BREAKING OF A SIMPLE WAVE IN THE KINETICS OF A RAREFIED PLASMA

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Stability of self-similar solutions of the kinetic equation in a rarefied quasineutral plasma is investigated. The existence of a new branch of undamped ion-acoustic oscillations related to the specific form of the self-similar distribution function is demonstrated. The behavior of the solution in the vicinity of a singularity arising on breaking of a simple wave front is investigated. It is shown that if the finiteness of the Debye radius is taken into account an oscillatory structure develops in the plasma behind the overturning point. A class of distribution functions is indicated for which stationary solitary waves (solitons) exist even if allowance is made for the thermal motion of the ions. In this case an equation of the Korteweg-de Vries is valid near the breaking point.

1. INTRODUCTION

In a preceding paper[1] we found a rather extensive class of exact solutions of the collisionless kinetic equation for a quasineutral plasma. This class is analogous to simple waves in ordinary hydrodynamics. It was noted in[1] that the obtained solutions have, generally speaking, a singular point with respect to the time. When the solution is formally continued beyond this point, the distribution function becomes a multiply-valued function of x and t. The purpose of the present paper is to investigate the solution near the singular point.

In hydrodynamics, this point is the point where the front of a simple wave breaks. Beyond this point, the flow tends to become one with multiple velocities. The density becomes in this case a multiply-valued function of x. In single-velocity hydrodynamics, such solutions are impossible. Therefore a strong discontinuity arises beyond the singular point and a shock wave is produced. In kinetics, however, there is an essential departure from hydrodynamics. First, the kinetic theory has in essence a multiple-velocity character. Therefore the tendency to produce a multiple-velocity flow beyond the singular point means here only the occurrence of a region where the particle velocity distribution function experiences a rapid change, and no fundamental difficulties arise here. Moreover, even if regions where f is not uniquely dependent on x and v arise, the distribution function can be imparted a perfectly defined meaning. (This circumstance was first noted by Stoker[2]).

Let us consider for concreteness a case of importance for the following exposition, when the distribution function f becomes a multiply-valued function of the coordinate x. In other words, if the plot of f(x) has the form 1a at the initial instant, it acquires the form 1b in the course of time. To explain the physical meaning of the "multiply-valued" plot b we note that the area bounded by the curve f(x) is the total number of particles $N = \int f dx$. It is easy to prove that this is valid also for the area bounded by the "multiply-valued" curve b, since the area is in general constant in time. But then the number of particles in the interval dx is, on the one hand, by definition f(x)dx, and on the other hand it is the area of the shaded strips in Fig. 1b. We see therefore that

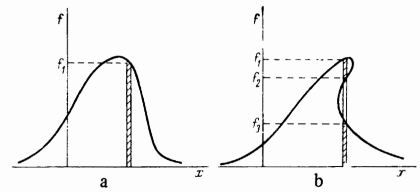

FIG. 1

$$f(x) = f_1(x) + f_3(x) - f_2(x), \tag{1}$$

where f_1 and f_3 are the largest and smallest values of the distribution function in the region where the distribution function is triply-valued, and f_2 is the intermediate value. The same rule applies also when f is a multiply-valued function not only of x but also of v. (To prove this it is necessary to consider the volume bounded by the surface x = f(x, v).)

We now proceed to our immediate problem—the investigation of a simple wave near the turning point. We recall first some information from earlier papers[1,3,4].

The equation for the dimensionless ion distribution function is[1)]

$$\frac{\partial g}{\partial t}+u\frac{\partial g}{\partial \xi}-\frac{1}{2}\frac{\partial g}{\partial u}\frac{\partial \psi}{\partial \xi}=0. \tag{2}$$

Here

$$\xi = x\left(\frac{M}{2T_e}\right)^{1/2}, \quad u = v\left(\frac{M}{2T_e}\right)^{1/2}, \quad \psi = e\varphi/T_e, \quad g = \frac{f}{N_0}\left(\frac{2\pi T_e}{M}\right)^{1/2}.$$

