CONTRIBUTION TO THE THEORY OF PARAMETRIC EXCITATION OF SOUND IN FERROMAGNETS

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The parametric excitation of sound in a ferromagnet during ferromagnetic resonance (FMR) is considered. As a rule, the frequency of the excited sound Ω is much less than the FMR frequency ω, which, as shown in the paper, leads to the necessity of taking into account the oscillations $\omega_k \approx \omega + \Omega$ along with oscillations of the spin subsystem of eigenfrequencies $\omega_k \approx \omega - \Omega$. As a result, the pattern of the process may change fundamentally. In particular, a large number of spin resonances may be involved in the sound excitation process. The analysis includes the case of linear (if magnetostriction is neglected) spin subsystem and the case when spin-spin interactions occur concurrently with magnetoelastic interactions. It is shown that with the growth of spin-spin nonlinearity, the conditions for the excitation of low-frequency acoustic resonances ($\Omega \lesssim \gamma$, where γ is the spin oscillation relaxation frequency) change sharply. In the limit when $\Omega \ll \gamma$, magnetoacoustic instability may either be strongly suppressed or may appear before spin-spin instabilities even for very small values of magnetostriction and strong damping of the acoustic resonance. The effect of nonlinearity of the spin subsystem weakens with growth of Ω/γ. The characteristic dimensions of the effects are determined and the limits of applicability of various approximations considered in the literature are discussed. Another purpose of the investigation is to draw attention to a number of features of the parametric excitation of low-frequency oscillations in nonlinear dispersive media.

In the simplest scheme of investigation of decay processes that develop in a complex dynamical object like a solid body under the action of an intense external perturbation of frequency ω, only oscillations of frequencies close to ω and oscillations with combination frequencies, minimally necessary for decay to occur, are retained out of the totality of oscillations of the system. For example, in the case of a decay process of the lowest order, which will be the subject of discussion below, oscillations of frequencies ω, Ω, and $\omega - \Omega$ are retained. However, and attention is not always paid to this, we certainly should not restrict ourselves to these three oscillations in the presence of low-frequency instabilities ($\Omega \ll \omega$).

The point is that as a result of the nonlinearity, the small oscillations Ω generate in a high-frequency system the oscillations $\omega - \Omega$ and $\omega + \Omega$ of, generally speaking, commensurate amplitudes, since both of these harmonics are results of one and the same lowest-order nonlinear process. If the system in which the oscillations having the side frequencies—with respect to ω—are realized (we shall call it the system A) is sufficiently selective, then the optimum conditions for decay correspond to the conditions when the system is "tuned" to the frequency $\omega - \Omega$. The frequency $\omega + \Omega$ is then filtered out and the customary three-frequency approximation is valid. In the opposite case, in the analysis of transient processes and the thresholds of instabilities, it is necessary to take into consideration both harmonics since the frequency $\omega + \Omega$ also falls in the transmission band of the system A. The pattern of decay may then fundamentally change in the quantitative sense. In the particular case when the subsystem A is represented by a single mode of oscillations, the criterion for applicability of the various approximations is the parameter Ω/γ, where γ is of the order of the relaxation frequency of the mode A. If, on the other hand, as may be the case in a distributed medium, the oscillation Ω effectively interacts with a large aggregate of resonances with density $N(\omega_k)$ and with close lying eigenfrequencies ω_k, then the role of the criterion is played by the parameter $\Omega \partial N/\partial\omega$.

Before proceeding directly to the subject of the paper, the main point of which is the consideration of the indicated circumstances, let us note again that the necessity for taking both the components $\omega \pm \Omega$ into consideration follows regardless of the fact that the oscillations $\omega - \Omega$ are enhanced while the oscillations $\omega + \Omega$ are weakened in the course of development of a low-frequency instability. In general, beyond the threshold of parametric excitation, the degree of suppression of the various combination frequencies is determined by the selective properties as well as by the nature of nonlinearity of the system. Also important is the converse: it does not follow that because these or other harmonics are suppressed in a steady post-threshold regime, they may be neglected in the calculation of the instability threshold, since the onset of decay and the post-threshold behavior are determined by different nonlinear mechanisms.

It is known that as a result of the large magnetoelastic coupling constants and the high acoustic Q of the samples, the resonances of the spin system of a ferromagnet under conditions of an intense external variable field are often accompanied by the excitation of low-frequency sound. Since the damping of the sound usually increases with the frequency, the excitation of low-frequency sound is the most effective in a ferromagnet.

As a rule, to the resulting acoustic oscillation Ω corresponds one of the principal modes of elastic oscillations of the sample, so that $\Omega/2\pi \sim v/2L$, where v is the velocity of sound and L is a characteristic dimension of the sample. For $L \sim 1$ mm the frequency of the sound is a few MHz, i.e., of the same order as the spin oscillation relaxation frequency γ for the best samples. Depending on the dimensions and quality of the sample, a considerable difference, in either direction, between Ω and γ is possible. A resonance directly excited by the field lies, depending on the shape of the sample and the conditions of the experiment, somewhere in the magnetostatic spectrum band $\Pi \sim 10^2$ MHz. The precession modes, effectively interacting with the low-frequency sound, generally speaking, cover the whole band Π, and their density may be very large. Nevertheless, in almost the entire extensive literature devoted to the parametric excitation of sound (we mention, for example, the earliest papers,[1,2] the reviews[3] and a number of the most recent ones,[4,5]), the analysis of the problem and the comparison with experiment were practically carried out on the basis of the model with three participating oscillations: ω, Ω, and $\omega-\Omega$. The only exception, apparently, are the papers [6,7] (we shall discuss the connection between these papers and the present paper later).

Depending on the magnitude of the magnetizing field and other conditions of the experiment, other nonlinear interactions may arise along with magnetoelastic interactions. We shall first (see Part I) dwell at length on the simplest situation when the amplitudes of oscillation of the intensity of magnetization at FMR, which correspond to the sound excitation threshold, are small in the sense that spin-spin and other nonlinearities are not important. Analysis of the problem under conditions when spin-spin nonlinearities are important is presented in Part II.

