2)$ should be positive, it follows that $\tau^2 \leq 1$.

The value of τ at which the Gibbs potential has a minimum is determined by the conditions

$$\left(\frac{\partial \tilde{g}}{\partial \tau}\right)_v = 0, \quad \left(\frac{\partial^2 \tilde{g}}{\partial \tau^2}\right)_v \geqslant 0. \tag{4.14}$$

Expressing u in terms of v and τ in accordance with (4.13), we can represent the equation $(\partial\tilde{g}/\partial\tau)_v = 0$ in the form

$$\left[\frac{\tau(1-\tau^2)}{1+q^2 v\tau} - v\right]\frac{1 - 3\tau^2 - 2\tau^3 q^2 v}{(1+q^2 v\tau)^2}$$
$$+ \frac{p}{2}\left[\frac{(1-\tau^2)[2\tau + q^2 v(1+\tau^2)]}{1+q^2 v\tau} - v\right] = 0. \tag{4.15}$$

In this equation, the unknown quantity is $\tau \backsim k$, while $v \backsim H_e$ and q^2 are given parameters. It is more convenient, however, to solve this equation first with respect to v. The function $v(\tau)$ should be bounded and non-negative in the interval $0 \leq \tau \leq 1$. In addition, using (4.15), it can be shown that the inequality $(\partial^2\tilde{g}/\partial\tau^2)_v \geq 0$ is equivalent to the condition

$$\partial v / \partial \tau \geqslant 0. \tag{4.16}$$

For samples having macroscopic dimensions we have $R \gg \lambda$, so that the parameter $\alpha \gg 1$ (see (4.10)). Therefore the parameter $p \ll 1$ ($\Delta \gg 4\sigma^{-1}\alpha^{-2}$) practically always. From (4.15) we get in this case

$$v \approx \frac{\tau(1-\tau^2)}{1+q^2 v\tau}, \tag{4.17}$$

meaning that u = v (see (4.10) and (4.13)), i.e.,

$$\omega R = 2H_{e\varphi}/H_{ez}. \tag{4.18}$$

The physical meaning of this equation becomes understandable if we rewrite it in the form

$$\frac{(B_\varphi)_{av}}{(B_z)_{av}} = \frac{H_{e\varphi}}{H_{ez}},$$

where

$$(B_\varphi)_{av} = \frac{1}{Rl}\int_0^l dz \int_0^R B_\varphi d\rho = \frac{\omega R}{2}\frac{1}{\pi R^2}\int_0^R B_z 2\pi\rho\, d\rho = \frac{\omega R}{2}(B_z)_{av}$$

Substituting the value $\omega R = 2H_{e\varphi}H_{ez}^{-1}$ in (4.6), we see that the tangential components of the vector H are discontinuous in this case on the surface of the sample: a current practically equal in magnitude and opposite in direction to the total current flows along the surface.

From (4.17) and the inequality (4.16) it follows that the equilibrium current states exist only so long as the parameter v is smaller than a certain definite value v_{max}. When $p \ll 1$ and $q \ll 1$ (i.e., $\alpha^{-1} \gg \Delta \gg 4\alpha^{-2}\sigma^{-1}$) $v_{max} = 2/3\sqrt{3}$, and consequently

$$(H_{e\varphi})_{max} = \frac{\alpha\Delta^{3/2}}{3\sqrt{3}} = \frac{\varkappa H_{c2}R}{3\lambda\sigma\sqrt{3}}\left(1 - \frac{H_{ez}}{H_{c2}}\right)^{3/2}. \tag{4.19}$$

In the region $1 \gg \Delta \gg \alpha^{-1}$ we have $q \gg 1$ and we get $v_{max} = 1/q\sqrt{2}$ and

$$(H_{e\varphi})_{max} = {}^1\!/_2\sqrt{H_{ez}(H_{c2} - H_{ez})}. \tag{4.20}$$

If $p \gg 1$ then, as can readily be concluded from (4.10), we always have $q^2 \ll 1$. In this case we get from (4.15) and (4.13)

$$v \approx 2\tau(1-\tau^2) \approx 2u. \tag{4.21}$$

Here $\omega R = H_{e\varphi}H_{ez}^{-1}$, so that the tangential components of H are continuous on the surface. Equilibrium is possible when $v \leq 4/3\sqrt{3}$, hence

$$(H_{e\varphi})_{max} = \frac{2\alpha\Delta^{3/2}}{3\sqrt{3}} = \frac{2\varkappa H_{c2}R}{3\lambda\sigma\sqrt{3}}\left(1 - \frac{H_{ez}}{H_{c2}}\right)^{3/2}. \tag{4.22}$$

This result was obtained earlier by Boyd[4] by another method.