In the quasineutral case, for electrons having a Boltzmann distribution,

$$\psi = \ln\frac{N_i}{N_0} = \ln\left(\int_{-\infty}^{\infty} g\frac{du}{\sqrt{\pi}}\right) = \ln n. \tag{3}$$

It is shown in[1] that the system (2)–(3) has a solution in the form

$$g = g_a[u, \tau(\xi, t)], \tag{4}$$

where τ is connected with ξ and t by the relation

[1)]For an explanation of the obvious notation see [1].

$$\xi = \tau t + p(\tau), \tag{5}$$

$p(\tau)$ is an arbitrary function, and $g_a(\tau)$ satisfies the equation

$$(u-\tau)\frac{\partial g_a}{\partial \tau} - \frac{1}{2}\frac{\partial g_a}{\partial u}\frac{d}{d\tau}\left(\ln \int g_a \frac{du}{\sqrt{\pi}}\right) = 0. \tag{6}$$

Dividing this equation by $u - \tau$ and integrating with respect to du, we obtain the important identity

$$n_a = \int g_a \frac{du}{\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}} \int \frac{1}{u-\tau}\frac{\partial g_a}{\partial u} du. \tag{7}$$

The solution of (6) is determined by the boundary values at $\tau \to \pm\infty$:

$$g_a \to \begin{cases} g_{\mathrm{I}}(u), & \tau \to -\infty, \\ g_{\mathrm{II}}(u), & \tau \to +\infty. \end{cases}$$

In what follows, a uniquely distinguished case is one in which one of the functions, for example g_{II}, is equal to zero. The corresponding solution $g_a(u, \tau)$ at $\tau = \xi/t$ describes self-similar expansion of plasma in vacuum. It was investigated in detail in[3]. The solution for $g_{\mathrm{I,II}} \neq 0$ (flow of a plasma into a plasma) was considered in[4].

The density n depends only on the parameter τ, which is defined by relation (5). Thus, the dependence of n on x and t is exactly the same as for a simple wave in hydrodynamics (see[5], Sec. 94). It is known that in the course of time the front of the simple wave in hydrodynamics becomes steeper and steeper and at a certain instant the wave breaks. At this instant a singular point ($|\partial n/\partial x| \to \infty$) appears in the density distribution. This singular point arises in the solution of (4) and (5) at the value $\tau = \tau_0$, at which $p''(\tau_0) = 0$, at the instant of time $t_0 = -p'(\tau_0)$. The solution near this point is

$$g \approx g_a(\tau_0) + \tau^* \frac{dg_a}{d\tau}(\tau_0), \quad n \approx n_a(\tau_0) + \tau^* \frac{dn_a}{d\tau}(\tau_0), \tag{8}$$

where $\tau^* = \tau - \tau_0$ satisfies the approximate equation

$$\begin{gathered} \zeta = \xi^* - \tau_0 t^* = \tau^* t^* - \alpha\tau^{*3}, \\ t^* = t - t_0, \quad \xi^* = \xi - \xi_0, \quad \xi_0 = t_0\tau_0 + p(\tau_0), \\ \alpha = -{}^1/_6 p'''(\tau_0). \end{gathered} \tag{9}$$

It is seen directly from (9) that at $t^* > 0$ the solution becomes multiply-valued—the same values of ξ^* and t^* correspond to three values of τ^*. One must not think, however, that at $t^* > 0$ it is possible simply to use formula (1). The point is that by virtue of the nonlinearity of Eq. (2), the sum (1) will no longer be a solution, even if each of the terms satisfies the equation. The solution at $t^* > 0$ will no longer have the character of a simple wave and the continuation of the solution beyond this point is a separate problem. We shall show later, however, that in some cases formula (1) gives a correct approximation of the solution. Another difficulty which we encounter is that the plasma can no longer be regarded as quasineutral near the singular point. Indeed, it is seen from (9) that at $t^* = 0$ we have $\tau^* \sim \xi^{*1/3}$, so that $|\partial n/\partial \xi| \sim \xi^{*-2/3} \to \infty$. Therefore at small ξ^* and t^* one can no longer neglect the second derivatives in the Poisson equation and use the quasi-neutral approximation for the potential. In other words, it is necessary here to take into account the finite length of the Debye radius r_D. The appearance in the equations of a new parameter with the dimension of length, r_D, is the reason why oscillating density distributions can be produced beyond the point where the wave breaks.

