SOVIET PHYSICS JETP VOLUME 33, NUMBER 5 NOVEMBER, 1971

NATURE OF THE NONLINEAR CONCENTRATION DEPENDENCE OF THE RESISTANCE OF METALS CONTAINING IMPURITIES

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Submitted November 6, 1970

Zh. Eksp. Teor. Fiz. 60, 1832–1844 (May, 1971)

It is shown that the resistance of a metal depends significantly on the anisotropy of the nonequilibrium electron distribution function. Such anisotropy mainly arises as a result of umklapp processes in electron-phonon interaction, and also as a result of anisotropy of the phonon spectrum and exists in the case of a spherical Fermi surface. Allowance for the anisotropy of the distribution function, whose role becomes apparent on going beyond the limits of the usual single-moment approximation, may lead to a change (decrease) in the resistance by an order of magnitude compared with its standard value. It turns out that the introduction of impurities sharply reduces the effect of the anisotropy of the distribution function due to purely elastic scattering. The latter therefore begins to play an important role in the temperature-dependent part of the resistance $\rho(T)$ and this leads to a nonlinear dependence of $\rho(T)$ on the impurity concentration c at extremely small values of c for which the mutual effect of the impurities may certainly be neglected. The results permit us to explain the previously observed experimental anomalies in the behavior of the resistance of metals: a) the nonlinear character of the dependence on concentration of the impurity part $\Delta\rho(T, c)$ of the resistance of a metal in a wide temperature range; b) the strongly nonmonotonic character of the dependence of $\Delta\rho(T, c)$ on c and its large magnitude; c) the strong dependence on concentration of the coefficient of T^5 in the total resistance at extremely low impurity contents.

1. INTRODUCTION

THE temperature dependence of the resistance of metals containing nonmagnetic impurities was investigated in [1]. It was found that at a low concentration of impurity atoms there is an incoherent inelastic scattering of electrons by randomly distributed dynamical perturbation regions connected with the change in the character of the vibration of the impurity atom itself as well as the surrounding atoms of the matrix. Allowance for this scattering revealed a strong temperature dependence of the impurity part of the resistance $\Delta\rho(T)$ with a very distinctive mode of behavior in a wide range of temperature; in particular, the asymptotic behavior of the total resistance ρ for $T \to 0$ was changed.

The theory led to the linear dependence of $\Delta\rho$ on the concentration c, which seemed perfectly reasonable in view of the absence of any overlap between the perturbed regions at sufficiently low values of c. However, it was discovered in experiments[2,3] performed after this work that at very low values of c, when the c^2 terms are certainly negligibly small, the dependence $\Delta\rho(c)$ has a nonlinear character at intermediate temperatures. It was found at the same time that the residual resistance and the impurity part of the resistance at high temperatures linearly depend on the concentration. Similar results were obtained in [4,5]. It must be said that the nonlinear dependence of $\Delta\rho$ on c was observed in earlier investigations (see, for example, [6,7]), but no fundamental importance was attached to this since it could be said that the impurity concentration was not sufficiently low (c ~ 1%).

In a recently published work by Tsoĭ[8] in which he measured the resistance of pure samples of indium in a limited interval of very low temperatures, the Bloch law for total resistance $\rho = AT^5$ was found to hold, with A, however, showing a significant dependence on the impurity concentration (in the range of $c \sim 10^{-4}$–10^{-5} at.%). A similar effect had previously been discovered in tin,[9,10] cadmium and indium.[9] The manifestation by the coefficient before T^5 of concentration dependence was one of the results obtained in [1]. However, this dependence was linear and in the indicated range of very low concentrations it must have led only to a slight variation of A. Thus, we again return to the question of strong nonlinear concentration effects at extremely low c. Analysis of the nature of this nonlinearity and of the general character of the behavior of resistance as a function of T and c is the aim of the present paper.

In a perfect metal the distribution function $f_{\mathbf{k}}$ for electrons in an electric field together with a term of the same symmetry as the field term in the kinetic equation, in the general case, should contain anisotropic terms which reflect the symmetry of the lattice. It turns out that this anisotropy of the distribution function, as a rule, is strongly expressed even in the case of a spherical Fermi surface, which is connected with the scattering anisotropy in electron-phonon interaction, primarily as a result of umklapp processes (see below). Allowance for the anisotropy of the distribution function leads to a significant decrease in the resistance of a perfect metal as compared with the standard approximation for $f_{\mathbf{k}}$, which is practically equivalent to the approximation for isotropic space. We emphasize at once that we are here dealing with such strong effects as the change by several factors of the resistance in a definite temperature region (see below).

In the presence of impurities, elastic scattering on the impurities substantially suppresses the anisotropic part of the distribution function without its being generated back as a result of the nonequilibrium caused by the electric field. As a result, the scale of the anisotropy of $f_{\mathbf{k}}$ is determined by the competition between inelastic scattering on phonons and elastic scattering on impurities. Since at any small concentrations we can, by decreasing the temperature, make the two scattering mechanisms equal in order of magnitude, then this competition should always appear.

