THE EFFECT OF INITIAL CONDITIONS ON THE BEHAVIOR OF ELECTRON DENSITY INHOMOGENEITY IN A PLASMA

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The effect of the thermal motion of the electrons of an inhomogeneity and of the motion of the inhomogeneity as a whole on the behavior of the latter in a collisionless plasma is considered. It is shown that in a cold plasma the thermal motion of the electrons of an inhomogeneity leads to pulsations that are superimposed on the ordinary plasma oscillations. In a hot plasma the inhomogeneity partially spreads out and a stationary distribution of its density is asymptotically established. Correspondingly, a stationary macroscopic electric field arises, whose magnitude is proportional to the initial temperature of the inhomogeneity electrons. It is found that an inhomogeneity may move in the plasma as a whole only when a definite relation exists between its dimensions and the parameters of the plasma. At a sufficiently high initial velocity an inhomogeneity moves as a whole without exciting in the first approximation plasma oscillations and without spreading out. A small initial velocity leads only to oscillatory shifts of the density along the direction of the initial velocity. It is shown that an uncompensated electron (or ion) beam of low density cannot excite a beam instability.

1. INTRODUCTION

THE need to know the behavior of inhomogeneities in a plasma arises in many problems. Inhomogeneities play the principal role in a number of physical processes occurring in a cosmic plasma: in the ionosphere, in the interplanetary medium, in clouds of ionized interstellar gas, etc. They determine, for example, the process of scattering of radio waves and streams of charged particles, the evolution of meteor trails, the structure of ionospheric inhomogeneities, etc. (see, in particular, [1]). The character of the behavior of an inhomogeneity is important for investigations into the properties of laboratory and natural plasmas, for example, for the analysis of the possibility of an instability in the presence of a charged beam.

We discuss below the behavior of mild inhomogeneities, i.e., inhomogeneities whose densities are considerably less than the density of the plasma (the background).

An inhomogeneity will behave in different ways, depending on whether it was initially charged or whether it was quasineutral. The behavior of a charged inhomogeneity in the case when the electron collision frequency ν_e is much higher than the plasma frequency ω_e ($\nu_e \gg \omega_e$), was considered, in particular, by A. V. Gurevich.[2] Here a rapid resorption of the uncompensated charge takes place and a quasineutral state is established which is an initial condition for ambipolar diffusion. The later process proceeds at a considerably slower rate and leads to a complete spreading of the inhomogeneity. On the whole the rate of the two processes is determined by the collision frequency.

The picture is totally different when $\omega_e \gg \nu_e$. The problem of the behavior of an electron inhomogeneity of dimensions x_0 greater than the Debye radius r_d ($x_0^2 \gg r_d^2$) was first considered for this case by Vlasov.[3] He deduced on the basis of the dispersion properties of the medium alone that an inhomogeneity oscillating with frequency ω_e completely diffuses out in space with a characteristic time $\tau_V = \omega_e^{-1}(x_0/r_d)^2$ (the time for doubling the size of an inhomogeneity). The spreading takes place owing to the emission of plasma waves which carry away energy of the electric field of the initial charge.

We show in the present paper that important features of the behavior of an inhomogeneity in the collisionless mode are determined not only by the dispersion properties of the medium, but by the initial conditions characterizing the inhomogeneity itself: its temperature, dimensions, and velocity of motion. Thus, the thermal motion of the electrons of the inhomogeneity leads, in addition to the oscillations, to pulsations of the density of the inhomogeneity according to the law $\sin^2(\omega_e t/2)$. A hot inhomogeneity does not diffuse out completely: a stationary distribution of the charge, whose magnitude is proportional to the initial temperature of the inhomogeneity, is established asymptotically in time. The motion of the inhomogeneity as a whole is then impossible at velocities considerably less than $\omega_e x_0$. If, however, the initial velocity is appreciably larger than $\omega_e x_0$ then the inhomogeneity continues to move with this velocity. In this case the inhomogeneity does not, in the first approximation, diffuse out and no oscillations in its density occur. Analysis of this situation shows the impossibility of a beam instability in a plasma with charged beams.