On the other hand, in the vicinity of the singular point we can use the simplifying circumstance that the distribution function is close to $g_a(u, \tau_0)$, and small deviations from this value move with a velocity close to τ_0 (see (9)). Therefore it is very important to investigate the dynamics of such perturbations. This problem, as will be shown below, is of independent interest in connection with the problem of the stability of self-similar solutions[6].

2. PROPAGATION OF SMALL DEVIATIONS FROM A SELF-SIMILAR DISTRIBUTION FUNCTION

In accordance with the foregoing, we put

$$g(u, x, t) = g_a(u, \tau_0) + g_1(x, u, t) \equiv g_a^0(u) + g_1 \tag{10}$$

and, assuming that $|g_1| \ll g_a^0$, we linearize the system (2)—(3) relative to g_1. We obtain the linear equation

$$\frac{\partial g_1}{\partial t} + u\frac{\partial g_1}{\partial \xi} - \frac{1}{2}\frac{\partial g_a^0}{\partial u}\frac{\partial \psi_1}{\partial \xi} = 0, \tag{11}$$

$$\psi_1 = \frac{1}{n_a^0}\int g_1 \frac{du}{\sqrt{\pi}} = \frac{n_1}{n_a^0}, \quad n_a^0 = n_a(\tau_0). \tag{12}$$

We now seek a solution in the form of plane waves[2)]

$$g_1 \sim \exp(i[k\xi - \omega t]). \tag{13}$$

Proceeding in the usual manner, we obtain the dispersion equation

$$n_a^0 = \frac{1}{2\sqrt{\pi}}\int \frac{k}{ku - \omega - i\delta}\frac{\partial g_a^0}{\partial u} du, \quad \delta \to +0,$$

or, introducing the phase velocity of the wave $s = \omega/k$,

$$\varepsilon^0(s, k) = -\frac{1}{2\sqrt{\pi}n_a^0}\int \frac{\partial g_a^0}{\partial u}\frac{du}{u - s - i\sigma}, \quad \sigma = \delta k, \quad \delta \to +0. \tag{14}$$

(Actually, $\epsilon^0(s, k)$ depends only on the sign and not on the value of k.)

Comparing this equation with the identity (7), we see that (14) will be satisfied if we substitute τ_0 for s. In other words, in a plasma described by a distribution function $g_a(u, \tau_0)$ there is an undamped branch of ion sound propagating with a velocity equal to τ_0. We emphasize that this, generally speaking, is precisely a new branch. It exists even when the distribution function is close to Maxwellian almost everywhere (as is the case for expansion in vacuum at $\tau_0 \ll -1$). In this case the usual branch, as is well known, has a damping on the order of unity.

The result has a simple physical meaning. Indeed, if we subject the boundary functions g_{I} and g_{II} in (7) to an arbitrarily small change, then we obtain a solution in which the particle density differs from the initial one by an arbitrary small function $n_1(x/t)$. Any such function should therefore be a solution of the system (11)—(12). On the other hand, such a function satisfies the equation

2)Of course, the representation of g in the form (10) is meaningful only at t and ξ close to t_0 and ξ_0. We can therefore consider in any case only sufficiently large k and ω. It is necessary to have $k\xi_0 \gg 1$ and $\omega t_0 \gg 1$.

$$t\frac{\partial n_1}{\partial t}=x\frac{\partial n_1}{\partial x} \quad \text{or} \quad \frac{\partial n_1}{\partial t}=\tau\frac{\partial n_1}{\partial x}, \tag{15}$$

so that it can be interpreted as a density wave propagating with velocity τ. We note also that in Eq. (7) the rule of circuiting is immaterial, since[3)]

$$\partial g_a / \partial u = 0 \text{ at } u=\tau. \tag{16}$$

This equation has different meanings in different problems, If we are dealing with expansion in vacuum, then the distribution function is equal to zero below the separatrix, which passes above the line $u = \tau$ everywhere in the (u, τ) plane[3]. In this case, (16) is satisfied identically in a finite velocity region near $u = \tau$. In all the remaining cases g_a has at $u = \tau$ either a maximum or a minimum. In the concrete example of plasma flow into a plasma, considered in[4], g_a has at $u = \tau$ a maximum for $\tau < \tau_1 \approx 0.28$ and a minimum for $\tau > \tau_1$[4)].