Such a distinctive "interference" between elastic and inelastic electron scattering clearly entails the appearance of strong nonlinear concentration effects, whose scale of influence on the resistance depends on the relative scale of the anisotropy of the function $f_{\mathbf{k}}$ in a perfect metal.

The manifestation of anisotropy by the distribution function may be seen only by going beyond the limits of the standard single-moment approximation in the solution of the kinetic equation. In this connection, we should mention that again Sondheimer and Wilson[11] (see also [4]) drew attention to the possibility of the appearance of a similar "interference" in scattering on taking the characteristics of the Fermi surface into account in the so-called two-zone approximation when just the two-moment approximation is used (see also the more general article [12]). However, a systematic quantitative estimate of the role of the anisotropy of the Fermi surface which, moreover, requires that we take at the same time the change in the electron wave function into consideration, has up till now not been done. On the face of it, it seems that in the transition metals where the model of quasi-free electrons works well, for quantities which are determined by integrals over the Fermi surface (as is the case for resistance), the distinctive features of the structure of the energy surface should exert less influence as compared with the strong effects connected with the scattering anisotropy and, in any case, will not lead to any qualitative results. In the present paper, the Fermi surface is everywhere assumed spherical and all the results are, thus, wholly dictated by the anisotropy of the distribution functions which is connected with the scattering anisotropy.

2. ANISOTROPIC SCATTERING OF ELECTRONS AND THE RESISTANCE OF A PERFECT METAL

In the absence of drag processes, the quasi-classical equation for electrons in metals has the form

$$\left(-\frac{\partial f_{\mathbf{k}}^{(0)}}{\partial \varepsilon_{\mathbf{k}}}\right) e\mathbf{v}_{\mathbf{k}}\mathbf{E} = \frac{1}{T}\int d\mathbf{k}' W_{\mathbf{k}\mathbf{k}'}^{(0)}(\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}'}) f_{\mathbf{k}}^{(0)} (1 - f_{\mathbf{k}'}^{(0)}), \tag{2.1}$$

where $\varphi_{\mathbf{k}}$ is a nonequilibrium correction to the electron distribution function $f_{\mathbf{k}}$:

$$f_{\mathbf{k}} = f_{\mathbf{k}}^{(0)} - \frac{\partial f_{\mathbf{k}}^{(0)}}{\partial \varepsilon_{\mathbf{k}}}\varphi_{\mathbf{k}},$$

$W_{\mathbf{k}\mathbf{k}'}^{(0)}$ is the probability of transition of an electron from the state with the wave vector **k** to the state with the wave vector **k′**.

Usually, $\varphi_{\mathbf{k}}$ is sought in the form

$$\varphi_{\mathbf{k}} = A(\varepsilon_{\mathbf{k}})(\mathbf{k}\varkappa), \quad \varkappa = \mathbf{E}/|\mathbf{E}|, \tag{2.2}$$

which is rigorous for an isotropic medium. In the single-moment approximation, A = const and we have the usual solution. Allowance for the expansion of $A(\epsilon_{\mathbf{k}})$ in powers of $\epsilon_{\mathbf{k}} - \epsilon_F$ in the framework of the variational procedure (or of the method of moments) does not yield any new qualitative results, while quantitatively, it only leads to small corrections not exceeding a few parts in a hundred (see, for example, [13,14]).

The symmetry of (2.2) corresponds to the symmetry of the field term in (2.1). For a spherical Fermi surface the collision operator does not change this symmetry if the transition probability $W_{\mathbf{k}\mathbf{k}'}^{(0)}$ depends only on the angle between **k** and **k′**. In that case, the solution strictly has the form (2.2). However, if $W_{\mathbf{k}\mathbf{k}'}^{(0)}$ depends on the absolute orientation of the vectors **k** and **k′**, then higher harmonics in **k** should appear in (2.2). Such a situation exists practically for all metals as a result of the anisotropy of the scattering of electrons on phonons, which is connected with umklapp processes and the anisotropy of the vibrational spectrum of metals. In this case, on the assumption that the Fermi surface is spherical, we have

$$\varphi_{\mathbf{k}} = \varphi_{\mathbf{k}\alpha}\varkappa_\alpha, \quad \varphi_{\mathbf{k}\alpha} = \sum_n A_{\alpha\alpha_1\ldots\alpha_n}^{(n)} k_{\alpha_1}\ldots k_{\alpha_n}. \tag{2.3}$$

Let us consider a crystal of cubic symmetry and let us choose a rigid system of coordinates along the 4-fold symmetry axes. Restricting ourselves to a simple approximation corresponding to the retention in (2.3) of the first two terms of the expansion (concerning the weak role of the corrections connected with the subsequent terms in the expansion, see below), we obtain

$$\varphi_{\mathbf{k}\alpha} = A^{(1)}\overset{(2)}{\varphi}_{\mathbf{k}\alpha} + A^{(2)}\overset{(2)}{\varphi}_{\mathbf{k}\alpha}, \tag{2.4}$$

$$\overset{(1)}{\varphi}_{\mathbf{k}\alpha} = k_\alpha, \quad \overset{(2)}{\varphi}_{\mathbf{k}\alpha} = k_\alpha{}^3 - {}^3/{}_5 k_\alpha k^2 \tag{2.5}$$

(in the selected form $\varphi^{(2)}$ is orthogonal to $\varphi^{(1)}$ with the weight function $\partial f_{\mathbf{k}}^{(0)}/\partial\epsilon_{\mathbf{k}}$).

