NONLINEAR WAVE FORMATION IN AN ELECTRON BEAM-PLASMA INTERACTION

V. P. KOVALENKO

Institute of Physics, Ukrainian Academy of Sciences

Submitted November 16, 1970

Zh. Eksp. Teor. Fiz. 60, 2122–2127 (June, 1971)

An approximate analytic solution is obtained for the problem of the interaction between a modulated electron beam and a plasma in the nonlinear stage up to the wave state close to the "breaking" of the wave. It is shown that along with the growth of amplitude the profile of the traveling wave is strongly distorted; the character of this distortion is determined by the dispersion of the beam-plasma system. It is shown, in particular, that under certain conditions the amplitude of the second harmonic can exceed that of the fundamental wave. The results are consistent with experiments on the anomalous scattering of a modulated electron beam in a plasma.

ONE of the strongest nonlinear effects arising during the propagation of longitudinal waves in a plasma is the distortion of the wave profile.[1] This process has heretofore been studied theoretically only for stable systems. There has been practically no analysis of the nonlinear development of waves in unstable plasma systems, particularly in beam-plasma systems. However, numerical studies[2-5] and laboratory experiments[6,7] indicate that the nonlinear distortion of wave profiles plays a decisive role in the dynamics of beam-plasma interactions. It has become necessary to analyze analytically the characteristics of nonlinear wave formation in these interactions. The present work obtains an approximate solution describing the behavior of a beam-plasma system in the nonlinear stage and shows the role played by the dispersion of this system in the nonlinear distortion of the wave profile.

We shall limit our analysis to the spatial evolution of a wave in a collisionless plasma consisting of electrons and a fixed ionic background. The electron beam passing through the plasma will be considered as velocity modulated at its entrance into the system:

$$v|_{z=0} = v_1 \sin \omega t, \quad j_b|_{z=0} = 0, \tag{1}$$

where v is the variable component of the beam velocity and j_b is the variable component of the beam current density. A unique description of the system can be obtained only if definite boundary conditions are imposed. The conditions (1) are not basic requirements of the nonlinearity here considered and were chosen only because they can be fulfilled easily in experiments. The electrons in both the plasma and the beam are assumed to be cold ($T_e = 0$).

From Maxwell's equation for an irrotational field of longitudinal oscillations

$$j_p + j_b + \frac{1}{4\pi}\frac{\partial E}{\partial t} = 0,$$

the expression for the plasma current density

$$j_p = \rho_p v_p$$

and the equation of motion of the plasma electrons

$$\frac{dv_p}{dt} = \frac{e}{m}E$$

in the approximation where the plasma oscillations (but not the beam oscillations) are linear we obtain an equation for the electric field E:

$$\frac{\partial^2 E}{\partial t^2} + \Omega^2 E = -4\pi\frac{\partial j_b}{\partial t}, \tag{2}$$

where Ω is the plasma frequency. We note that the indicated approximation is justified by the facts that the waves have a greater velocity than the plasma electrons and that the beam electron concentration can be chosen as much smaller than the plasma concentration.

We shall assume that the variable component of the beam current density can be expanded in a Fourier series of traveling waves:

$$j_b = \sum_{n=1}^{\infty} A_n(z)\sin n\omega\left(t - \frac{z}{v_0}\right) + \sum_{n=1}^{\infty} B_n(z)\cos n\omega\left(t - \frac{z}{v_0}\right), \tag{3}$$

where v_0 is the constant component of the beam velocity. From (2) we then obtain

$$E = 4\pi\sum_{n=1}^{\infty}\frac{n\omega}{\Omega^2 - n^2\omega^2}\left[B_n(z)\sin n\omega\left(t - \frac{z}{v_0}\right) - A_n(z)\cos n\omega\left(t - \frac{z}{v_0}\right)\right]. \tag{4}$$