So far we have written the dispersion equation using the quasi-neutrality condition (12), a procedure which is correct in the limit as $k \to 0$. To take deviations from quasineutrality into account, we replace (12) by the Poisson equation

$$\frac{\partial^2\psi}{\partial\xi^2}=-\frac{8\pi N_0 e^2}{M}(n-e^{\psi})$$

or

$$a^2\frac{\partial^2\psi_1}{\partial\xi^2}=e^{\psi_1}-1-\frac{n_1}{n_a^0}, \tag{17}$$

where $a^2 = M/8\pi N_0 e^2 n_a^0$, $n_a^0 = n_a(\tau_0)$, $n_1 = n - n_a^0$. Linearizing, we have

$$a^2\frac{\partial^2\psi_1}{\partial\xi^2}=\psi_1-\frac{n_1}{n_a^0}, \tag{18}$$

$(2a^2)^{-1}$ is the plasma frequency of the ions at the point τ_0.

From (11) and (18) we get the dispersion equation

$$\varepsilon(s,k)=1+a^2k^2-\frac{1}{2\sqrt{\pi}\,n_a^0}\int\frac{\partial g_a^0}{\partial u}\frac{du}{u-s-i\sigma}=1+a^2k^2$$
$$-\frac{1}{2\sqrt{\pi}\,n_a^0}\left[\int\frac{\partial g_a^0}{\partial u}\frac{du}{u-s}+\pi i\left(\frac{\partial g_a^0}{\partial u}\right)_{u=s}\frac{k}{|k|}\right]=0. \tag{19}$$

At small k, the root of (19) is close to τ_0. Introducing

$$s^*=s-\tau_0$$

and expanding in terms of s^*, we obtain

$$\varepsilon(s,k)=(ak)^2-\left(\mu+i\gamma\frac{k}{|k|}\right)s^*, \tag{20}$$

where[5)]

$$\mu=\frac{1}{2\sqrt{\pi}\,n_a^0}\int\frac{\partial g_a^0}{\partial u}\frac{du}{(u-\tau_0)^2} \tag{21}$$

$$\gamma=\frac{\sqrt{\pi}}{2n_a^0}\left(\frac{\partial^2 g_a}{\partial u^2}\right)_{u=\tau_0}$$

[3)]Actually, according to formula (16) of [3], the equation of the characteristics for $u \approx \tau$ has the form $(u-\tau)^2 = F(\bar\tau)(\tau-\bar\tau)$ ($\bar\tau$ is constant), and therefore

$$\frac{\partial g_a}{\partial u}=\frac{dg_a}{d\bar\tau}\frac{\partial\bar\tau}{\partial u}\approx\frac{dg_a}{d\bar\tau}\frac{2(u-\tau)}{F(\bar\tau)}\to 0,$$

as $u \to \tau$.

[4)]τ_1 is the value of τ at which the characteristic $u_0 = 0$ crosses the line $u = \tau$ (see [4], Fig. 2).

[5)]We recall that by virtue of (16) the integrand in (21) has a pole of only first order.

Equating (20) to zero, we obtain

$$s^*=\frac{(ak)^2}{\mu+i\gamma k/|k|},$$

$$\omega=k\tau_0+\frac{(ka)^2}{\mu^2+\gamma^2}(\mu k-i\gamma|k|).$$

We see therefore that the sign of γ coincides with the sign of $(\partial^2 g_a/\partial u^2)_{u=\tau_0}$. But a negative sign of γ denotes instability. Therefore the self-similar solutions in question are stable if the distribution function has a minimum at $u = \tau$, and unstable if g_a has a maximum at $u = \tau$.