I. LINEAR SPIN SUBSYSTEM

1. Formulation of the Problem

We shall assume that for the description of nonlinear magnetoelastic phenomena, the model of a ferromagnet as an ensemble of interacting magnetic and elastic oscillations described by the Hamiltonian:

$$\mathscr{H} = \sum_i \omega_i c_i^* c_i + \sum_\nu \Omega_\alpha d_\nu^* d_\nu + \Big\{ \sum_{i,k,\nu} \Psi_1^{ik\nu} c_i^* c_k d_\nu + \text{c.c.} \Big\} + \mathscr{H}_1 \quad (1)$$

is applicable. Here $c_i(t)$ and $c_i^*(t)$ are canonically conjugate variables, so that their product $c_i^* c_i$ determines, according to (1), the strength of small spin oscillations; the mode i is characterized by its spatial shape $\mathbf{f}_i(\mathbf{r})$ and its eigenfrequency ω_i, which is determined from the solution of the boundary-value problem. Similarly d_ν, Ω_ν, and $\mathbf{F}_\nu(\mathbf{r})$ characterize the corresponding mode of elastic oscillations of the sample. The braces enclose the first nonvanishing nonlinear term of the magnetoelastic interaction (linear interaction with an elastic system, an electromagnetic field and others, as is well-known, may, in principle, be taken into account by a judicious choice of the normal oscillations). Other nonlinear interactions (they are contained in $\mathscr{H}_1$) are not taken into consideration in this section. The coefficients Ψ_1 are proportional to the integrals of the combinations of the corresponding eigenfunctions $\mathbf{f}_i(\mathbf{r})$, $\mathbf{f}_k(\mathbf{r})$, and $\mathbf{F}_\nu(\mathbf{r})$ taken over the volume of the sample. It is not, in the general case, possible—and hardly ever useful—to evaluate these integrals. In the case of short-wave excitations in the system, when $\mathbf{f}_i \sim \exp\{j\mathbf{k}_i\cdot\mathbf{r}\}$, $\mathbf{F}_\nu \sim \exp\{j\mathbf{k}_\nu\cdot\mathbf{r}\}$ (j is the imaginary unit), the coefficients $\Psi_1^{ik\nu}$ differ from zero when $\mathbf{k}_i = \mathbf{k}_k + \mathbf{k}_\nu$ and the order of their magnitudes for a ferromagnet with cubic symmetry is given, for example, in [8]. Notice that Ψ_1 may be determined directly from experiments on nonlinear FMR. For example, by measuring the correction $\Delta\omega_i$ to the resonance frequency ω_i for the deformation d_ν, we may determine $\Psi_1^{ii\nu}$, since according to (1), $\Delta\omega_i = \Psi_1^{ii\nu} d^\nu$.

Let an FMR "transverse" pumping regime be realized; the alternating field of frequency ω excites one or at once several magnetic resonances of eigenfrequencies close to ω. From (1), bearing in mind the introduction of dissipative terms, we obtain equations of motion which describe the interaction of the sound d with the reservoir of spin oscillations:

$$j(d/dt+\gamma_k)c_k = \omega_k c_k + \sum_i \Psi_1^{ik} c_i (d+d^*) + h_k e^{-j\omega t} \quad k=0,1,2,\ldots,$$
$$j(d/dt+\Gamma)d = \Omega d + \sum_{i,k} \Psi_1^{ik} c_i^* c_k. \quad (2)$$

Here γ_k and Γ are spin-oscillation and sound relaxation frequencies. The coefficients h_k, which determine the degree of excitation of the mode k by the field $\mathbf{h}(\mathbf{r})\exp -j\omega t$, are proportional to the overlap integrals $\int \mathbf{f}_k(\mathbf{r})\mathbf{h}(\mathbf{r})d\mathbf{r}$ taken over the volume of the sample. It follows from (2) that in the linear regime ($d \equiv 0$) of FMR, the power W_k absorbed by the oscillations k at resonance, $\omega = \omega_k$, is related to h_k through the formula $h_k^* h_k = \gamma_k W_k / 2\omega_k$. As a rule, under given conditions the excitation threshold for one of the sound vibrations is minimal. The influence of other elastic oscillations is insignificant if their eigenfrequencies Ω_ν are sufficiently spread out. Here and henceforth, an elastic subsystem is represented by one oscillation d, with the index ν omitted.

For $\Omega \ll \omega$ a multimode system, similar to (2) but of different physical nature, has previously been investigated by one of the authors,[9] and we shall be guided by the results obtained there.

2. Conditions for Stability

In the absence of sound vibrations (the time derivative $\dot{d} \equiv 0$) those magnetic resonances k for which $h_k \neq 0$ and having amplitudes[1)]

$$a_k = h_k / (\omega - \omega_k + j\gamma_k) \quad (3)$$

[1)]Notice that when $\dot{d} \equiv 0$, the stationary solution of the system (2) contains a nonvanishing static deformation

$$d_0 = -\frac{1}{\Omega}\sum_{i,k} \Psi_1^{ik} \langle c_i^* c_k \rangle,$$

where $\langle \ldots \rangle$ denotes time averaging. The presence of d_0, in its turn, leads to a FMR anharmonicity, analogous to spin-spin interactions (see Part II). Usually, the excitation of sound begins considerably earlier than the appearance of the indicated nonlinearity, the criterion for its insignificance being (see [9]) the smallness of the quantity Γ/Ω (or for $\Omega \lesssim \gamma$, the quantity $\Gamma\gamma/\Omega^2$).

are excited in the steady regime. With the growth of the amplitude of the alternating field, the forced periodic regime (3) becomes unstable with respect to the excitation of sound.

