INTERNAL SYMMETRY OF THE MAXWELL "FISH-EYE" PROBLEM AND THE FOCK GROUP FOR THE HYDROGEN ATOM

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As a special case of the focusing potentials studied previously, a system is obtained which Maxwell referred to as the "fish-eye" problem. The wave equation for this problem can be reduced to the Laplace equation on a four-dimensional sphere. The symmetry group for this problem is, therefore, the same as that found by Fock for the hydrogen atom in the case of the discrete spectrum. It is shown that the Maxwell potential has focusing properties in the wave sense as well, i.e., when diffraction is rigorously taken into account. The Green function is found, and generalizations of the fish-eye problem are discussed.

1. INTRODUCTION

In a previous paper[1] we considered a sphere of radius R with refractive index

$$n(r)=\frac{2n'}{1+(r/R)^2} \qquad (1)$$

where $n(r) = n'$ when $r > R$. Let us denote the surface of this sphere by S_3. It was shown earlier that all rays leaving a point on S_3 in the inward direction were focused at the point diametrically opposite on S_3. Let us now suppose that Eq. (1) is valid throughout space. In the equivalent problem of classical mechanics one considers the motion of a particle with zero energy[1)] (i.e., such that its velocity at an infinitely distant point is zero) in a potential

$$U_1(r)=-\frac{2v}{(R^2+r^2)^2}, \qquad (2)$$

where $v > 0$ is an arbitrary parameter. In this case, all the trajectories are circles (or straight lines) intersecting S_3 at diametrically opposite points. A striking feature is that the refractive index given by Eq. (1) will exactly focus rays leaving any point in space and not merely points confined to S_3. All the particle trajectories (or rays) leaving a point **r** are found to pass through the point $\mathbf{r}R^2/r^2$. The image of an object in such an optical system is the result of its inversion in the sphere S_3 and reflection at the origin of coordinates. This transformation is conformal.

The refractive index given by Eq. (1) was first found by Maxwell in 1854.[2] Maxwell demanded that all the rays in a spherically symmetric optical system were circles. He obtained Eq. (1), and then established the focusing properties of the system which he called the fish-eye problem.

The reason for the particular properties of the fish eye was pointed out by Caratheodory[3] (see also[4]). To demonstrate it, let us perform a stereographic projection of the three-dimensional space onto a four-dimensional sphere S_4 of radius R. A point (x, y, z) is then projected onto the point $(R\xi_1, R\xi_2, R\xi_3, R\xi_4)$, where the point $(\xi_1, \xi_2, \xi_3, \xi_4)$ lies on a unit sphere in the four-dimensional space

$$\xi_1=2Rx/(R^2+r^2),\ \xi_2=2Ry/(R^2+r^2),\ \xi_3=2Rz/(R^2+r^2),\quad \xi_4=(R^2-r^2)/(R^2+r^2). \qquad (3)$$

The trajectories transform into the major circles of the sphere S_4, and all these circles which pass through a certain point on S_4 intersect at a diametrically opposite point on the four-dimensional sphere, which explains the focusing phenomenon in the original problem.

The Maxwell fish-eye problem is closely related to the Coulomb problem. When an electron of energy ϵ moves on an ellipse in the Coulomb potential $U_C(r) = -Z/r$, the family of orbits in momentum space[5,6] is identical with the trajectories which we have considered for the potential given by Eq. (2) if[2)]

$$R^2=-2\varepsilon. \qquad (4)$$

For a monotonic, spherically symmetric potential U(r) the action function in momentum space

$$T(\mathbf{p},\mathbf{p}',\varepsilon)=\int_{\mathbf{p}'}^{\mathbf{p}}\mathbf{r}\,d\mathbf{p}$$

(the integral is evaluated in momentum space over the classical trajectory joining the points **p** and **p′**) satisfies the equation

$$\left(\frac{\partial T}{\partial p_x}\right)^2+\left(\frac{\partial T}{\partial p_y}\right)^2+\left(\frac{\partial T}{\partial p_z}\right)^2-\left[r\left(\varepsilon-\frac{p^2}{2}\right)\right]^2=0, \qquad (5)$$

where r(U) is the inverse of U(r).[7] In the case of the Coulomb potential $r(U) = 2Z/(U^2 - 2\epsilon)$, Eq. (5) becomes identical with the Hamilton-Jacobi equation (with the momenta replaced by the coordinates) for the Maxwell problem, provided both Eq. (4) and the following condition are satisfied:

$$Z^2=v. \qquad (6)$$

This will be investigated below for the corresponding wave problems.