If g_a corresponds to expansion in vacuum, then $g_a = 0$ at $u \approx \tau$, $\gamma \equiv 0$, and we can omit the principal-value symbol in μ and integrate by parts, so that

$$\mu=\frac{1}{\sqrt{\pi}\,n_a^0}\int g_a^0\frac{du}{(u-\tau_0)^3}, \quad \mu>0.$$

We note also that a gap of finite width Δ is produced between the speed of sound τ_0 and the separatrix in the problem involving expansion of plasma in a vacuum. The distribution function g_a below the separatrix is identically equal to zero. The presence of the gap Δ leads to definite singularities in the development of small perturbations. The most important of them is the possible existence of stationary solitary waves—solitons. It is known[7] that solitons exist in a plasma with cold ions, $T_i = 0$. Such a plasma is described by the equations of isothermal hydrodynamics[6]. In kinetics, the ion temperature is finite, $T_i \neq 0$, and solutions of the type of stationary solitary waves are generally speaking impossible because some of the ions are reflected from the soliton. In our case, however, the distribution function has an important singularity—it is identically equal to zero in the region of finite width Δ in the vicinity of the speed of sound. Because of this, not only are waves of infinitesimally small amplitude undamped, but perturbations of finite amplitude, propagating with velocity close to that of sound, likewise do not reflect ions. This means in fact that solutions of the type of solitary waves can exist. Indeed, we shall find stationary solutions for the kinetic equation for ions (2) jointly with the Poisson equation for the field (17). From (2) we have

$$g(u,\xi)=g_0[(u^2-\psi)^{1/2}],$$

where in our case $g_0 = g_a(\tau_0, u)$. It follows therefore that the stationary density of the ions in a coordinate system moving together with the wave is

$$n=\frac{1}{\sqrt{\pi}}\int_{\tau_0}^{\infty}\frac{g_a(\tau_0,u)\,du}{[1-\psi_1/(u-\tau_0+u_0)^2]^{1/2}}. \tag{22}$$

Here u_0 is the velocity of the stationary wave under consideration relative to the speed of sound τ_0 ($u_0 > 0$ denotes that the wave moves in a direction opposite to the growth of τ), and ψ_1 is the potential of the wave relative to the potential $\psi_a(\tau_0)$.

Substituting (22) in the Poisson equation (17), we obtain

$$\frac{d^2\psi_1}{d\zeta^2}=e^{\psi_1}-\frac{1}{\sqrt{\pi}\,n_a^0}\int_{\tau_0+\Delta}^{\infty}\frac{g_a(\tau_0,u)\,du}{[1-\psi_1/(u-\tau_0+u_0)^2]^{1/2}}.$$

Here $\zeta = \xi/a$ and $\tau_0 + \Delta$ is the separatrix ($g_a \equiv 0$ at

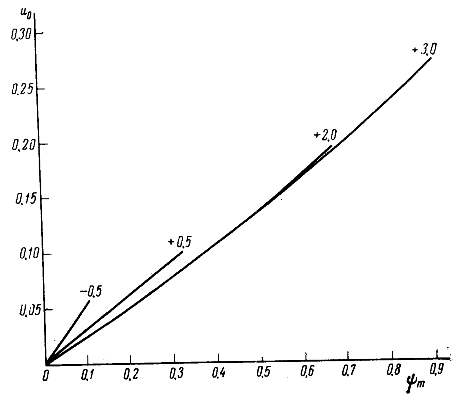

FIG. 2

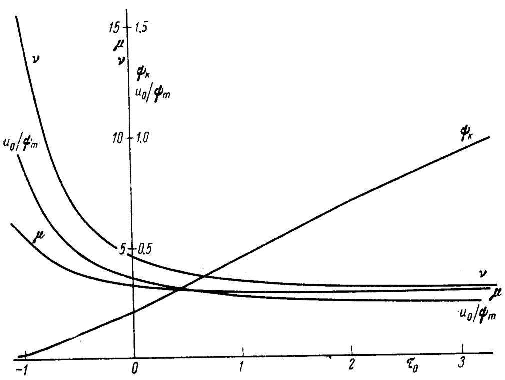

FIG. 3

$u < \tau_0 + \Delta$). Integrating this equation once, we rewrite it in the form