The presence in the system of the oscillations d of frequency Ω is accompanied by the appearance of spin oscillations with frequencies $\omega \pm n\Omega$, $n = 1, 2, \ldots$, on both sides of ω. These oscillations, in turn, act back on the oscillation d. The work done on the oscillations d at combination frequencies smaller than ω is positive, while at frequencies greater than ω it is negative. It follows from the system (2) that close to the steady regime (3) and for small oscillations d, the negative damping introduced into the elastic subsystem, in the first order of smallness d, is equal to

$$\Gamma_1 = \sum_{i,k,l} \Psi_1^{ik} \Psi_1^{kl} a_i^* a_l [\chi_k''(\omega - \Omega) - \chi_k''(\omega + \Omega)], \qquad (4)$$

where $\chi_k''(\nu) = \gamma_k/[(\nu - \omega_k)^2 + \gamma_k^2]$ is the imaginary part of the susceptibility of the resonance k with respect to the field h_k at the frequency ν. The difference $\chi_k''(\omega - \Omega) - \chi_k''(\omega + \Omega)$ is positive when $\omega > \omega_k$ and negative when $\omega < \omega_k$. When $\Gamma_1 \geq \Gamma$, the regime (3) becomes unstable and sound is excited. In the particular case when the external field excites one resonance (for definiteness, let this be the oscillations of the uniform precession c_0; the excitation of some other oscillation c_k does not affect the analysis in any way), we obtain from (4) the following condition for the threshold of magnetoelastic instability:

$$a_0^* a_0 \geq \Gamma \left\{ \sum_k |\Psi_1^{0k}|^2 [\chi_k''(\omega - \Omega) - \chi_k''(\omega + \Omega)] \right\}^{-1}. \qquad (5)$$

Depending on the relation between the values of the spin oscillation damping γ_k, and also between the coefficients Ψ_1 of the magnetoelastic coupling with the given oscillation d, some terms or other will be the important ones in the sums (4) and (5). In the simplest case, the oscillations with the side-frequencies—with respect to ω—accompanying the sound are realized only in one of the spin oscillations (in particular, the resonance directly excited by the field may play the role of the oscillations k). The remaining spin reservoir is not excited. This is precisely the idealized formulation in which the problem has been considered in the literature up to the present. Let us analyze here the stability conditions which follow from (5) in this approximation. From this will follow, as limiting cases, already considered models. Retaining in the sum in (5) one vibration k, we obtain for the threshold

$$a_0^* a_0 \geq \frac{\Gamma}{|\Psi_1^{0k}|^2} [\chi_k''(\omega - \Omega) - \chi_k''(\omega + \Omega)]^{-1}. \qquad (6)$$

Depending on the relation between Ω and γ_k, we may here distinguish several cases.

If $\Omega \gg \gamma_k$, then the expression in the square brackets in (6) is a maximum when the oscillation k is tuned to the resonance $\omega_k = \omega - \Omega$, when the term $\chi_k''(\omega + \Omega)$ is small. Close to the optimal tuning, and neglecting in (6) the term $\chi_k''(\omega + \Omega)$, we obtain for the threshold amplitude

$$a_0^* a_0 \geq P_k [1 + (x_k - \Omega/\gamma_k)^2]^2 \qquad (7)$$

and for the threshold field

$$h_0^* h_0 \geq P_k \gamma_0^2 (1 + x_0^2) [1 + (x_k - \Omega/\gamma_k)^2]^2, \qquad (8)$$

where

$$P_k = \frac{\Gamma \gamma_k}{|\Psi_1^{0k}|^2}, \quad x_0 = \frac{\omega - \omega_0}{\gamma_0}, \quad x_k = \frac{\omega - \omega_k}{\gamma_k}.$$

If $\Omega \lesssim \gamma_k$, then it is necessary to take into account in (6) both $\chi_k''(\omega - \Omega)$ and $\chi_k''(\omega + \Omega)$, and for $\Omega \ll \gamma_k$ the thresholds are respectively equal to

$$a_0^* a_0 \geq P_k \frac{\gamma_k}{\Omega} \frac{(1 + x_k^2)^2 + 4\Omega^2/\gamma_k^2}{4x_k}, \qquad (9)$$

$$h_0^* h_0 \geq P_k \frac{\gamma_k}{\Omega} \gamma_0^2 (1 + x_0^2) \frac{(1 + x_k^2)^2 + 4\Omega^2/\gamma_k^2}{4x_k}. \qquad (10)$$

The right hand sides of (9) and (10) have their minimum values on the slope $x_k > 0$ in the region of greatest slope $\chi_k''(\nu)$; when $x_k < 0$ instability is impossible. The FMR frequency detuning x_0 may be of either sign regardless of the relation between Ω and γ, and the optimum threshold corresponds to $x_0 = 0$.

If the most important coupling in the system is the direct coupling of sound with the uniform precession (Ψ_1^{00} is large), then the sound vibrations may be accompanied by a modulation of the oscillations of the uniform precession without the participation of other spin modes. The given expressions remain valid in this case if we set in (7)–(10) $x_k = x_0$, $\gamma_k = \gamma_0$, and $\Psi_1^{0k} = \Psi_1^{00}$. In this simplest of all cases, only one tuning parameter x_0 enters into the instability condition, instability being possible only when the FMR frequency detuning $x_0 > 0$. The optimum thresholds then turn out to be equal to

$$h_0^* h_0 = P_0 \gamma_0^2 (1 + \Omega^2/\gamma_0^2) \text{ when } \Omega \gg \gamma_0, \qquad (8a)$$

$$h_0^* h_0 = P_0 \gamma_0^2 (\gamma_0/\Omega) \text{ when } \Omega \ll \gamma_0, \qquad (10a)$$

from which it can be seen that as the sound frequency Ω changes, the other conditions remaining unchanged, the threshold assumes its minimum value at $\Omega \sim \gamma_0$.[2)]

Beginning from [1], in which the excitation of sound at FMR in a disk, with the magnetizing field oriented normal to the plane of the disk, was investigated, almost all the subsequent papers dealing with the analysis of nonlinear magnetoelastic phenomena took into consideration only three oscillations: $\omega \approx \omega_0$, Ω, $\omega_k \approx \omega - \Omega$. As has been shown above, this corresponds to the limiting case $\Omega \gg \gamma$. Another limiting case has been considered in [6], using as an example the excitation of sound directly by the uniform precession in a disk which has been magnetized in a direction parallel to its plane; this example was analyzed in [7] for an arbitrary relation between Ω and γ. Except for the notation, the expressions obtained in [1,6,7] for the thresholds go over into the expressions given here. In light of the analysis carried out here, the unified aspect of the above-mentioned papers as different limiting cases of a simple model becomes clear.