The coefficients $A^{(i)}$ in (2.4) depend on k^2. However, this dependence, as in the case (2.2), has a very weak effect and we may limit ourselves to the zero-order terms in the expansion of $A^{(i)}$ in powers of k^2. For definiteness, we shall seek the solution in the form of (2.4); we then arrive at the following system of equations:

$$j = P_{11}^{(0)}A^{(1)} + P_{12}^{(0)}A^{(2)}, \quad 0 = P_{12}^{(0)}A^{(1)} + P_{22}^{(0)}A^{(2)}. \tag{2.6}$$

Here,

$$\begin{gathered} P_{ik}^{(0)} = \frac{1}{6T}\iint d\mathbf{k}\,d\mathbf{k}'(\overset{(i)}{\varphi}_{\mathbf{k}\alpha} - \overset{(i)}{\varphi}_{\mathbf{k}'\alpha})(\overset{(k)}{\varphi}_{\mathbf{k}\alpha} - \overset{(k)}{\varphi}_{\mathbf{k}'\alpha}) f_{\mathbf{k}}^{(0)}(1 - f_{\mathbf{k}'}^{(0)}) W_{\mathbf{k}\mathbf{k}'}^{(0)}, \\ j = \int d\mathbf{k}\, e\mathbf{v}_{\mathbf{k}}\overset{(1)}{\varphi}_{\mathbf{k}\alpha}\varkappa_\alpha. \end{gathered} \tag{2.7}$$

The index zero determines the quantities pertaining to metals with no defects. (In (2.6) we made use of the hermiticity of the collision operator.) Solving the system (2.6) and substituting (2.4) in the usual expression for current, we obtain directly for the resistance

$$\rho^{(0)} = \frac{P_{11}^{(0)}}{j^2}(1 - \eta), \quad \eta = \frac{P_{12}^{(0)\,2}}{P_{11}^{(0)}P_{22}^{(0)}}. \tag{2.8}$$

Let us analyze the temperature dependence of the quantity η. In the region of low temperatures when the umklapp processes are exponentially small, $P_{12}^{(0)}$ differs from zero only within the limits of the anisotropy of the long-wave phonons, i.e., within the limits of the anisotropy of sound in a metal. Then,

$$P_{11}^{(0)} = \gamma_{11}T^5, \quad P_{12}^{(0)} = \gamma_{12}T^5 \quad P_{22}^{(0)} = \gamma_{22}T^5 \tag{2.9}$$

and

$$\eta(T \to 0) = \text{const.} \tag{2.10}$$

If, for small q, the surfaces of constant frequency are nearly spherical, then γ_{12} is small and the constant in (2.10) is close to zero.

In the high temperature region $T \gtrsim \Theta_D$, we have $P_{ik} \sim T$ and again η assumes a constant value. This limiting value is, as a rule, small and this is connected with the smallness of the off-diagonal matrix elements of the collision operator, if the integral in (2.7) is taken over all phase space.

However, at intermediate temperatures, $\eta(T)$ may have values comparable with unity—which sharply changes the values of $\rho^{(0)}$. In order to make this assertion clear, we reproduce here results of the computation of the function $\eta(T)$ obtained for the simple cubic lattice model with a phonon spectrum corresponding to nearest neighbor interactions. We recall that this spectrum consists of three branches of equal frequency with polarization along the cubic axes and with the dispersion law

$$\omega_{\mathbf{q}}^2 = {}^1\!/_2\omega_0^2[1 - \xi\cos(q_x a_x) - {}^1\!/_2(1-\xi)(\cos(q_y a_y) + \cos(q_z a_z))] \quad (2.11)$$

when the polarization of the oscillations is along the X axis, and similarly for the other two polarizations (the equivalent Debye temperature $\Theta_D = 1.1\,\omega_0$). For the transition probability $W^{(0)}_{\mathbf{kk}'}$, we have used the expression given in [1], choosing the scattering amplitude of an electron by an ion $a_n^{(0)}(q)$ to correspond to a screened Coulomb interaction with $a^{(0)}(q = 2k_F)/a^{(0)}(q = 0) = 0.25$ (k_F is the radius of the Fermi sphere).

Figure 1 shows the curves $\eta(T)$, corresponding to $k_F/q_0 = 0.9$ and 1.45 ($q_0 = \pi b_{\min}$, where $b_{\min}$ is the nearest vector of the reciprocal lattice) and $\xi = \frac{1}{2}$ in (2.11). A few facts are at once conspicuous. The first one is the existence of a very large change in the magnitude of the resistance at relatively low temperatures as compared to the value obtained in the standard approximation corresponding to (2.2). Allowance for the anisotropy of the distribution, i.e., transition from (2.2) to (2.4), changes the value of the resistance $\rho^{(0)}$ severalfold in some temperature range.