$$\frac{1}{2}\left(\frac{d\psi_1}{d\zeta}\right)^2 = e^{\psi_1} + \frac{2}{\sqrt{\pi}\, n_a^0}\int\limits_{\tau_0+\Delta}^{\infty}(u-\tau_0+u_0)^2\left[1-\frac{\psi_1}{(u-\tau_0+u_0)^2}\right]^{1/2} \times g_a(\tau_0, u)\,du + C. \qquad (23)$$

We seek a solution in the form of a solitary wave. In this case $d\psi_1/d\zeta \to 0$ as $\psi_1 \to 0$. It then follows from (23) that

$$C = -1 - \frac{2}{\sqrt{\pi}\, n_a^0}\int\limits_{\tau_0+\Delta}^{\infty} g_a(\tau_0, u)(u-\tau_0+u_0)^2 du.$$

In addition, $\partial\psi_1/\partial\zeta = 0$ at the maximum of the solitary wave, i.e., at $\psi_1 = \psi_m$:

$$e^{\psi_m} - 1 + \frac{2}{\sqrt{\pi}\, n_a^0} \times \int\limits_{\tau_0+\Delta}^{\infty}(u-\tau_0+u_0)^2\left[\left(1-\frac{\psi_1}{(u-\tau_0+u_0)^2}\right)^{1/2} - 1\right] g_a(\tau_0, u)\,du = 0. \qquad (24)$$

This condition determines the dependence of the velocity of the solitary wave u_0 on its amplitude ψ_m. The result of a numerical solution of Eq. (24) is shown in Fig. 2 for different values of τ_0 for g_a corresponding to flow of an isothermal plasma with $T_i = T_e$ into vacuum. We see that at small soliton amplitudes $\psi_m \to 0$, the velocity $u_0 \to 0$, i.e., as $\psi_m \to 0$ the soliton moves with the velocity of undamped sound τ_0. The velocity $u_0 > 0$, i.e., the soliton always moves towards decreasing values of τ. $\psi_m = \psi_k = (u_{0k} + \Delta)^2$—is the maximum possible amplitude of the soliton. When $\psi > \psi_k$, the reflection of the ions from the soliton begins and the soliton "breaks." The dependence of the maximum soliton amplitude ψ_k on τ_0 is shown in Fig. 3. At large positive values $\tau_0 \gg 1$ we have $\psi_k \to 1.3$, just as in hydrodynamics[7]. This should be the case, for when $\tau_0 \gg 1$ the ion distribution function degenerates into a δ function[3]. With decreasing τ_0, the value of ψ_k decreases rapidly. At negative τ_0 the gap width Δ is small. In this case $\psi_k \approx \Delta^2$, i.e., there exist only solitons of very small amplitude.

If the soliton amplitude is small, $\psi_m \ll \psi_k$, then $u_0 \ll \Delta$, and, expanding the exponential and the radicand of (24) in a series, we can readily find a solution of (24) in the form

$$u_0 = \frac{1}{3}\frac{\nu-1}{\mu}\psi_m,$$

$$\mu = \frac{1}{\sqrt{\pi}\, n_a^0}\int\limits_{\tau_0+\Delta}^{\infty}\frac{g_a(\tau_0, u)}{(u-\tau_0)^3}du, \qquad \nu = \frac{3}{4\sqrt{\pi}\, n_a^0}\int\limits_{\tau_0+\Delta}^{\infty}\frac{g_a(\tau_0, u)}{(u-\tau_0)^4}du.$$

The parameters μ and ν and the ratio u_0/ψ_m as a function of τ_0 are shown in Fig. 3[6)]

3. BREAKING OF THE FRONT WHEN PLASMA FLOWS INTO A PLASMA

In order to take into account the influence of the finite nature of the Debye radius on the properties of the solution near the singular point, we shall use the linearized system of equations (11) and (18). We have to find for this system a solution that goes over far from the singular point into the usual simple wave (4)–(5)[7)].