2. THE GENERAL SOLUTION OF THE PROBLEM WITH INITIAL CONDITIONS

The behavior of a mild inhomogeneity in the collisionless mode may be described by a collisionless kinetic equation with a self-consistent field. Let us consider the one-dimensional problem whose solution corresponds to the situation along the magnetic field. We have in this case

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial f_0}{\partial v}=0, \qquad (1)$$

$$\frac{\partial^2\varphi}{\partial x^2} = -4\pi e \int f\,dv, \qquad (2)$$

where

$$\int f_0(v)\,dv = n_0, \qquad \int f(v,x,t)\,dv = n(x,t),$$

φ is the potential of the electric field of the inhomogeneity, f(x, v, t) the distribution function of the electrons of the inhomogeneity, and $f_0(v)$ the equilibrium distribution function.

It is convenient to seek the solution of Eqs. (1) and (2) with the given initial conditions by the method of the L-transformation (the Laplace transformation with respect to time). Applying again an F-transformation (Fourier transformation) with respect to the coordinate x, we find for the LF-transform of the density of an inhomogeneity (see [4])

$$n_{kp} = n_k\left[1 - \frac{4\pi e^2 n_0 i}{km}\int_{-\infty}^{\infty}\frac{df_0}{dv}\frac{dv}{p+ikv}\right]^{-1}\int\frac{g(v)\,dv}{p+ikv}, \qquad (3)$$

where n_k is the F-transform of the initial density of the inhomogeneity, g(v) the initial velocity distribution of the electrons of the inhomogeneity normalized to unity, and f_0 is also here normalized to unity.

We shall, for simplicity, assume that the initial distribution n(x, 0) is symmetric about x = 0 and decreases sufficiently slowly from this point. Let x_0 be then a characteristic dimension of the inhomogeneity. Accordingly, n_k is a sufficiently rapidly decreasing function of k, so that on performing the F^{-1}-transformation of the expression (3) we have the main contribution to the Fourier integral being made by the small values of k in the region $k < k_0$, where $k_0 \sim x_0^{-1}$.

The assumed restrictions on the form of the initial distribution are not very stringent. They allow the consideration of a broad class of initial conditions which are of physical interest. At the same time, under the assumptions made with respect to n(x, 0), the analysis gets simplified considerably since it is then sufficient to consider in formula (4) (see below) only the region of small k.

3. ESTABLISHMENT OF A STATIONARY DISTRIBUTION OF CHARGE

In what follows we shall, for definiteness, understand by f_0 a Maxwellian distribution function with dispersion v_T. Let g(v) also be a Maxwellian distribution function with dispersion $\tilde{v}_T$.

Let us expand the integrand in (3) in a series in powers of kv/p and let us limit ourselves to the first two nonvanishing terms. Carrying out the integration with respect to v, we obtain

$$n_{kp} = \frac{n_k}{p}\left(1 - \frac{k^2\tilde{v}_T^2}{p^2}\right)\left[1 + \frac{\omega_e^2}{p^2}\left(1 - 3\frac{k^2 v_T^2}{p^2}\right)\right]^{-1}, \qquad (4)$$

where $\omega_e^2 = 4\pi e^2 n_0/m$. The solution (4) is clearly valid for $k^2 \ll (\omega_e/v_T)$, $(\omega_e/\tilde{v}_T)^2$.[1)] By taking into account what we said at the end of the last section, we can write these conditions in the form $\epsilon \equiv (r_d/x_0)^2 \ll 1$, $\tilde{\epsilon} \equiv (\tilde{r}_d/x_0)^2 \ll 1$ where $\tilde{r}_d = \tilde{v}_T/\omega$. Notice that exactly the same limitations on the shape of the inhomogeneity were used in [3].

In a cold plasma, i.e., for $v_T = 0$, we have from (4)

$$n(x^*,t^*) = n(x^*,0)\cos t^* + 2\tilde{\varepsilon}\sin^2\left(\frac{t^*}{2}\right)\frac{d^2n(x^*,0)}{dx^{*2}}, \qquad (5)$$

where $x^* = x/x_0$, $t^* = \omega_e t$.