The second peculiarity of these curves is the existence of a sharp asymmetric peak with the maximum at $T = T^* \ll \omega_0$ and this may have a significant effect on the form of the dependence $\rho^{(0)}(T)$, in particular, the range of T, where the Bloch law is obeyed, drastically narrows down.

Finally, we draw attention to the fact that the limiting values of η for $T \to 0$ and at large temperatures are, in fact, very small ($\eta(T = \Theta_D) = 0.022$ for $k_F/q_0 = 0.9$ and $\eta(T = \Theta_D) = 0.023$ for $k_F/q = 1.45$), especially if we take the values of η at the maximum into consideration.

We have selected for illustration curves which correspond to the parameters k_F/q_0 that are characteristic of real metals. All the calculations with other values of $k_F/q_0 > 0.9$ gave a qualitatively similar picture. A discrepancy arose only when this parameter was drastically decreased, when the weak role of the umklapp processes led to the decrease of the quantity η. Thus, when $k_F/q_0 = 0.6$, the maximum value of η did not exceed 0.02.

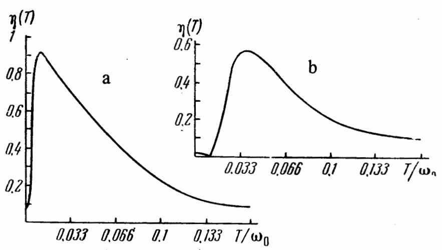

FIG. 1. η(T) curves for different anisotropies of the probability for the scattering of an electron: a—$k_F/q_0 = 1.45$, b—$k_F/q_0 = 0.9$.

In view of the importance of the obtained result, especially for the impurity problem (see the following section), it was important to verify that it was not connected with the form of the chosen approximation (2.4)–(2.5). The analysis, which was carried out, showed that all the results are decidedly connected with the allowance made for the anisotropy of the distribution function $\varphi_{\mathbf{k}}$ and that, qualitatively, these results change only slightly when the form of the approximate solution is changed. For the purpose of illustrating this assertion we show in Fig. 2 the $\eta(T)$ dependence (curve 2), obtained in the three-moment approximation, that corresponds to taking one more term in the expansion (2.3) into consideration:

$$\varphi^{(3)}_{\mathbf{k}\alpha} = k_\alpha(k_x^4 + k_y^4 + k_z^4 - {}^3\!/_5 k^4) \quad (2.12)$$

(for such a choice, $\varphi^{(3)}$ is orthogonal to $\varphi^{(1)}$); all the parameters of the problem were kept constant ($k_F/q_0 = 0.9$). For comparison, we show in the same figure the curve 1, which was constructed using the approximation (2.4) with the functions (2.5) (see Fig. 1) and also the curve 3, corresponding to (2.4) but with $\varphi^{(3)}$ substituted for $\varphi^{(2)}$. It can be seen that there is little difference—even purely quantitative—between the curves.

To demonstrate the relative role of the anisotropy of the phonon spectrum, we show in Fig. 3 two curves of $\eta(T)$ corresponding to $\xi = \frac{1}{2}$ and $\xi = \frac{1}{3}$ (isotropic sound) and $k_F/q_0 = 0.9$. These curves, as, incidentally, the results of all the other computations with $k_F/q_0 \gtrsim 0.9$, do attest to the fact that it is the umklapp processes that determine the shape of the sharp peak of the curve $\eta(T)$.

3. NONLINEAR CONCENTRATION DEPENDENCE OF THE RESISTANCE OF A METAL CONTAINING IMPURITIES

In the presence of defects, the electrons in a metal undergo additional scattering and the total transition probability $W_{\mathbf{kk}'}$, appearing in the collision integral (2.1), assumes the form

$$W_{\mathbf{kk}'} = W_{\mathbf{kk}'} + W^{(1)}_{\mathbf{kk}'}. \quad (3.1)$$

The expression for $W^{(1)}_{\mathbf{kk}'}$—the scattering probability on impurities in the regions of dynamical perturbation

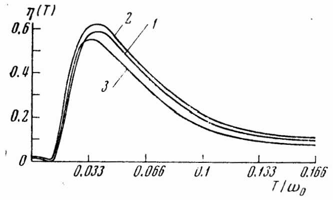

FIG. 2. η(T) curves for different trial functions (see the text).

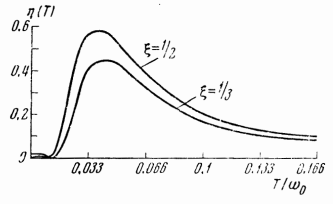

FIG. 3. η(T) curves for different phonon dispersion laws.

caused by these impurities—for a substitutional alloy has been obtained in [1] (for the case when the change in the force constants in the vibration problem is taken into consideration, see [15]).