To find this solution, we note that at $t = 0$ the simple wave satisfies the initial condition

$$g(u, \xi, t=0) = g_a[u, \bar{\tau}(\xi)],$$

where $\bar{\tau}(\xi)$ is a function inverse to $p(\tau)$:

$$\xi = p(\bar{\tau}).$$

Accordingly, we shall solve the system (11) and (18) with the initial condition

$$g_1(u, \xi, t=0) = g_a[u, \bar{\tau}(\xi)] - g_a^0(u) \equiv \bar{g}_1(u, \xi). \qquad (25)$$

The obtained solution will first follow (4) and (5), but near the singular point the left-hand side of (18), i.e., the finite character of the Debye radius, will come into play.

The general solution of the linear problem with initial conditions, as is well known, was obtained by Landau[9]. To simplify the notation, we shall use, unlike[9], not the Laplace transformation, but the unilateral Fourier transformation, introducing the corre-

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6)We note that in the present kinetic analysis of the solitary waves we have made essential use only of the fact that the self-similar distribution function of the ions is cut off in the vicinity of the speed of sound. This analysis is therefore valid not only for self-similar but also for arbitrary cut-off distribution functions. Such distribution functions arise in reality, for example, in Q-machines [8].

7)At the same time we must be sufficiently close to the point τ_0 to make the linearization meaningful. It may seem strange that the solution of the linear system (11) and (18) has a point where the front breaks, while in hydrodynamics such a phenomenon is connected with nonlinearity. It must be borne in mind, however, that linearization of the kinetic equation has a meaning different from the linearization of the equations of hydrodynamics. Thus, the linear kinetic equation for free motion of particles $\partial f/\partial t + v\partial x/\partial x = 0$ is equivalent at $f = \delta(v - v_1)$ to the nonlinear hydrodynamic equation $\partial v_1/\partial t + v_1\partial v_1/\partial x = 0$.

sponding components by means of the formulas (see, for example,[6])

$$g_{k\omega}(u)=\int_0^\infty dt\int_{-\infty}^{\infty} d\xi g_1(u,\xi,t)\,e^{i(\omega t-k\xi)},$$

$$\psi_{k\omega}=\int_0^\infty dt\int_{-\infty}^{\infty} d\xi \psi_1(\xi,t)\,e^{i(\omega t-k\xi)},\qquad \omega\to\omega+i\delta.$$

The inverse transformation can be written in the form

$$\psi_1(\xi,t)=\frac{1}{(2\pi)^2}\int dk\,d\omega\,\psi_{k\omega}e^{-i(\omega t-k\xi)}.$$

We obtain in place of (11) and (18)

$$i(ku-\omega)g_{k\omega}-\frac{1}{2}ik\psi_{k\omega}\frac{\partial g_a^0}{\partial u}=\bar{g}_k(u),$$
$$-(ka)^2\psi_{k\omega}=\psi_{k\omega}-\frac{1}{n_a^0}\int g_{k\omega}\frac{du}{\sqrt{\pi}},\tag{26}$$

where $\bar{g}_k(u)$ is the three-dimensional Fourier transform of the initial function (25). From (26) we obtain

$$\psi_{k\omega}=\frac{1}{\varepsilon(k,\omega/k)}\frac{1}{i\sqrt{\pi}}\int\frac{\bar{g}_k du}{ku-\omega-i\delta}.\tag{27}$$

where ϵ is the "dispersion" function (19). If we put a = 0, i.e., we return to the quasineutral case, then we obtain

$$\psi_{k\omega}^{(H)}=\frac{1}{\varepsilon^0(k,\omega/k)}\frac{1}{i\sqrt{\pi}}\int\frac{\bar{g}_k\,du}{ku-\omega-i\delta}.\tag{28}$$

Taking the inverse Fourier transform, we can show by means of straight-forward but rather cumbersome calculations that the solution in (ξ, t)-space has in this case the form

$$\psi^{(H)}(\xi,t)=\psi_1^{(a)}(\tau)\qquad\text{for } t<t_0,\tag{29a}$$

$$\psi^{(H)}(\xi,t)=\psi_1^{(a)}(\tau_1)-\psi_1^{(a)}(\tau_2)+\psi_1^{(a)}(\tau_3)\qquad\text{for } t>t_0\tag{29b}$$