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2)This may be understood from the following considerations. For $\Omega \ll \omega$, the deformation d assumes the role of a slow parameter modulating the frequency ω_0. If $\Omega \gg \gamma_0$, then an almost adiabatic passage through the magnetic resonance occurs and practically no energy enters the slow system. For $\Omega \ll \gamma_0$, the magnetic resonance is passed through quasistatically, the retardation of the magnetostriction forces is small, and the increment Γ introduced is small.

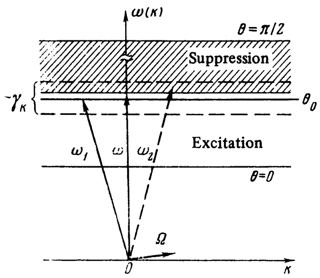

FIG. 1. Conditions for the parametric excitation of sound.

3. Discussion

Let us try and examine to what extent the various approximations can be realized in experiments. In particular, let us discuss the applicability of the simplest model when the spin reservoir is represented by only two (or one) magnetic resonances. Let us first consider the pattern of sound excitation in sufficiently extended ferromagnets, assuming that the plane-wave approximation works. The latter permits the use of the customary[2,3] dispersion diagram in the (ω, k) plane. In Fig. 1 the solid arrows represent the process of decay of the oscillation $\omega(k = 0)$ into sound $\Omega(k_1)$ and the spin oscillation $\omega(-k_1)$. The lines $\theta = 0$ and $\theta = \pi/2$ represent the limits of the spectrum of the spin waves, θ being the angle of propagation of the waves. Because of the law of conservation of momentum, an acoustic wave propagating in the direction $-\theta_0$ interacts only with spin oscillations that correspond to only one dispersion spin branch θ_0. Interacting with the sound, the spectrum of spin oscillations $\omega_{\theta_0}(k) < \omega$ introduces, according to (4), a negative damping into the elastic subsystem while the entire spectrum from $\omega_{\theta_0}(k) > \omega$ introduces a positive damping. Depending, therefore, on the relation between ω and $\omega_{\theta_0}(-k_1)$, sound is either excited ($\omega > \omega_{\theta_0}(-k_1)$) or is suppressed ($\omega < \omega_{\theta_0}(-k_1)$).[3)] This is the state of affairs if $\Omega \gg \gamma_k$. Allowance for the finite lifetime of the magnons leads to a broadening of the dispersion branch $\omega_{\theta_0}(k)$ (this is arbitrarily represented in Fig. 1 by the dashed lines). And if $\Omega \lesssim \gamma_k$, then simultaneously with the process of decay of the oscillation ω into the phonon Ω and the magnon $\omega_1 = \omega - \Omega$ there effectively proceeds the process of fusion of the phonon Ω and the magnon ω into the magnon $\omega_2 = \omega + \Omega$—a process which is taken into account here (in Fig. 1 it is represented by the dashed arrow).

As it is converted into excitations of longer wavelength, the sound propagating in the direction θ_0 begins to effectively interact, by virtue of the finite dimensions of the sample, with some band of the dispersion branch $\omega(\mathbf{k})$ near θ_0. And, in the limit, the possibility of interaction of the sound with the entire band of spin oscillations arises. In that case ($\mathbf{k}$ is small) the spectrum is discrete, but when it is not too highly discrete, many spin resonances are nonlinearly coupled with the sound, i.e., the parameters Ψ_1 for many oscillations in the entire band of the spectrum are of the same order. Indeed, as a result of the sharp selectivity in the parameters $\chi''(\omega - \Omega)$ and $\chi''(\omega + \Omega)$, only oscillations lying on bands of width $\sim\gamma$ near the frequencies $\omega \pm \Omega$ are selected from the whole ensemble of resonances. Depending on the density of distribution of the resonances ω_k on these bands, we have either excitation or suppression of sound. In particular, at FMR in a disk magnetized in the direction perpendicular to its plane (precisely such a problem has been considered in [1]), the frequency of the uniform precession lies at the bottom of the spectrum of spin oscillations and the conditions for the excitation of sound are extremely difficult. This, apparently, is the cause of the unexpected failure of the experiments, described in [10], on the excitation of sound in a disk in a resonant magnetizing field normal to the plane of the disk. Sound should be easiest to excite in a disk magnetized in a direction parallel to its plane for then the frequency of the uniform precession lies near the upper limit of the magnetostatic section of the spectrum.

It is clear from the foregoing that calculations of the thresholds and the regions of magnetoelastic instabilities (and even more so, of transient processes and post-threshold behavior) in the framework of the customary model in which only three oscillations: $\omega, \Omega, \omega_k \approx \omega - \Omega$ are taken into consideration, are, generally speaking, of limited applicability. Such a model may be realized in a sample of small dimensions when, as compared with the relaxation frequencies, the γ-spectrum of the spin oscillations is highly discrete, or in an extended ferromagnet, when for some reason the excitation of low-frequency sound is difficult and a comparatively high-frequency sound ($\Omega \gg \gamma$) appears.