We may still retain the expression for the resistance, introducing in place of $P_{ik}^{(0)}$, the value

$$P_{ik} = P_{ik}^{(0)} + P_{ik}^{(1)}, \tag{3.2}$$

where $P_{ik}^{(1)}$ is obtained from (2.7) by substituting $W^{(1)}$ for $W^{(0)}$. The quantity $P_{ik}^{(1)}$, like $W^{(1)}$, linearly depends on the impurity concentration c ($c \ll 1$).

Let us separate in $P_{ik}^{(1)}$ in an explicit form the elastic (e) part corresponding to static scattering on an impurity itself and the part corresponding to inelastic (i) scattering by phonons in the dynamical perturbation regions:

$$P_{ik}^{(1)}(T) = cR_{ik}^{(e)} + cR_{ik}^{(i)}(T). \tag{3.3}$$

In conformity with the fact that in elastic scattering on impurities, $W_{\mathbf{kk}'}^{(1)}$ depends only on the square of the transferred momentum $\mathbf{q} = \mathbf{k} - \mathbf{k}'$,

$$R_{ik}^{(e)} = 0, \quad i \neq k. \tag{3.4}$$

Let us substitute (3.2)–(3.4) in the expression (2.8) and take into consideration the fact that in the temperature range from 0 to $T \sim \Theta_D$, $R_{ik}^{(i)}(T) \ll R_{ik}^{(e)}$ almost always. Then, we have approximately

$$\rho \approx \frac{1}{j^2}\left[P_{11}^{(0)}(T) + cR_{11}^{(e)} + cR_{11}^{(i)}(T) - \frac{P_{12}^{(0)}(T)^2}{P_{22}^{(0)}(T) + cR_{22}^{(e)}}\right]. \tag{3.5}$$

Hence we find for the temperature dependent impurity part of the resistance $\Delta\rho_T$

$$\Delta\rho_T \equiv \rho(T) - \rho^{(0)}(T) - \rho(T=0) \approx \frac{1}{j^2}$$
$$\times\left[cR_{11}^{(i)}(T) + P_{11}^{(0)}(T)\,\eta(T)\frac{cR_{22}^{(e)}}{P_{22}^{(0)}(T) + cR_{22}^{(e)}}\right]. \tag{3.6}$$

Notice that to the customary single-moment approximation correspond the expressions (3.5) and (3.6) excluding the last terms in the square brackets (the corresponding values $R_{11}^{(i)}(T)$ and $R_{11}^{(2)}(T)$ were found in [1]).

Let us analyze the expressions obtained. In the region of extremely low temperatures, when

$$P_{22}^{(0)}(T) \ll cR_{22}^{(e)}, \tag{3.7}$$

we obtain by taking (2.9) into consideration

$$\rho \approx [P_{11}^{(0)}(T) + cR_{11}^{(e)} + cR_{11}^{(i)}(T)]/j^2. \tag{3.8}$$

The resistance and, in particular, the residual resistance

$$\rho_0 = cR_{11}^{(e)}/j^2 \tag{3.9}$$

retain, in this case, the linear dependence on concentration.

In the high temperature region when the opposite inequality

$$P_{22}^{(0)}(T) \gg cR_{22}^{(e)},$$

is valid, we find for $\Delta\rho_T$

$$\Delta\rho_T = \frac{c}{j^2}\left[R_{11}^{(i)}(T) + R_{22}^{(e)}\frac{P_{12}^{(0)}(T)^2}{P_{22}^{(0)}(T)^2}\right]. \tag{3.10}$$

Thus, in both limiting cases, allowance for the anisotropy of the distribution function and deviation from the single-moment approximation do not lead to the violation of the linear concentration dependence.

The picture becomes completely different when we consider the intermediate temperature range for which

$$P_{22}^{(0)}(T) \sim cR_{22}^{(e)}. \tag{3.11}$$

As can be seen from (3.5) and (3.6), there is in this case a strong nonlinear dependence of the resistance on concentration. The magnitude of the impurity resistance itself turns out to be large in this region:

$$\Delta\rho_T \sim \rho_0\eta, \tag{3.12}$$

i.e., if for temperatures corresponding to (3.11), $T \gtrsim T^*$ (T^* is the temperature corresponding to the maximum of the function $\eta(T)$), then $\Delta\rho_T$ and, consequently, the deviation from Matthiessen's rule may be comparable in magnitude with the residual resistance. It is precisely this circumstance that makes the second term in (3.6), in the temperature region under consideration, large in comparison with the first term which contains the small—with respect to the residual resistance—additional factor $\sim\langle u^2\rangle/a^2$, i.e., the ratio of the mean square displacement of the ions to the square of the interatomic distance.[1] (For small values of k_F/q_0, when η is small in the entire temperature range, the decisive role is played by the first term in (3.6) and $\Delta\rho_T$ should be appreciably smaller in magnitude than ρ_0.)

The cause of the appearance of the nonlinear dependence lies in the isotropization of the distribution function as a result of the elastic scattering of electrons by the impurities. It is significant that the latter does not lead only to a relaxation of any part of the distribution function that does not have the symmetry of (2.2). As a result, scattering on the impurities in a definite temperature range "eats up" the second term in (2.8) (the last term in (3.5)) which was due to the anisotropy of the distribution function and which significantly changed the resistance of a perfect metal (see the preceding section). Hence, the measure of the effects when impurities are introduced is large in magnitude.