[1)]We shall be using the atomic system of units, and will set the particle mass equal to unity.

[2)]The motion of the imaging point over a major circle on S_4 is, in both cases, nonuniform and is different for the two problems. Uniformity can be achieved by introducing a new time parameter, and this is done for the Coulomb problem in [7].

2. THE WAVE PROBLEM

The fact that the classical trajectories are closed is found to be related to the additional symmetry of the corresponding wave problem and the possibility of separating the variables in a number of coordinate systems. The particular feature of our case is that we must consider the wave problem and the classical problem for a fixed energy $E = 0$:

$$\left[-\frac{1}{2}\nabla^2 - \frac{2v}{(R^2+r^2)^2}\right]\psi(\mathbf{r}) = 0. \tag{7}$$

Accordingly, the variables in Eq. (7) (and in the Hamilton-Jacobi equation for the classical problem) can be separated both in the bispherical and toroidal coordinate systems (as well as in the spherical system) provided the poles of the coordinate system lie on S_3. These two systems differ by the fact that the Laplace equation is separable for them, but the Schroedinger equation with nonzero energy is not.[8] The solution in the spherical set of coordinates will be given in Sec. 4 for a more general case.

Equation (7) is invariant under inversion in a sphere centered at an arbitrary point $\mathbf{b}$ if it cuts S_3 on a major circle (i.e., if its radius is $\sqrt{R^2+b^2}$). These inversions form a group. Therefore, if we have a solution $\psi(\mathbf{r})$ of Eq. (7), the function

$$\frac{R}{|\mathbf{r}-\mathbf{b}|}\psi\left(\frac{\mathbf{r}-\mathbf{b}}{(\mathbf{r}-\mathbf{b})^2}(R^2+b^2)+\mathbf{b}\right) \tag{8}$$

will also satisfy Eq. (7). By performing successive inversions in spheres centered on the points $x = b$ and $x = b + \delta b$ on the x axis, we obtain an infinitesimal transformation. The corresponding infinitesimal operator acting on the wave functions is

$$A_x' = {}^1/_2 R^{-1}[-ix + (R^2 - r^2)p_x + 2x(\mathbf{rp})],$$

where $\mathbf{p}$ is the momentum operator. The quantities A_y' and A_z' are defined by analogy. The components of the angular momentum operator $\mathbf{L}$ and the operator $\mathbf{A}'$ form the algebra of the O(4) group of four-dimensional rotations:

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, \quad [L_i, A_j'] = i\varepsilon_{ijk}A_k',$$
$$[A_i', A_j'] = i\varepsilon_{ijk}L_k, \quad \mathbf{LA}' = \mathbf{A}'\mathbf{L} = 0. \tag{9}$$

The commutator of the Hamiltonian H with the operator $\mathbf{A}'$, i.e.,

$$[H, \mathbf{A}'] = -2i\frac{\mathbf{r}}{R}H, \quad H = -\frac{1}{2}\nabla^2 - \frac{2v}{(R^2+r^2)^2}, \tag{10}$$

yields zero when it operates on a function satisfying Eq. (7). The irreducible representations of the O(4) group are realized on such functions, and this leads to additional degeneracy of the energy levels of the problem $(H - E)\psi = 0$ at the limit of the continuous spectrum. These levels exist only for certain definite values of v [see Eq. (14)] and their degree of degeneracy is n^2 $(n = 1, 2, 3, \ldots)$, which is analogous to the electron energy levels in a Coulomb field, and contrasts with the degeneracy of $(2l + 1)$ for the usual spherically symmetric potential, where l is the orbital quantum number. Therefore, the symmetry group for the fish-eye problem is identical with that found by Fock[9] for the discrete spectrum of the hydrogen atom.

We note that solutions of Eq. (7), obtained by separating the variables in bispherical coordinates with the z axis, are the eigenfunctions of the operators H, $A_x'^2 + A_y'^2 + L_z^2$, and L_z^2, and in toroidal coordinates they are the eigenfunctions of H, $A_z'^2$, and L_z^2, where the corresponding eigenvalues are the separation constants.