The coefficients

$$A_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} j_b \sin n\omega\left(t - \frac{z}{v_0}\right)dt, \tag{5}$$

$$B_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} j_b \cos n\omega\left(t - \frac{z}{v_0}\right)dt$$

are obtained by using the continuity equation of the beam current density in the Lagrangian form

$$j_0 dt_0 = j dt, \tag{6}$$

where j_0 is the current density of the beam at its entrance into the system, j is the beam current density at a point z, t_0 is the time when an electron enters the system, and t is the time of its arrival at the point z. The variables t_0 and t are related by

$$t = t_0 + \tau = t_0 + \int_0^z \frac{dz}{v_0 + v(z, t_0) + v_1 \sin \omega t_0}$$

where τ is the time of electron flight from the point

$z = 0$ to a given point z; $v(z, t_0)$ is the variable component of the velocity resulting from the action of the plasma's electric field. We determine $v(z, t_0)$ from the expression for the increment of the electron's kinetic energy W when it proceeds from 0 to z:

$$\Delta W = m[v_0 + v_1 \sin \omega t_0 + v(z, t_0)]^2/2 - m(v_0 - v_1 \sin \omega t_0)^2/2 = e\int_0^z E(z, t_0)\,dz.$$

Assuming $v(z, t_0) + v_1 \ll v_0$, we obtain approximately

$$v(z, t_0) = \frac{e}{mv_0}\int_0^z E(z, t_0)\,dz,$$

$$t = t_0 + \frac{z}{v_0} - \frac{e}{mv_0^3}\int_0^z dz\int_0^z E(z, t_0)\,dz - \frac{v_1 z}{v_0^2}\sin \omega t_0. \qquad (7)$$

Using the notation

$$g(z, t_0) = \frac{e\omega}{mv_0^3}\int_0^z dz\int_0^z E(z, t_0)\,dz + \frac{v_1 z\omega}{v_0^2}\sin \omega t_0, \qquad (8)$$

from (5)–(7) we obtain

$$A_n = \frac{\omega j_0}{\pi}\int_0^{2\pi/\omega} \sin n[\omega t_0 - g(z, t_0)]\,dt_0, \qquad (9)$$

$$B_n = \frac{\omega j_0}{\pi}\int_0^{2\pi/\omega} \cos n[\omega t_0 - g(z, t_0)]\,dt_0.$$

By twice integrating both parts of (4) from 0 to z and using (7)–(9), we obtain the following equation for $g(z, t_0)$:

$$g = \frac{\omega_b^2\omega^3}{\pi v_0^2}\sum_{n=1}^{\infty}\frac{n}{\Omega^2 - n^2\omega^2}\cdot\int_0^z dz\int_0^z\cdot\Big[\sin n(\omega t_0 - g)\int_0^{2\pi/\omega}\cos n(\omega t_0 - g)\,dt_0$$

$$- \cos n(\omega t_0 - g)\int_0^{2\pi/\omega}\sin n(\omega t_0 - g)\,dt_0\Big]\,dz + \frac{v_1 z\omega}{v_0^2}\sin \omega t_0, \quad (10)$$

where $\omega_b = \sqrt{4\pi e^2 n_b/m}$ is the natural frequency of the beam. Knowledge of $g(z, t_0)$ enables us to determine the profiles and harmonics of the beam current density, the electric field, and the beam velocity, and also to determine the coordinate at which the "breaking" of the wave occurs. By using (7) and (8) to calculate the wave slope we obtain

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial t_0}\frac{\partial t_0}{\partial t} = \frac{\partial v/\partial t_0}{1 - \omega^{-1}\,\partial g/\partial t_0}. \qquad (11)$$

If at some distance the oscillatory amplitude of $\omega^{-1}\partial g/\partial t_0$ equals unity the beam velocity becomes discontinuous.