Notice that the temperature range, where the nonlinear concentration effects are most vividly manifested, changes slightly with concentration. In fact, in conformity with (3.11) and (2.9), when $c \ll 1$ we have for the characteristic temperature $\tilde{T}$ ($P_{22}^{(0)}(\tilde{T}) = cR_{22}^{(e)}$) the law

$$\tilde{T} \propto c^{1/5}. \tag{3.13}$$

Because of this, the nonlinear concentration dependence can be observed in a wide range of low concentrations. The nonlinearity will disappear only at those extremely low concentrations for which $\tilde{T} \ll T^*$. Estimates show that, as a rule, this corresponds in multivalent metals to such extremely low values of the concentration as 10^{-7}–10^{-8}.

For relatively large values of c (but, naturally, $c \ll 1$) $\tilde{T}$ begins to much more strongly depend on the concentration ($P_{22}^{(0)}(T)$ exhibits a weaker temperature dependence) and relatively rapidly shifts with the growth of c to a region of higher temperatures where $\eta(T)$ sharply decreases. Therefore, the nonlinear effects are weak in "dirty" samples and, moreover, the scale of deviation from Matthiessen's rule (in units of ρ_0) is itself much smaller.

Let us consider now the general character of the temperature dependence. At sufficiently low temperatures, when (3.8) is true, the term quadratic in the temperature, which is due to the incoherent scattering of electrons by the vibrations of the impurity atoms,[1] predominates in the impurity part of the resistance, as a result of which

$$\rho \approx \rho_0 + \gamma_1 T^5 + \gamma_2 c T^2 \qquad (3.14)$$

(for the relation between γ_1 and γ_2, see [1]).

When T ~ T* the terms connected with the anisotropy of the distribution function begin to play a significant role. This is most vividly exhibited on the example of $\Delta\rho_T$. It is easy to deduce from (3.10) that at sufficiently low impurity concentrations the impurity resistance will have a sharp maximum, whose position shifts with concentration according to the law (3.13). The relation $\Delta\rho_T/\rho_0$ will then grow as the concentration is reduced.

The behavior of the resistance is described at high temperatures by the relation (3.10). One of the main results of [1] was the proof that $R_{11}^{(i)}$ linearly depends on T in the classical temperature region and that the sign of this quantity is determined by the sign of the difference between the scattering amplitudes of an electron on an impurity and on an ion of the matrix (in the simple approximation—by the sign of the difference between the valencies of the ions of the impurity and the matrix). As for the second term in (3.10), it does not, in general, depend on temperature. The magnitude of this constant is small, since in this limit

$$P_{12}^{(0)}(T)^2 / P_{22}^{(0)}(T)^2 \ll 1.$$

Hitherto, we have practically ignored in the discussion the question of the role of the rearrangement of the phonon spectrum on the introduction of impurity atoms, in particular, in the presence of quasilocalized levels (an explicit change in the phonon spectrum is taken into account in $R_{11}^{(i)}(T)$[1]). Such a rearrangement will play a decisive role in a wide range of temperature in those cases, when the umklapp processes are hindered and $\eta(T)$ is small in the entire temperature range. All the results presented in [1] will then be correct. In those cases, however, when the role of the umklapp processes is large, even at comparatively low temperatures the rearrangement of the phonon spectrum proves to be important at relatively high values of c, for which $\tilde{T}$ is appreciably higher than T*. At extremely low concentrations a specific behavior of $R_{11}^{(i)}(T)$ plays a significant role only when T ≪ T* and when T ≫ T*.

The described pattern of temperature and concentration behavior may be followed on the example of the dependence curves of $\Delta\rho_T/\rho_0$ on T for different values of c, shown in Figs. 4 and 5. The same model used in the preceding section was used here with $a^{(0)}(q = 2k_F) \times 1/a^{(0)}(q = 0) = 0.25$, $\xi = \frac{1}{3}$. The parameter k_F/q_0 was was equal to 0.9 (Fig. 4) and 1.45 (Fig. 5). The parameters, characterizing the variation from one lattice site to another of the mass M and charge Z, were varied in the following way (see [1]):

$$\frac{\Delta M}{M_0} = \frac{M' - M_0}{M_0} = 0.6, \quad \frac{\Delta a(0)}{a^{(0)}(0)} = \frac{Z' - Z_0}{Z_0} = \frac{\Delta Z}{Z_0} = +1, -1.$$