The above representations of the O(4) group are not unitary with respect to the scalar product defined in the usual way, and the operator $\mathbf{A}'$ is not Hermitian because the transformation given by Eq. (8) does not conserve the normalization of the wave function. The representations are unitary with respect to the scalar product with weight $(R^2 + r^2)^{-2}$. The introduction of the scalar product is equivalent to the reformulation of the problem, given below. Unitary representations can be obtained if we consider a new formulation of the problem, which is analogous to the Sturm problem for the hydrogen atom.[10] If we use the wave function

$$\varphi(\mathbf{r}) = (R^2 + r^2)^{-1}\psi(\mathbf{r}),$$

Eq. (7) becomes the eigenvalue problem:

$$(\tilde{H} - v)\varphi(\mathbf{r}) = 0, \tag{11}$$

where $\tilde{H} = {}^1/_4(R^2 + r^2)\nabla^2(R^2 + r^2)$. The Hermitian operator

$$\mathbf{A} = (R^2 + r^2)^{-1}\mathbf{A}'(R^2 + r^2) = \frac{1}{2R}[-3i\mathbf{r} + (R^2 - r^2)\mathbf{p} + 2\mathbf{r}(\mathbf{rp})],$$

commutes with $\tilde{H}$, and as $\mathbf{A}'$ it satisfies relationships of the form given by Eq. (9), i.e., it is a generator of the O(4) group, where

$$\tilde{H} = R^2(A^2 + L^2 + {}^3/_4). \tag{12}$$

If we now transform to the coordinates ξ_i given by Eq. (3), and introduce the wave function $\Phi(\xi_1, \xi_2, \xi_3, \xi_4) = (R^2 + r^2)^{3/2}\varphi(\mathbf{r})$, then by using Eq. (12) we find that we can write Eq. (11) in the form of the Laplace equation on the four-dimensional sphere

$$R^2(D - {}^1/_4)\Phi = v\Phi, \quad D = -{}^1/_2\sum_{\substack{i,k=1\\ i\neq k}}^{4} M_{ik}M_{ki} + 1. \tag{13}$$

In this expression M_{ik} is the generator of an infinitesimal rotation on the $\xi_i\xi_k$ plane of the four-dimensional space, and $1-D$ is the angular part of the four-dimensional Laplace operator. The expressions for the operators M_{ik} in terms of the coordinates x, y, z are given, for example in [11,12]. The eigenfunctions of Eq. (13) are the four-dimensional spherical harmonics Ψ_{nlm}[9], and the eigenvalues are degenerate in the quantum numbers l, m:

$$v_n = R^2(n^2 - {}^1/_4). \tag{14}$$

The Schrödinger equation for the discrete spectrum of the hydrogen atom in the Fock representation[9] takes the form

$$K\Phi = \frac{\sqrt{-2\varepsilon}}{Z}\Phi, \tag{15}$$

where K is the integral operator

$$K\Phi(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{1}{2\pi^2}\int\frac{\Phi(\xi_1',\xi_2',\xi_3',\xi_4')\,d^4\Omega'}{(\xi_1-\xi_1')^2+(\xi_2-\xi_2')^2+(\xi_3-\xi_3')^2+(\xi_4-\xi_4')^2} \qquad (16)$$

In this expression $d^4\Omega'$ is an area element on the unit four-dimensional sphere over which the integral is evaluated, $\Phi(\xi_1, \xi_2, \xi_3, \xi_4) = (R^2 + r^2)^2\varphi_C(\mathbf{r})$ (we are omitting normalizing factors), ξ_i can be expressed in terms of x, y, z with the aid of Eq. (3), and $\varphi_C(\mathbf{r})$ is the wave function for the hydrogen-like atom in the momentum representation in which **p** is replaced with **r**.

The functions Ψ_{nlm} form a complete set of eigenfunctions for the operators K and D, and comparison of the eigenvalues

$$K\Psi_{nlm}=n^{-1}\Psi_{nlm},\qquad D\Psi_{nlm}=n^2\Psi_{nlm}$$

leads to the result

$$K^2=D^{-1}. \qquad (17)$$

Therefore, the wave equation for the fish-eye problem reduces to a differential equation for the four-dimensional spherical functions Ψ_{nlm}, while the Schrödinger equation for the hydrogen atom in the Fock representation is an integral equation for Ψ_{nlm}. The wave functions for the original problems can be expressed in terms of each other

$$\psi(\mathbf{r})=(R^2+r^2)^{-3/2}\varphi_C(\mathbf{r}). \qquad (18)$$

The wave equation with the potential $-2v/(R^2 - R^2)^2$ (with a singularity at $r = R$) is invariant under inversions in spheres which cut S_3 at right-angles. The internal symmetry of the problem is described by the Lorentz group, and so is the continuous spectrum in the Coulomb problem with which this potential is connected in a similar way to that in which the Maxwell potential is related to the discrete spectrum.