We see from this that the thermal motion of the electrons of an inhomogeneity leads only to additional pulsations, but not to the escape of these electrons from the initial volume.

Let us now determine the asymptotic behavior of $n(x^*, t^*)$ for $v_T \neq 0$. After the L^{-1}-transformation of the formula (4), in the F^{-1}-transformation appears an integral of the type

$$\int_{-\infty}^{\infty} e^{-ik^*x^*} n_{k^*}\cos\left[\left(1+\frac{3}{2}\varepsilon k^{*2}\right)t^*\right]dk^*,$$

where $k^* = kx_0$. At large t^* ($t^* \gg \epsilon$) the behavior of this integral is determined, in the main, by the rapidly oscillating function $\cos[(1 + \frac{3}{2}\epsilon k^{*2})t^*]$. The function n_{k^*}, on the other hand, can, in the first approximation, be assumed to be constant, setting, say, $n_{k^*} = \frac{1}{2}n^0 x_0$, where $n^0 = n(0, 0)$, and can be taken outside the integration sign.

Bearing this in mind, we obtain from formula (4) for $\tilde{v}_T = 0$

$$n(x^*,t^*) = n^0\sqrt{\frac{\pi}{12\varepsilon t^*}}\left\{\cos t^*\left(\cos\frac{x^{*2}\varepsilon}{4t^*} + \sin\frac{x^{*2}\varepsilon}{4t^*}\right) - \sin t^*\left(\cos\frac{x^{*2}\varepsilon}{4t^*} - \sin\frac{x^{*2}\varepsilon}{4t^*}\right)\right\}. \qquad (6)$$

This result attests to the fact that asymptotically a cold inhomogeneity completely diffuses out in a hot plasma with a characteristic time $t^* \sim \epsilon$. This agrees with Vlasov's result[3] cited above. The density of the inhomogeneity decreases as $\sqrt{t^*}$.

Allowance for the thermal motion of the inhomogeneity electrons leads in the F^{-1}-transformation of (4) to an additional term of the form

$$\int_{-\infty}^{\infty} e^{-ik^*x^*}k^{*2}n_{k^*}\sin^2\left[\left(1+\frac{3}{2}k^{*2}\varepsilon\right)\frac{t^*}{2}\right]dk^*.$$

Let us apply the preceding arguments to this integral, using the formula $2\sin^2(\alpha/2) - \cos\alpha$. As a result, we obtain an expression for $n(x^*, t^*)$ consisting of two parts: one part asymptotically decays like (6), while the second is not dependent upon t^*. And the density after the time determined by the condition $\sqrt{t^*} \gg (\epsilon)^{-3/2}$ will be given by the time independent expression

$$n(x^*, t^*\to\infty) = -\frac{\tilde{\varepsilon}}{2}\frac{d^2n(x^*,0)}{dx^{*2}}. \qquad (7)$$

Corresponding to this

$$\varphi(x^*, t^*\to\infty) = -2\pi e\tilde{r}_d^2 n(x^*,0). \qquad (8)$$

Thus, a hot inhomogeneity does not spread completely. The density and the electric field in the stationary state being established are proportional to the initial temperature of the inhomogeneity.

Obviously, the distribution (7) will subsequently spread under the action of collisions and, in this sense, it is quasistationary.

1) We should bear in mind that (4) has practically only two poles $p_{1,2} \approx \pm(\omega_e + \frac{3}{2}kv_T)$ (see [3,4]).

4. A MOVING INHOMOGENEITY

Let an inhomogeneity with a Maxwellian velocity distribution have along the x-axis an initial velocity v_0:

$$g(v)=\frac{1}{\tilde{v}_T\sqrt{2\pi}}\exp\left\{-\frac{(v-v_0)^2}{2\tilde{v}_T^2}\right\}.$$

Expanding again the integrand in (3) in powers of kv/p and restricting ourselves to the first two terms, we obtain after integrating with respect to v

$$n_{kp}=n_k p^2[(p+ikv_0)(p^2+\omega_e^2)]^{-1}. \tag{9}$$

The L^{-1}-transform of formula (9) yields

$$n_\varkappa(t^*)=\frac{n_\varkappa}{\varkappa^2-1}(\varkappa^2 e^{i\varkappa t^*}+i\varkappa\sin t^*-\cos t^*), \tag{10}$$

where $\kappa = kv_0/\omega_e$.