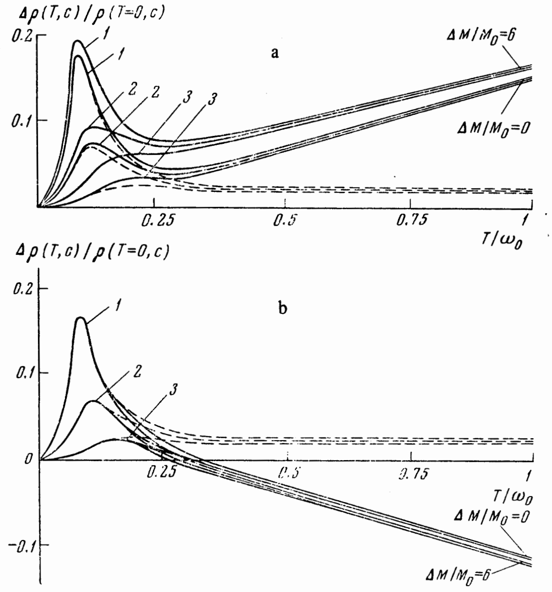

FIG. 4. Temperature dependence of the impurity part of the resistance for $k_F/q_0 = 0.9$, in the cases: a—$(Z'-Z_0)/Z_0 = 1$, b—$(Z'-Z_0)/Z_0 = -1$ and for different concentrations: 1—$c = 10^{-4}$, 2—$c = 10^{-3}$, 3—$c = 10^{-2}$.

Here, the prime denotes the quantities pertaining to the impurity (changes in the force constants of the vibration problem were neglected).

It can be seen from the figures that $\Delta\rho_T/\rho_0$ rises very sharply as the concentration is reduced, the value of $\Delta\rho_T$ at the maximum exceeding the residual resistanc when $k_F/q_0 = 1.45$ and $c \lesssim 10^{-4}$! Therefore, there is

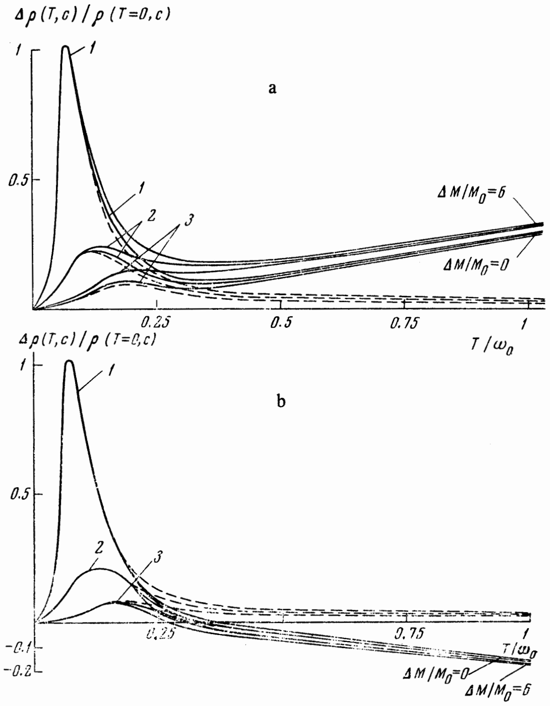

FIG. 5. Temperature dependence of the impurity part of the resistance for $k_F/q_0 = 1.45$ in the cases: a—$(Z'-Z_0)/Z_0 = 1$, b—$(Z'-Z_0)/Z_0 = -1$ and for different concentrations: 1—$c = 10^{-4}$, 2—$c = 10^{-3}$, 3—$c = 10^{-2}$.

direct evidence of large nonlinear effects and a most drastic deviation from Matthiessen's rule. It is interesting to note that at c ~ 0.01 all the effects are already very feebly marked and, consequently, this concentration, for the problem under consideration, is large.

The difference between curves corresponding to different values of the parameter $\Delta M/M_0$ in the region of the maximum of $\Delta\rho_T$, is also noteworthy. This difference is explained by the appearance in the phonon spectrum when $\Delta M/M_0 = 6$ of a quasilocal frequency $\omega_p \ll \omega_0$, scattering by which leads already at $T \gtrsim \omega_p/2$ to the dependence $R_{11}^{(i)} \sim T$,[1] whereas in the case when $\Delta M/M = 0$, $R_{11}^{(i)} \sim T^5$ in the region under consideration.

Further, the broken lines in the figures represent the values corresponding to the second term in (3.10). It can be seen that at high temperatures, the decisive contribution is made by $R_{11}^{(i)} \sim T$. Notice that the curves for different $\Delta M/M$ have in the classical temperature region one and the same slope, although they do not coincide with each other—which is connected with the allowance made in the residual resistance for the zero-point vibrations through the Debye-Waller factors (see [1]).

Let us return now to the general expression for the resistance (3.5). We should like to point out that circumstance that the presence of the last term in (3.5) can imitate, in a limited range of low temperatures, a temperature dependence which is similar to the Bloch law but with a coefficient that significantly depends on concentration. This is connected with the structure of this term and the characteristic behavior of $P_{12}^{(0)}$ as a function of T. In fact, it is possible to show (and this can be seen from the nature of the curve $\eta(T)$) that in a comparatively narrow range of temperature near T*, where $P_{11}^{(0)} \sim T^5$, $P_{12}^{(0)}(T)$ alters the temperature dependence from a stronger than T^5 to a weaker dependence. It is easy then to verify that at a sufficiently low concentration there is always a range of T in which the last term is proportional to T^5. As a result, $\rho - \rho_0 \approx A(c)T^5$ in this range, with A decreasing as the concentration decreases. We emphasize that this temperature range is bounded from above as well from below (at lower temperatures the same temperature dependence is manifested but with A = const).