3. FOCUSING IN THE WAVE PROBLEM AND THE GREEN FUNCTION

The transformations given by Eq. (8) of the symmetry group of Eq. (7) enable us to find the Green function quite readily. Separating the variables in terms of the spherical coordinates, we can find the spherically symmetric solutions of Eq. (7) with a singularity of the form r^{-1}:

$$\psi_1(\mathbf{r})=\frac{\sqrt{R^2+r^2}}{rR}\exp\left(i\alpha\tan^{-1}\frac{R}{r}\right)\Big/ i\sin\frac{\pi\alpha}{2},$$

$$\psi_2(\mathbf{r})=\mathrm{Re}[\psi_1(\mathbf{r})]=\frac{\sqrt{R^2+r^2}}{rR}\sin\left(\alpha\tan^{-1}\frac{R}{r}\right)\Big/ \sin\frac{\pi\alpha}{2},$$

where $\alpha = \sqrt{1 + 4v/R^2}$. If we apply the inversion given by Eq. (8) to $\psi_1(\mathbf{r})$, substitute $\mathbf{r}' = -\mathbf{b}R^2/b^2$, and multiply the result by $(R^2 + r'^2)^{-1/2}$, we obtain the following solution of Eq. (7), which is symmetric in **r** and **r**′:

$$\Gamma(\mathbf{r},\mathbf{r}')=\frac{1}{i\sin(\pi\alpha/2)}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{(R^2+r^2)(R^2+r'^2)}{(R^2+r^2)(R^2+r'^2)-|\mathbf{r}-\mathbf{r}'|^2R^2}\right]^{1/2}$$
$$\times\exp\left\{i\alpha\,\mathrm{arctg}\frac{[(R^2+r^2)(R^2+r'^2)-|\mathbf{r}-\mathbf{r}'|^2R^2]^{1/2}}{|\mathbf{r}-\mathbf{r}'|R}\right\}, \qquad (19)$$

which has singularities of the form $|\mathbf{r} - \mathbf{r}'|^{-1}$ and $|\mathbf{r} + \mathbf{r}'R^2/r'^2|^{-1}$, and describes a source placed at the point **r**′ and its point image (sink) at the point $-\mathbf{r}'R^2/r'^2$. The existence of this solution means that the Maxwell potential will ensure perfect focusing in a wave sense as well, i.e., when diffraction is rigorously taken into account.

Applying the transformation given by Eq. (8) to $\psi_2(\mathbf{r})$, we find that

$$G(\mathbf{r},\mathbf{r}')=\mathrm{Re}\,[\Gamma(\mathbf{r},\mathbf{r}')], \qquad (20)$$

which has a singularity of the form $|\mathbf{r} - \mathbf{r}'|^{-1}$. It is clear that

$$[\nabla^2+4v/(R^2+r^2)^2]G(\mathbf{r},\mathbf{r}')=4\pi\delta(\mathbf{r}-\mathbf{r}'),$$

i.e., we have found the Green function for Eq. (7).[3)]

In[6] the Coulomb Green function in the momentum representation $\tilde{F}_C(\mathbf{p}, \mathbf{p}', \epsilon)$ was found in the phase-integral approximation as the sum over all classical trajectories joining the points **p** and **p**′ in the momentum space of the components whose wave behavior is determined by the factor $\exp[-iT(\mathbf{p}, \mathbf{p}', \epsilon)]$. In view of the above connection between the two problems we can replace the momenta **p**, **p**′ in the expression for $\tilde{F}_C(\mathbf{p}, \mathbf{p}', \epsilon)$ by the coordinates **r**, **r**′ and then by multiplying it by $(R^2 + r^2)^{-3/2}(R^2 + r'^2)^{-3/2}$ and using Eqs. (4) and (6) we obtain the quasiclassical Green function for the Maxwell potential. It is interesting that this differs from the exact Green function given by Eq. (20) only by the fact that $\alpha = \sqrt{1 + 4v/R^2}$ has been replaced by $2\sqrt{v}/R$.