It is easily seen that the solution $g(z, t_0)$ of (10) is an odd function of t_0, so that the solution may be sought in the form of the Fourier series

$$g(z, t_0) = \sum_{n=1}^{\infty} X_n(z)\sin n\omega t_0.$$

However, our subsequent analysis will show that certain limitations will leave only the first term of this expansion in the desired solution. When we assume $X_1^{(0)}(z) \ll 1$, the solution of (10) in first approximation becomes

$$g^{(0)}(z, t_0) = X_1^{(0)}(z)\sin \omega t_0. \qquad (12)$$

Inserting (12) into (10) and expanding $\sin n(\omega t_0 - g)$ and $\cos n(\omega t_0 - g)$ in series of Bessel functions, to the first order in $X_1^{(0)}$ and far from higher harmonics $\Omega = n\omega$ $(n \neq 1)$ we obtain

$$X_1^{(0)} = \gamma_1^2\int_0^z dz\int_0^z X_1^{(0)}\,dz + \frac{v_1 z\omega}{v_0^2}, \qquad \gamma_1 = \frac{\omega_b}{v_0\sqrt{(\Omega/\omega)^2 - 1}}. \quad (13)$$

The solution of (13) is

$$X_1^{(0)} = \frac{\operatorname{sh}\gamma_1 z}{\gamma_1 S_k} \qquad (14)$$

where $S_k = v_0^2/v_1\omega$. For $\omega < \Omega$, $X_1^{(0)}$ increases exponentially with distance. We shall henceforth consider only this case of growing oscillations.

After inserting (12) and (14) into (10) we can evaluate the error incurred when this solution is used at distances where $X_1^{(0)}$ is not very small. When the modulation frequency lies within narrow limits near the plasma frequency such that $|a_2^2/a_1^2| \ll 1$, where $a_n^2 = [(\Omega/n\omega)^2 - 1]^{-1}$, the error is small even for $X_1^{(0)} = 1$. In the case the error is about $1/8$.

When the modulation frequency differs considerably from the plasma frequency, (12) and (14) become unsatisfactory, primarily because the second harmonic appears in $g(z, t_0)$. We therefore seek a more exact solution of (10) in the form

$$g^{(1)}(z, t_0) = X_1^{(1)}(z)\sin \omega t_0 + X_2^{(1)}(z)\sin 2\omega t_0, \qquad (15)$$

assuming $X_1^{(1)} \ll 1$, $X_2^{(1)} \ll 1$. We consider the region of modulation frequencies far from the resonances $\Omega = n\omega$ $(n \geq 3)$. Inserting (15) into (10) and neglecting terms of the orders $X_1^{(1)}X_2^{(1)}$ and $[X_2^{(1)}]$ as compared with $X_1^{(1)}$ and $X_2^{(1)} + X_1^{(1)}$, we obtain

$$X_1^{(1)} = \frac{v_1 z\omega}{v_0^2} + \gamma_1^2\int_0^z dz\int_0^z X_1^{(1)}\,dz,$$

$$X_2^{(1)} = \frac{\gamma_2^2 - \gamma_1^2}{2}\int_0^z dz\int_0^z [X_1^{(1)}]^2\,dz + \gamma_2^2\int_0^z dz\int_0^z X_2^{(1)}\,dz, \qquad (16)$$

where $\gamma_2 = \omega_b a_2/v_0$. The solution of (16) is

$$X_1^{(1)} = \frac{\operatorname{sh}\gamma_1 z}{\gamma_1 S_k},$$

$$X_2^{(1)} = \frac{\gamma_2^2 - \gamma_1^2}{4\gamma_1^2 S_k^2}\left[\frac{\operatorname{ch}2\gamma_1 z}{4\gamma_1^2 - \gamma_2^2} - \frac{4\gamma_1^2\operatorname{ch}\gamma_2 z}{\gamma_2^2(4\gamma_1^2 - \gamma_2^2)} + \frac{1}{\gamma_2^2}\right] \qquad (17)$$

At sufficiently large values of z the expression for $X_2^{(1)}$ retains only the most rapidly growing terms. We thus obtain

$$X_2^{(1)} \approx \frac{a_2^2 - a_1^2}{2(4a_1^2 - a_2^2)}[X_1^{(1)}]^2 \quad \text{for} \quad a_2^2 < 0, \qquad (18)$$

$$X_2^{(1)} \approx \frac{a_2^2 - a_1^2}{2(a_2^2 - 4a_1^2)}[X_1^{(1)}]^2\left[\frac{4a_1^2}{a_2^2}(2\gamma_1 S_k X_1^{(1)})^{\gamma_2/\gamma_1 - 2} - 1\right] \quad \text{for} \quad a_2^2 > 0. \qquad (19)$$