The temperature dependence of $\rho - \rho_0$ is shown in Fig. 6, in which T^5 has been plotted along the abscissa. The parameters are the same as in the case of Fig. 4 with $\Delta M/M = 0$ and $\Delta Z/Z = 1$ and, for definiteness, the temperature was assumed to be $\Theta_D = 300$. It can be seen that in the region of temperatures $T > T^*$ we have a clear-cut linear dependence but that the slope changes appreciably in the range of such low concentrations as 10^{-3}–10^{-5}.

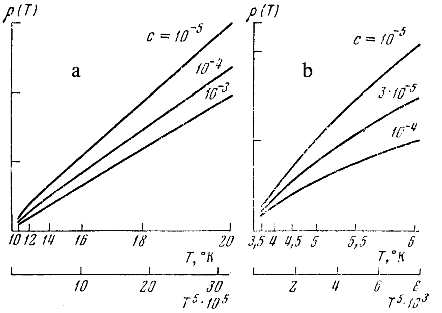

FIG. 6. Temperature dependence of the total resistance (in arbitrary units) in the cases: a—$k_F/q_0 = 0.9$, b—$k_F/q_0 = 1.45$ and for different impurity concentrations.

4. CONCLUDING REMARKS

The results obtained in the present paper for the dependence of the impurity part of the resistance $\Delta\rho_T$ on temperature and concentration are in good agreement with the experimental results which have been obtained in recent years from the study of the resistance[3-5] of alloys with very small impurity contents ($c \ll 10^{-2}$). We are dealing here with the coincidence of all the qualitative peculiarities of the curves (a dependence of the form $\rho \sim cT^2$ as $T \to 0$; nonlinear concentration dependence; the existence of a maximum of the function $\Delta\rho_T/c$, which increases in magnitude and shifts to the left as the impurity concentration is reduced; the nature of the temperature dependence at high T), as well as with a purely quantitative explanation of the scale of variation of $\Delta\rho_T$ which causes a most violent violation of Matthiessen's rule. We note that these results turn out to be stable with respect to variation of the parameters in a comparatively wide interval corresponding to the parameters of the multivalent metals. All this does not leave in doubt the fact that the experimental results pertaining to the intermediate temperature region are connected, to a decisive degree, with the isotropization of the electron distribution function as a result of elastic scattering on impurities, whereas the nature of the behavior of $\Delta\rho_T$ when $T \ll T^*$ and $T \gg T^*$ is determined by scattering by the dynamical disordering near the impurities.

The strong dependence of $\Delta\rho_T$ on concentration has long before been experimentally observed in a series of investigations[6,7] in which the resistance of comparatively "dirty" alloys was investigated. The relatively high concentration of the second constituent (~1%) left the field wide open for every kind of speculation about the nature of such a concentration dependence. However, in the light of the analysis carried out above, it is now clear that this nonlinearity is, in the main, connected with the same isotropization of the electron distribution function. It is precisely because of this that the ratio $\Delta\rho_T/\rho_0$ turned out in the experiment to be much smaller in the case of the "dirty" alloys than for the alloys with a lower concentration (see the curves for different c in Figs. 4 and 5).

Special attention must be paid to the behavior of $\Delta\rho_T$ in the region of high temperatures where this behavior is of a general character for "dirty" and "clean" alloys. It was found in [1] that the predicted theoretical linear temperature dependence and the connection of the sign of $\Delta\rho_T$ with the difference between the scattering amplitudes of electrons on the ions of the impurity and the matrix are in good agreement with all the then known experimental results beginning from the classical investigations of Linde (see the corresponding discussion in [1]). It must be said that recent precision measurements[2,3] have totally confirmed this interesting physical result.

As regards the total resistance, we must mention at once that the family of curves shown in Fig. 6 is qualitatively very similar to the curves obtained in Tsoĭ's work (as well as to the results of the earlier article[9] in which the discovery of a significant concentration dependence of the coefficient before T^5 in Bloch's law in the region of extremely low concentrations was reported). This is, primarily, the consequence of the above-described distinctive "interference" between inelastic scattering on phonons and elastic scattering on impurities.

Notice that the drastic change in the electron distribution function as a result of the anisotropy of scattering in a perfect metal (the role of $\eta(T)$) and the drastic decrease of its influence even at extremely low values of the impurity concentration are the reasons why it is so difficult to elucidate the Bloch law in metals or why the corresponding temperature range turns out to be so narrow. In fact, in a perfect metal this range corresponds to the temperature $T \ll T^* \ll \Theta_D$. At extremely low concentrations one more narrow range of temperature appears, where the Bloch law is valid (see Fig. 6), the law being violated in the gap between the two ranges. In "dirty" alloys the role of the term $cR_{11}^{(i)}(T)$ in (3.5) turns out to be extremely important and the temperature dependence is given by a varying power law with an exponent smaller than 5.

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Translated by A. K. Agyei
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