II. NONLINEAR SPIN SUBSYSTEM

Experiments on the study of highly excited states of a spin system, to which ever-increasing attention is being paid in recent years, have shown that different nonlinear regimes of FMR under conditions of "transverse" as well as "longitudinal" pumping, as a rule, are accompanied by sound excitation (see, for example, [11,12]). Apart from the obvious circumstances, connected directly with the growth of the amplitudes of the spin oscillations, this may be favored by a number of distinctive features arising as a result of the nonlinearity of the spin subsystem. In particular, even without taking into consideration the magnetoelastic interaction, a low-frequency instability of the FMR—automodulation[12]—may arise as a result of the nonlinearity; the periodic magnetostriction forces that appear here may, as discussed in [12], give rise to intense elastic oscillations of the ferromagnet at the automodulation frequency. Along with such a nonparametric mechanism for the excitation of sound in nonlinear FMR, we may have the usual magnetoacoustic resonance (MAR) with its own threshold. For example, in the experiments of [10], the regions of automodulation and MAR do not overlap, although the thresholds of both instabilities are close.

We wish here to draw attention to another peculiarity of the magnetoelastic interaction under conditions of a

3)Notice that allowance for the electromagnetic field leads to a deformation of the dispersion law $\omega_{\theta_0}(k)$ at small k. As a result, both the oscillations $\omega-\Omega$ and $\omega+\Omega$ in this branch can simultaneously effectively interact with the acoustic wave. This circumstance should lead to an appreciable selectivity in the excitation (or amplification) of sound.

highly nonlinear spin system. This peculiarity may be understood from the following considerations (they have been previously advanced in [7,9]). Let us consider the stationary regime of high-frequency oscillations of the magnetization at FMR in the absence of automodulation. If a slow deformation of the sample d arises close to the position of equilibrium, then this leads to a change in the FMR regime, and, as a result, the squares of the magnetostriction forces, averaged over the spin oscillations F(d), change. The derivative $\partial F/\partial d$ has the meaning of rigidity introduced by the magnetoelastic interaction into the acoustic system and is, depending on the FMR detuning, positive or negative. In response to the deformation d(t), the spin oscillations are established not at once but over some interval of time τ— which causes a retardation of the forces $F = F[(d - \tau)]$. As a result, the force F does nonvanishing work on the elastic system during a periodic deformation of the sample. At a sufficiently low frequency of the sound Ω ($\Omega\tau \ll 1$), a damping $\tau\partial F/\partial d$ of either sign is introduced into the acoustic system as a result of the retardation. When the changes introduced into the attenuation and rigidity of the mechanical oscillations become commensurate with their initial values, magnetoelastic instabilities appear in the system. They are most likely to be found at the points where $\partial F/\partial d$ is large, and this is realized when small changes in the parameters lead to a sharp change in the stationary FMR regime. The latter is, in fact, unique for a highly excited spin system. In particular, the dependence of the eigenfrequencies of spin resonances on the amplitude leads to such a deformation of the resonance curves that the appearance on them of very steep sections becomes possible. At points close to a vertical slope we get $\partial F/\partial d \to \infty$ and we can expect sound to be excited with extremely small magnetoelastic couplings. Correspondingly, the regions of magnetoelastic instability may be very narrow.

The effectiveness of such a kind of nonlinear mechanism of the intensification of the interaction between fast and slow oscillatory subsystems is well illustrated by the example of the magnetostrictive instability of a magnetic-tape parametron.[13] The parametrically excited oscillations of the intensity of magnetization in the thin ($\sim 10^{-4}$ mm) film were accompanied by the excitation of intense flexural oscillations of the thick (~ 1 mm) backing on which the film had been deposited; the region of sound excitation turned out to be much narrower than the region of parametric oscillations.

It is significant that the considered mechanism of the intensification of the magnetoelastic interaction is effective only for low-frequency acoustic resonances. This is connected with the fact that for $\Omega \gg 1/\tau$ the state of the spin system does not have time to change appreciably and, in such a near-adiabatic regime, practically no energy enters the slow acoustic system in a period $2\pi/\Omega$. The characteristic time for the establishment of spin oscillations τ in the nonlinear FMR regime depends, generally speaking, on the pumping power. As a rough measure of the retardation we may take the relaxation frequency γ of the spin oscillations.

In view of the complexity of the problem, the theory of magnetoelastic phenomena at FMR, under conditions when other nonlinear interactions are present at the same time, has scarcely been worked out. Apart from the papers already cited, we mention again [4,5]. In [4] the case of the interaction of a parametrically excited spin-wave resonance with sound is considered, but the nonlinearity of the spin system which, in fact, determines the steady state of the parametric resonance is completely not taken into account and the problem is practically solved in the framework of the applicability of the customary model,[1] when the spin subsystem is linear. As far as we know, the first attempt to take into consideration the effect exerted on the parametric excitation of sound by spin-spin interaction leading to the amplitude dependence of the resonance frequencies of the oscillations of the magnetization was made in [5]. However, the specific character of the excitation of low-frequency sound was not taken into consideration in the analysis in [5] and hence the mechanism of the strengthening of the magnetoelastic coupling was missed missed.

1. Formulation of the Problem

We shall describe nonlinearly interacting spin and elastic oscillations of a ferromagnet by the Hamiltonian (1) where, as in [5],

$$\mathscr{H}_1 = \sum_{i,k,l,m} \Psi_2^{iklm} c_i^* c_k^* c_l c_m + \text{c.c.} \tag{1a}$$

(for details about the coefficients Ψ_2 determining the spin-spin interactions, see [5]). Let, as in Part I, a regime of "transverse" FMR pumping be realized and let the alternating field excite only one resonance c_0. We consider the case when the frequency $\omega/2$ lies below the bottom of the spin oscillation spectrum, so that conditions ar difficult for the parametric excitation of the spin resonances $\omega_k \approx \omega/2$,[14] for which the terms in the Hamiltonian which are cubic in c are responsible. These terms are therefore dropped, and their presence is reevaluated as corrections to the coefficients Ψ_2.