We have seen that (12) and (14), which were obtained for small $|g|$, comprise a satisfactory solution when $|g| \sim 1$ only within a narrow range of modulation frequencies, since this solution neglects the second harmonic in $g(z, t_0)$. The solution represented by (15) and (17) is shown to be sufficiently accurate for $|g| \sim 1$ in a considerably broader range that includes frequencies near $\omega = \Omega/2$. The entire analysis does not pertain, of course, to the cases of the resonances when $\omega = \Omega$ and $\Omega/2$.

When (15) and (17) as a solution for $g(z, t_0)$ are substituted into (9), the harmonics of beam current density and, correspondingly, of the electric field are expressed

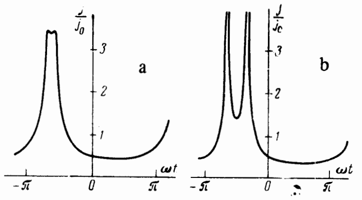

FIG. 1

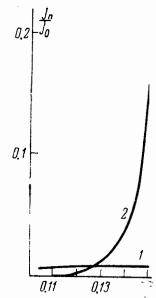

FIG. 2

FIG. 1. Wave profiles of beam current density for $X_1 = 1$: a—$\omega = 0.91\,\Omega$; b—$\omega = 0.6\Omega$.

FIG. 2. Dependences of the first (1) and second (2) harmonics of beam current density on the coordinate $L = \gamma_1 z/\text{arc sh}\,\gamma_1 S_k$ for $\omega = 0.45\Omega$, $\gamma_1 S_k = 100$.

through combinations of Bessel functions of the first kind having the arguments $nX_1^{(1)}$ and $nX_2^{(1)}$. By analyzing the spatial dependences of the amplitudes of these harmonics it is shown that when ω lies near the plasma frequency the growth of the wave of beam current density is accompanied by a strong distortion of its shape, but the electric field practically retains its sinusoidal time dependence. When the modulation frequency differs considerably from the plasma frequency the nonlinear distortion of the wave profile of beam current density is accompanied by distortion of the electric field wave profile. Figure 1 shows time-dependent wave profiles calculated on the basis of (6)–(8).

The character of the nonlinear distortion of the wave profiles has thus been shown to depend strongly on the modulation frequency. Nevertheless, dense electron bunches are always formed, with a variation of only their "fine" structure.

A new effect appears when the modulation frequency ω is somewhat smaller than $\Omega/2$ and therefore $\gamma_2^2 \gg \gamma_1^2 > 0$. Equation (19) then shows that, beginning at a certain distance, $X_2^{(1)}$ exceeds $X_1^{(1)}$ and therefore the amplitude of the second harmonic exceeds that of the first harmonic (Fig. 2). "Breaking" of the wave can therefore occur twice within a modulation period.

Equations (15) and (17) comprise a satisfactory approximation that describes the beam-plasma interaction up to a state which is close to the "breaking" of the wave. This is confirmed by a comparison of our results with experimental work[6,7] where it was shown that "breaking" of the wave occurs in the "meniscus," a distinct scattering zone whose coordinate is determined by the condition $X_1 \approx 1$. Here occurs the maximum of the probe-detected oscillations, which are strongly nonsinusoidal with a large harmonic component. As the modulation frequency is varied both the location of the "meniscus" and the external appearance of the scattering zone are changed. This effect may possibly be associated with the frequency dependence of the nonlinear wave profiles that has been observed in the present work. It should also be noted that the "breaking" of the wave which occurs twice during a modulation period—an effect that follows from our solution—has also been observed experimentally.

The author wishes to thank M. D. Gabovich and É. A. Pashitskiĭ for discussions of the results.

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Translated by I. Emin
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