4. ~~LORENTZ~~ Lenz POTENTIALS AND ADDITIONAL DEGENERACY

The fish-eye problem can be generalized as follows. We shall require that all rays leaving a given point on the sphere S_3 and lying in a certain plane passing through the source and the center of symmetry will pass through some other point on S_3. Let π/μ be the angle which the two points subtend at the center. Therefore, rays lying in a given meridional plane will be focused, but the "foci" for different planes will lie on a circle. The angle of deflection is given by Eq. (17) in[1] for $b = 1$, $a = \mu^{-1} - 1$, and for $r < R$ the refractive index is found to be a special case of Eq. (18) in[1]:

$$n(r)=n'\frac{R}{r}\left[\mathrm{ch}\left(\mu\ln\frac{r}{R}\right)\right]^{-1}=\frac{R}{r}\frac{2n'}{(r/R)^\mu+(R/r)^\mu}. \qquad (21)$$

When $\mu = 1$ this result becomes identical with Eq. (1). If we can extend the validity of Eq. (21) to all space by analogy with what has been done for Eq. (1), we obtain the refractive index first found by Lenz[13] (see also[4, 14]), or the mechanical problem of the motion of a zero-energy particle in the potential

$$U_\mu(r)=-\frac{2v}{r^2R^2[(r/R)^\mu+(R/r)^\mu]^2}. \qquad (22)$$

Let us introduce the polar coordinates r, ϑ in the plane of a particular trajectory. The equation of the trajectory is

$$(r/R)^\mu=B\sin\mu(\vartheta-\vartheta_0)+\sqrt{B^2\sin^2\mu(\vartheta-\vartheta_0)+1}, \qquad (23)$$

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3) The expression for the Green function is valid for either sign of v.

where the constants ϑ_0 and B are determined from the initial conditions. If μ is a rational number, i.e., if it can be written in the form of the simple fraction $\mu = \nu_1/\nu_2$, where ν_1 and ν_2 are integers, then the function $r(\vartheta)$ in Eq. (23) is periodic and the trajectory closes after ν_2 revolutions around the center. In the opposite case, the trajectory is open and fills densely a plane region. All the rays leaving the point r are focused after $\nu_2/2$ revolutions: 1) at the point $-\mathbf{r}R^2/r^2$ if ν_1 and ν_2 are odd, 2) at the point $\mathbf{r}R^2/r^2$ if ν_1 is odd and ν_2 even, and 3) at the point $-\mathbf{r}$ if ν_1 is even and ν_2 odd.

The wave equation with the Lenz potential

$$[-{}^1\!/_2\nabla^2 + U_\mu(r)]\psi(\mathbf{r}) = 0 \tag{24}$$

where $v = v_N$ and

$$v_N = R^2\mu^2\left(N+\frac{1}{2\mu}\right)\left(N+\frac{1}{2\mu}-1\right), \tag{25}$$

has the following regular solutions which decrease at infinity:

$$\psi_{nlm}(\mathbf{r}) = \left(\frac{r}{R}\right)^l\left[\left(\frac{r}{R}\right)^{2\mu}+1\right]^{-(2l+1)/2\mu} C_{n_r}^{(2l+1)/2\mu+1/2}\left(\frac{R^{2\mu}-r^{2\mu}}{R^{2\mu}+r^{2\mu}}\right)Y_{lm}(\vartheta,\varphi), \tag{26}$$

where n_r is the radial quantum number ($n_r = 0, 1, 2, \ldots$), $n = n_r + l + 1$ is the principal quantum number, $N = n + (\mu^{-1} - 1)l$, and C_n^λ are the Gegenbauer polynomials. Therefore, the values of v for which energy levels exist at the limit of the continuous spectrum depend on the quantum numbers n and l, but only in the form of the combination $n + (\mu^{-1} - 1)l$. This leads to additional degeneracy of the zero-energy levels if ν is a rational number. The same reason (dependence of energy on only the linear combination of quantum numbers) is responsible for the degeneracy of the levels of the two-dimensional anisotropic harmonic oscillator. The possible degrees of degeneracy are equal to the squares of integers. If we adopt the definition of the symmetry group of a quantum-mechanical system given in[15], the Sturm problems for all the Lenz potentials with rational μ have O(4) symmetry. The infinitesimal operators of the group can be constructed by analogy with the case of the anisotropic oscillator.[15,16] but the finite transformations of the group are unknown.[4)] We note that, in the case of two dimensions, the solutions of the wave equation for the Lenz potential can be obtained from the solutions for the Maxwell potential with the aid of the conformal transformation used in[4] for the corresponding classical problem.