With the interaction (1a) taken into consideration, the behavior of the system is described by the equations

$$j(d/dt + \gamma_0)c_0 = \omega_0 c_0 + h_0 e^{-j\omega t} + \Psi_1^{00} c_0 (d + d^*) + 4\Psi_2^{00} c_0^* c_0^2, \tag{11}$$

$$j(d/dt + \gamma_k)c_k = \omega_k c_k + \Psi_1^{0k} c_0 (d + d^*) + 8\Psi_2^{0k} c_0^* c_0 c_k + 4\Psi_2^{0k} c_k^* c_0^2 \quad (k = 1, 2, 3, \ldots), \tag{12}$$

$$j(d/dt + \Gamma)d = \Omega d + \Psi_1^{00} c_0^* c_0 + \sum_k \Psi_1^{0k} (c_0^* c_k + c_0 c_k^*), \tag{13}$$

where $\Psi_2^{00} = \Psi_2^{00,00}$, $\Psi_2^{0k} = \Psi_2^{0k,0k} \equiv \Psi_2^{00,kk}$. In these equations only terms linear in the amplitudes c_k are retained since we intend to consider only the process of the development of instability and not post-threshold behavior.

The main difference between the above equations and those considered in [5] lies in the fact that we have retained terms which are important when there is coupling with low-frequency sound:

1) The coupling of sound with the resonance c_0 (the term $\sim \Psi_1^{00}$) is taken into account, since this coupling is important even for a uniform precession in a sample of finite dimensions.

2) Coupling with both the oscillations d and d* is taken into account in (11) and (12).

3) We retain in Eq. (12) the term $\sim c_k^* c_0^2$ which, generally speaking, is responsible for the parametric

excitation of the resonance k with frequencies $\omega_k \approx \omega$,[14] and, in our case, is just the term that leads to the strengthening of the magnetoelastic coupling.

The regions of applicability of the results obtained in [1, 5, 14] and also of the results in Part I, will be seen from the analysis.

2. Conditions for Stability

In the absence of elastic oscillations of the sample ($(d/dt)d \equiv 0$), the usual expression for the resonance curve of the nonlinear FMR follows from (11)–(13):

$$|a_0|^2 = \frac{|h_0|^2}{\gamma_0^2[1+(x_0-b_0)^2]^2} \qquad b_0 = \frac{4\Psi_2^{00}+\varepsilon_0}{\nu}|a_0|^2. \tag{14}$$

The correction

$$\varepsilon_0 = -\sum_\nu \frac{2}{\Omega_\nu}|\Psi_1^{00\nu}|^2$$

is due to the static deformation of the sample at FMR; the main contribution to the sum over ν may not necessarily be made by the chosen mode d. For the frequencies of the resonances k we have similar nonlinear corrections

$$\varepsilon_k = -\sum_\nu \frac{2}{\Omega_\nu}|\Psi_1^{0k\nu}|^2.$$

In order not to complicate the analysis and redesignate the coefficients Ψ_2, we shall neglect the corrections ϵ_0 and ϵ_k. For high-Q acoustic resonances ($\Gamma\gamma/\Omega^2 \ll 1$) the presence of ϵ_0 and ϵ_k does not affect the quantitative results.

Equation (14) determines the amplitude a_0 implicitly. As the field h_0 grows, the resonance FMR curve deforms and, beginning from some value of the field, becomes nonunique and a hysteresis develops on its passage. It follows from Eqs. (11)–(13) that, close to the stationary regime (14), the negative damping introduced as a result of the magnetoelastic interaction into the acoustic resonance d is, in the first approximation with respect to the smallness of the oscillation d, equal to

$$\Gamma_1 = |a_0|^2\sum_{k=0}^{\infty}|\Psi_1^{0k}|^2 A_k, \tag{15}$$

$$A_k = \frac{1}{2\gamma_k}\left[\frac{\Delta_k^*(\omega+\Omega)-\Delta_k(\omega-\Omega)+2jb_k}{\Delta_k^*(\omega+\Omega)\Delta_k(\omega-\Omega)-b_k^2}+\text{c.c.}\right],$$
$$b_k = \frac{4\Psi_2^{0k}}{\gamma_k}|a_0|^2,\quad \Delta_k(\nu) = 1+j\left(\frac{\omega_k-\nu}{\gamma_k}+2b_k\right). \tag{16}$$

Assuming that when h_0 grows the instability of the stationary regime (14) with respect to the excitation of the sound d is the first to set in, we obtain for the instability the criterion $\Gamma_1 \geq \Gamma$. This instability may compete with the Suhl instability[14] against excitation of the resonance k of frequency $\omega_k \approx \omega$ in the absence of sound. The threshold of this latter instability can be obtained from (11) and (12) when $d \equiv 0$; it is equal to

$$|a_0|^2 = g_1|\Delta_k(\omega)|, \tag{17}$$

and we must suppose that the process with the lower threshold is the one which will develop. The quantity $g_1 = \gamma_k/4|\Psi_2^{0k}|$ determines the minimum value of the threshold of the Suhl instability and corresponds to the FMR amplitude a_0 at which the nonlinear shift of the frequency of the resonance k assumes the half-width γ_k of the resonance.

Let us analyze the sound excitation threshold. To the case of a weakly nonlinear spin subsystem correspond those thresholds $|a_0|^2$, for which the nonlinear corrections to the resonance frequencies ω_k are small ($b_k \ll 1$). In that case

$$A_k \approx \chi_k''(\omega-\Omega)-\chi_k''(\omega+\Omega), \tag{18}$$

and we arrive at the approximation considered in Part I. The spin-spin nonlinearity gives rise to nonlinear corrections to the resonance eigenfrequencies ω_k and, what is more important, causes, for each mode k, interaction between the harmonics $\omega+\Omega$ and $\omega-\Omega$, which intensifies as the sound frequency Ω decreases in comparison with γ_k.