If $\eta \equiv v_0/x_0\omega_e \ll 1$, then we should expand (10) in a series in powers of κ. Limiting ourselves to the first two terms of the series and carrying out the F^{-1}-transformation, we obtain

$$n(x^*,t^*)=n(x^*,0)\cos t^*+\eta\frac{dn(x^*,0)}{dx^*}\sin t^*. \tag{11}$$

We see from this that the inhomogeneity does not move: the existence of an initial velocity leads only to oscillatory density shifts about the point x = 0.

In the opposite case when $\eta \gg 1$, we have by neglecting in (10) terms of the order of unity in comparison with κ, and after performing the F^{-1}-transformation,

$$n(x^*,t^*)=n(x^*-\eta t^*)-\eta^{-1}\sin t^*\int_0^{x^*}n(x^*,0)dx^*. \tag{12}$$

Thus an inhomogeneity may move in a plasma only with a sufficiently high velocity which depends on the dimensions of the inhomogeneity. In contrast to a stationary inhomogeneity, a moving inhomogeneity does not excite plasma oscillations and its density in the system of coordinates attached to it does not, in the first approximation, vary in time.

The expressions (11) and (12) have been obtained with no allowance for the thermal motion of the electrons of the inhomogeneity and the background. It can be shown, in complete analogy to the preceding section, that the thermal motion of the particles of the background leads, for $\eta \ll 1$, asymptotically to a stationary state similar to the expressions (7) and (8). For $\eta \gg 1$ the temperature of the background does not, in the first approximation in κ, influence the motion of the inhomogeneity, i.e., the first term in (12) does not change while the second term decays asymptotically. Thus, the motion of an initial distribution of charge with an initial velocity is asymptotically stationary.

5. THE QUESTION OF THE STABILITY OF A PLASMA WITH CHARGED BEAMS

Conditions often arise when an uncompensated beam of charged particles appears in a plasma. The beam may be introduced into the plasma artificially (electron beams in electron-beam amplifiers, ion beams in traps), may appear under the action of external fields ("runaway" electrons); the beams develop in certain processes in cosmic plasmas, etc.

At present beam instability due to quasineutral beams have been well investigated (see, for example, [5-7]). Can charged beams be called an instability of the same type?

The system plasma plus charged beam is not in the general case stationary to the same degree as the system plasma plus quasineutral beam. Therefore, the generally employed scheme of the theory of small perturbations is not, generally speaking, applicable to it. However, if it turns out that the characteristic times of the nonstationary processes are much larger than the time necessary for the development of an instability of the initial states, then this scheme may be applied, assuming that the initial state is, in the first approximation, stationary.

It was shown above that the motion of a charged beam is possible only if the dimensions in the direction of motion are bounded. Let us assume that the time taken by the beam to cross the volume it occupies is much larger than the time of development of an instability, i.e., $t = x_0/v_0 \gg \gamma^{-1}$ where γ is the negative damping of the instability. This, in our case, is the required condition for stationarity. On the other hand, a moving beam exists only at a velocity $v_0 \gg x_0\omega_e$. So, in order that a charged beam may give rise to an instability, it is necessary that the two conditions

$$v_0 \ll x_0\gamma \text{ and } v_0 \gg x_0\omega_e. \tag{13}$$

be simultaneously satisfied. Assuming that the first of the conditions of (13) is fulfilled, we obtain for γ the well-known expressions (see [5-7]) from which follows the inequality $\gamma \ll \omega_e$. As a result, we see that the conditions (13) are incompatible. Physically, this means that a charged beam can move in a plasma only so rapidly that it does not have time to excite an instability.

This is valid not only for electron beams, but also for ion beams. Thus, weak charged beams in a plasma do not lead to an instability.

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