5. CONCLUSION

The connection between the closed nature of the trajectories in classical mechanics and the additional degeneracy in the corresponding quantum-mechanical problem is well known. It is also known that there are only two central fields, namely, the Coulomb field and the harmonic field for which the classical trajectories are closed for arbitrary energy E (this occurs only for $E < 0$ in the Coulomb field). We have considered here a class of problems in which the trajectories are closed only for the fixed energy value $E = 0$, but with the coefficient of the potential energy were allowed to take arbitrary values. We could then choose a potential $U_\mu(r)$ such that the ratio of the periods of the radial and angular motion will be constant for all trajectories (when $E = 0$) and equal to any number given in advance. In the corresponding quantum-mechanical problem we have the additional degeneracy for $E = 0$, so that as the coefficient v in the potential increases, and the potential well becomes deeper, the new bound states appear in groups with the same value of the linear combination of the quantum numbers n_r and l, and the degeneracy is removed as the energy of these states decreases and departs from the value $E = 0$. In spite of the fact that we have confined our attention to $E = 0$, the fact that the coefficient in the potential has been arbitrary enables us to formulate the self-adjoint problem and obtain a complete set of orthogonal functions for each potential.

In the simplest potential of this kind, i.e., the fish-eye potential, the ratio of the radial and angular periods of the particle is unity, i.e., it is the same as in the case of motion in a Coulomb field. Accordingly, in the quantum case, we have the same degeneracy, i.e., the energy depends only on the linear combination $n_r + l$. Hence, it follows that the symmetry group for the two problems should be the same, and for the fish eye the existence of the group of four-dimensional rotations is even more useful than for the Coulomb problem because the stereographic projection in the coordinate space transforms the Schrödinger equation into the differential equation for the spherical functions, whereas in the Coulomb field we obtain the integral equation for the spherical functions after stereographic projection in momentum space.

The fact that the wave equation for the fish-eye problem can be written as a wave equation on the surface of a four-dimensional sphere provides a particularly useful and unique property of this problem, i.e., the fact that the perfect focusing occurs not only in the geometric but also in the wave approximation. In fact, waves leaving the source located on one pole of the sphere are obviously focused at the opposite pole.

The analysis given here shows that the focusing properties of the potential are quite general and are closely related to its internal symmetry. Analysis of this connection in a general form would be an interesting theoretical problem.

4)We note that Eq. (24) is invariant under the transformation defined by Eq. (8) when $\mathbf{b} = 0$ (inversion in the sphere S_3).

1 Yu. N. Demkov, V. N. Ostrovskiĭ, and N. B. Berezina, Zh. Eksp. Teor. Fiz. 60, 1604 (1971) [Sov. Phys.-JETP 33, No. 5 (1971)].

2 J. C. Maxwell, The Scientific Papers, New York, 1952, p. 74.

3 C. Caratheodory, Sitzungsberichte der Bayerischen Akad. Wiss., Math.-Naturwiss., Abt. No. 1 (1926); Geometrische Optik, Berlin, 1937.

4 R. K. Luneberg, Mathematical Theory of Optics, Berkeley, 1964.

5 A. Sommerfeld, Mechanics, New York, 1952.

6 M. C. Gutzwiller, J. Math. Phys. 8, 1979 (1967).

7 G. Györgyi, Nuovo Cimento 53A, 717 (1968).

[8]P. M. Morse and H. Feshbach, Methods of Theoretical Physics, 1953.
[9]V. A. Fock, Z. Phys. 98, 145 (1935).
[10]M. Rotenberg, Ann. Phys. 19, 262 (1962).
[11]Yu. A. Dobronravov, Vestnik LGU, No. 10, 5 (1957).
[12]M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966).
[13]W. Lenz, collection: Problems der modernen Physik, P. Debye, Leipzig, 1928.
[14]R. Stettler, Optik 12, 530 (1955).
[15]Yu. N. Demkov, Zh. Eksp. Teor. Fiz. 44, 2007 (1963) [Sov. Phys.-JETP 17, 1349 (1963)].
[16]L. A. Il'kaeva, Vestnik LGU, No. 22, 56 (1963).

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