We shall trace the pattern of the phenomena in the simple case when the spin reservoir is represented by one oscillation k while the coupling with other spin resonances during the excitation of the sound d is unimportant. We then obtain from (15) the instability condition

$$|a_0|^2 \geqslant \frac{\Gamma}{|\Psi_1^{0k}|^2}A_k^{-1}, \tag{19}$$

which implicitly determines the threshold $|a_0|^2$. Let us discuss the results for $\Omega \gg \gamma_k$ and $\Omega \ll \gamma_k$.

For $\Omega \gg \gamma_k$, as the analysis of (16) shows, the expression on the right hand side of (19) is a minimum when $\omega - \Omega = \omega_k + 2\gamma_k b_k$. Close to such a tuning of the system the spin oscillations at the frequency $\omega+\Omega$ are unimportant and

$$A_k = 1/\gamma_k|\Delta_k(\omega-\Omega)|^2. \tag{20}$$

We then arrive at the expression obtained for the instability threshold in [5]. The nonlinearity of the spin subsystem in such an approximation, as can be seen from (19) and (20), shifts and deforms the region of magnetoelastic instability in accordance with the variation of the resonance frequencies, but does not change the magnitude of the minimum threshold.

The effect of nonlinearity is felt most in the opposite case $\Omega \ll \gamma_k$. Correct to terms $\sim (\Omega/\gamma_k)^2$ the condition (19) then takes the form

$$|a_0|^2 \geqslant g_2\frac{[(x_k-b_k)(x_k-3b_k)+1]^2+4(\Omega/\gamma_k)^2}{4(x_k-b_k)},$$
$$g_2 = \frac{\Gamma\gamma_k}{|\Psi_1^{0k}|^2}\frac{\gamma_k}{\Omega}. \tag{21}$$

The quantity g_2 corresponds to the minimum threshold for sound excitation when $\Psi_2 \equiv 0$ (see (9)). The vanishing of the expression in the square brackets in the numerator of (21) coincides with the threshold for the Suhl instability (17).

Depending on the magnitude and sign of the spin-spin nonlinearity, the minimum threshold determined from (21) varies considerably. If $\Psi_2^{0k} < 0$ then $b_k < 0$ and it is not difficult to see from (21) that the minimum threshold increases with $|\Psi_2^{0k}|$. And conversely, if $\Psi_2^{0k} > 0$ then the threshold decreases. In the case when the nonlinearity of Ψ_2^{0k} is due to dipole-dipole interaction, Ψ_2^{0k} is a maximum and positive for resonances corresponding to the spin branch $\theta = 0$ (the wave vector is parallel to the constant field); for $\theta = 0$ the Suhl instability threshold is a minimum: $\omega_k \approx \omega$.[14] Depending on the direction of propagation θ and the shape of the sample, Ψ_2 may also have the opposite sign. For the uniform precession in a disk magnetized in the direction perpendicular to its plane, $\Psi_2^{00} > 0$, while when the direction of magnetization is parallel to the plane $\Psi_2^{00} < 0$. Crystallographic anisotropy may make a contribution of either sign to Ψ_2; for yttrium iron garnet this contribution is negative.

Let us see how the sound excitation threshold varies as the relation between the spin-spin and the magnetoelastic interactions changes. To do that it is convenient to introduce the parameter $\mu = g_1/g_2$ and consider the region of instability in the $(|b_k|, x_k)$-plane (see Figs. 2 and 3).

Let $\Psi_2^{0k} < 0$ (Fig. 2). The vertical shading corresponds to the region of the Suhl instability which, in this case, lies to the left of the straight line $x_k + |b_k| = 0$. According to (21) the excitation of sound is possible only in the region $x_k + |b_k| > 0$, i.e., the regions of the two instabilities are separated for all values of μ. As μ decreases the sound excitation region (in Fig. 2 it is obliquely shaded) becomes bounded from the top, decreases, shrinks to a point and vanishes. Total suppression of the sound occurs at $\mu \leq 2 + O(\Omega^2/\gamma_k^2)$. (We recall that we are considering here low-frequency sound $\Omega \ll \gamma_k$.) Thus, even if the minimum MAR threshold g_2 is below the Suhl instability threshold g_1 by almost a factor of two, sound is nevertheless not excited.

Let $\Psi_2^{0k} > 0$ (Fig. 3). In this case both the region of the Suhl instability (it is enclosed by the boundary 1 in Fig. 3), and the sound excitation region lie to the right of the straight line $x_k - |b_k| = 0$. For $\mu \gg 1$ the magnetoelastic instability threshold (the curve 2 in Fig. 3) lies far below the boundary 1 and the effect of the nonlinearity of the spin subsystem is insignificant. As μ decreases, the boundary of the sound excitation region rises towards the boundary of the Suhl instability. (If we assume the quantity g_2 to remain constant, then the sound excitation threshold decreases with actual growth of the nonlinearity of Ψ_2^{0k}, i.e., in terms of $|a_0|^2$ instead of b_k.) As $\mu \to 0$ the boundary of the sound excitation region intersects the curve 1 as it rises. In the region of detuning, $x_k \lesssim 3 - 5$, the two boundaries practically merge at some $\mu = \mu_0$. If $\mu > \mu_0$, sound is excited first, while when $\mu < \mu_0$ the Suhl instability occurs first; the quantity $\mu_0 \approx \Omega^2/\gamma_k^2$.

Close to $\mu \approx \mu_0$, the distance between the boundaries of the two instabilities along the straight line $x_k - 2|b_k|$

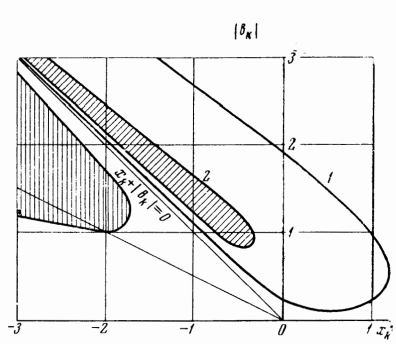

FIG. 2. Region of instability when $\psi^{0k} < 0$, $(\Omega/\gamma_k)^2 = 0.1$; for 1, $\mu = 10$ and for 2, $\mu = 3$.