If $H_{e\varphi} > (H_{e\varphi})_{max}$, there are no equilibrium current states in a cylinder made of an ideal superconductor of the second kind. It is easy to verify that inequalities (3.5) and (3.6), which we have used above, are well satisfied when $H_{e\varphi} < (H_{e\varphi})_{max}$.

5. DISCUSSION OF RESULTS

Thus, under macroscopic equilibrium, the induction distribution is determined by the equations

$$B_z = \frac{B}{H} H_z = \frac{B}{H} \frac{H_{ez} + \omega R H_{e\varphi}}{1 + \omega^2 \rho^2},$$
$$B_\varphi = \omega \rho B_z, \quad (5.1)$$
$$\frac{B}{H} = 1 + \frac{1}{\beta(2\varkappa^2 - 1)} - \frac{H_{c2}^* \sqrt{1 + \omega^2 \rho^2}}{\beta(2\varkappa^2 - 1)(H_{ez} + \omega R H_{e\varphi})}$$

which are obtained from (4.6) when relations (3.7) and (3.22) are taken into account. In formulas (5.1) it is necessary to substitute

$$\omega R = 2H_{e\varphi} H_{ez}^{-1} \text{ for } p \ll 1 \text{ and } \omega R = H_{e\varphi} H_{ez}^{-1} \text{ for } p \gg 1.$$

Inasmuch as in all cases $\tau \leq 1$ and $\Delta \ll 1$, we get

$$H_{c2}^* = H_{c2}(1 - \tau^2 \Delta) \approx H_{c2}.$$

Relations (5.1) are valid for a long ($l \gg 2\pi\omega^{-1}$) circular ($R \gg \lambda$) cylinder made of an extremely pure superconductor of the second kind placed in an external magnetic field H_{ez} close to H_{c2} ($\Delta \ll 1$) and carrying a transport current smaller than the critical value.

The magnetic field $H_{\varphi c}$ of the critical current on the surface of the cylinder cannot exceed the values $(H_{e\varphi})_{max}$ determined by formulas (4.19), (4.20), and (4.22). One cannot state, however, that at all values $H_{e\varphi} < (H_{e\varphi})_{max}$ the current states are stable. Indeed, if while varying the Gibbs potential (4.5) we assume that B_φ and B_z are independent parameters, we obtain

$$H_z = H_{ez}, \quad H_\varphi = RH_{e\varphi} / \rho. \quad (5.2)$$

With such a distribution of the magnetic field, the Gibbs potential would have an absolute minimum if the corresponding configuration of the vortex filaments could be in local equilibrium. The following picture of the phenomena occurring in the cylindrical superconductor in question when the current reaches the critical value is therefore very probable: Near the surface (in the region $R > \rho > r$) there occurs a nonequilibrium state in which $B_z \approx H_{ez}$, $B_\varphi \approx RH_{e\varphi}/\rho$. In the remaining region, the vortex filaments have, as before, equal pitches, so that formulas (5.1) are valid. Since, however, the entire current flows now in the region $0 < \rho < r$, the corresponding value of ω is

$$\omega(r) = 2RH_{e\varphi} / r^2 H_{ez}.$$

If such a realignment of the configuration of the vortex filaments does take place, then the separation boundary $\rho = r$ moves into the interior of the sample, since the Gibbs potential is decreased thereby. At a certain value of r the quantity $RH_{e\varphi}/r$ exceeds $(H_{e\varphi})_{max}$ and the sample goes over into the normal state.

A detailed analysis, which will be reported elsewhere, shows that such a twisting of the vortex filaments about the symmetry axis becomes possible when

$$H_{e\varphi} > H_{\varphi\kappa} = \sqrt{\gamma / \beta(2\varkappa^2 - 1)}\,(H_{c2} - H_{ez}), \quad (5.3)$$

where $\gamma \leq 1$. When $p < 1$ we have $H_{\varphi c} < (H_{e\varphi})_{max}$, as can be readily concluded from (4.19) and (4.20).

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Translated by J. G. Adashko
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