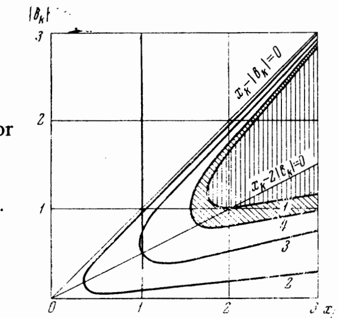

FIG. 3. Region of instability for $\psi_2^{0k} > 0$, $(\Omega/\gamma_k)^2 = 0.1$; 1 – boundary of the spin-spin instability, $\omega_k \approx \omega$, 2–$\mu = 10$, 3–$\mu = 1$, 4–$\mu = 0.2$.

$= 0$ is given by $\delta \approx (\mu - \mu_0)/2\mu$. For low-frequency sound μ_0 may be extremely small. This means that whenever it would appear that there should be a purely spin-spin instability, it would be preceded by sound excitation even though the magnetoacoustic instability thresholds are large ($g_2 \approx g_1/\mu_0 \gg g_1$).

Let us discuss again the simplest case which is realized when the coupling of the sound with the resonance c_0, directly excited by the external field, is the most important. Then, it is necessary to replace the index k by 0 everywhere in the above given conditions for instability. Now, the instability condition (17) implies not a parametric excitation of the resonance c_0 at the frequency ω, but a development of a region where the stationary states of the FMR curve (14) are not unique. On passage through resonance, the system of these states becomes stable jumpwise; the FMR curve manifests a hysteresis. For anharmonicity of the right sign, sound should be observed close to the points where the stationary regime collapses—if, of course, there are low-frequency resonances in the spectrum of the elastic system.

3. Discussion

On the dispersion diagram (Fig. 1), the processes may be represented in the following way. As has been said already, when $\Omega \lesssim \gamma$ the pairs of oscillations $\omega_1(-k_1)$ and $\omega_2(k_1)$ participate in the excitation of the sound. When $\Psi_2 \neq 0$ these same pairs are at the same time the results of a Suhl process of the second order: $2\omega(k = 0) = \omega_1(-k_1) + \omega_2(k_1)$, which leads to important effects as Ω/γ decreases. Notice that when the magnetoelastic interaction is neglected, the instability with respect to disintegration into two spin oscillations has the threshold (17) regardless of the density of the magnetic resonances in the band close to $\omega_k \approx \omega$, whereas the process of the development of low-frequency sound includes the entire aggregate of these resonances (the sum over k in (15)). Since the terms which are cubic in c in the energy of the spin-spin interaction most effectively generates a coupling between FMR oscillations of frequency ω and the oscillations $\omega_k \approx \omega/2$, while the coupling with the resonances $\omega_k \approx \omega$ appears only in second-order perturbation theory, the effect of this spin-spin nonlinearity on the process of the excitation of low-frequency sound in a ferromagnet appears, accordingly, in that same order of smallness. Therefore, under the conditions of the Suhl instability $\omega_k \approx \omega/2$, the two processes are not so strongly intertwined.

We dwelt at length on the case when the parametrically excited spin subsystem is represented by one oscillation k. We must, however, take into consideration the fact that, in reality, the minimum thresholds of the magnetoacoustic instabilities g_2 may correspond to one group of spin resonances G_2, connected with the acoustic resonance d_2, while the thresholds of the Suhl instability g_1 may correspond to another group $G_1 \not\equiv G_2$. It is clear that if $\mu \gg 1$, then the resonances G_2 are excited simultaneously with the sound d_2, while if $\mu \ll 1$, then the spin resonance G_1 and the acoustic resonance d_1 which is most strongly coupled with G_1 and is, in the general case, different from d_2, are excited.

The considered pattern is, generally speaking, valid only when we exceed the values of the threshold parameters by small amounts, and our analysis does not allow us to elucidate what happens when the thresholds of both types of instabilities are simultaneously exceeded. The establishment of oscillations in a parametric excitation is determined exclusively by the nonlinear properties of the system. Notice that the processes that occur in this are extremely complicated and the consistent theory of the steady FMR state, even without the consideration of the magnetoelastic interaction, is far from complete.

Thus, the presence of spin-spin nonlinearity may lead either to a strong suppression or to the excitation of low-frequency sound when the magnetoelastic coupling is extremely small and the acoustic damping is large. The effectiveness of the mechanism of the strengthening of the coupling with the acoustic system increases as the frequencies of the acoustic resonances in the system decrease. In particular, the effects are favored by increase in the dimensions of the sample and, as estimates show, even by "loading" the ferromagnet with a large additional mass, in spite of the decrease in the magnetoacoustic coupling Ψ_1 (for example, a ferromagnet diluted in a dielectric or a magnetic film on a backing).

The elucidation of the regions of applicability of the models which have hitherto been considered in the literature and in which the authors restricted themselves to the consideration of either only spin-spin or only the magnetoelastic nonlinearity, has shown the following. Usually, for the analysis of the pattern of nonlinear phenomena in a system, it is considered sufficient to determine the thresholds of each of the competing nonlinear processes separately, neglecting the rest and supposing that, in reality, the process with the lowest threshold is the one that develops. The present problem shows that this, generally speaking, is not the case if one of the competing processes is the excitation of a low-frequency system. In our example, the MAR threshold g_2 may exceed the Suhl instability threshold g_1, in principle, by many orders (by a factor of γ^2/Ω^2) and nonetheless the process of sound excitation develops (simultaneously with one of the resonances of the magnetization which correspond to the threshold g_1).

We should expect a similar intensification of the interaction of a high-frequency system with low-frequency resonances of different physical nature, owing to the nonlinearity of the high-frequency system.

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