(here $\psi_1^{(a)}=\psi_a(\tau)-\psi_a(\tau_0)$, and τ_1, τ_2, and τ_3 are three successive roots of Eq. (5) at $t>t_0$) in accordance with the general result (1). This could be expected beforehand, since by virtue of the linearity of the system (11) and (18), the sum (29b) is an exact solution of this system if each of the terms is a solution. When the nonlinearity is taken into account, this, naturally, will no longer be the case.

Comparing (27) and (28), we obtain the connection between $\psi_{k\omega}$ and $\psi_{k\omega}^{(H)}$:

$$\psi_{k\omega}=\frac{\varepsilon^0(k,\omega/k)}{\varepsilon(k,\omega/k)}\psi_{k\omega}^{(H)}.\tag{30}$$

Let us calculate $\psi_{k\omega}^{(H)}$. We have

$$\psi_{k\omega}^{(H)}=\int_0^\infty dt e^{i\omega t}\int_{-\infty}^{\infty}\psi^{(H)}(\xi,t)e^{-ik\xi}d\xi.\tag{31}$$

The calculation of the integral with respect to ξ in (31) is made difficult by the fact that $\psi^{(H)}(\xi)$ is multiply-valued at $t>t_0$. This difficulty, however, can be circumvented by integrating by parts and changing over from integration with respect to ξ to integration with respect to τ. This simplifies the matter, since ξ and ψ are single-valued functions of τ. It is important to emphasize, however, that integration by parts can be carried out only when ψ in the region of ambiguity is taken to have the meaning given by (29b) (or, what is the same, by (1)[8)]). Therefore the formulas obtained below already contain automatically the correct choice of the roots. Thus, we have

$$\psi_{k\omega}^{(H)}=-\frac{1}{ik}\int_0^\infty dte^{i\omega t}\int\psi^{(H)}(\xi,t)\,de^{-ik\xi}=\frac{1}{ik}\int_0^\infty dte^{i\omega t}\int e^{-ik\xi(\tau)}\,d\psi^{(H)}(\tau)$$
$$=\frac{1}{ik}\int_0^\infty dte^{i\omega t}\int_{-\infty}^{\infty}e^{-ik[\tau t+p(\tau)]}\frac{d\psi_1^{(a)}}{d\tau}d\tau=-\frac{1}{ik}\int_{-\infty}^{\infty}d\tau\frac{d\psi_1^{(a)}}{d\tau}\frac{e^{-ikp(\tau)}}{i(\omega-k\tau+i\delta)}.\tag{32}$$

Substituting (32) in (30) and taking the inverse Fourier transformation, we obtain after differentiation with respect to ξ

$$\frac{\partial\psi_1(\xi,t)}{\partial\xi}=-\frac{1}{(2\pi)^2}\int e^{i[k\xi-kp(\tau)-\omega\tau]}\frac{d\psi_1^{(a)}}{d\tau}\frac{\varepsilon^0(k,\omega/k)}{\varepsilon(k,\omega/k)}\frac{dk\,d\omega\,d\tau}{i(\omega-k\tau+i\delta)}$$
$$=-\frac{1}{(2\pi)^2}\int ds\,d\tau\int dke^{ik[\xi-p(\tau)-st]}\frac{|k|}{i(ks-k\tau+i\delta)}\frac{\varepsilon^0(k,s)}{\varepsilon(k,s)}\frac{d\psi_1^{(a)}}{d\tau}$$
$$=-\operatorname{Re}\frac{1}{2\pi^2}\int ds\,d\tau\int_0^\infty dk\frac{1}{i(s-\tau+i\delta)}\frac{\varepsilon^0(s)}{\varepsilon(k,s)}\frac{d\psi_1^{(a)}}{d\tau}$$
$$=-\operatorname{Re}\frac{1}{2\pi^2}\int dk\,d\tau I\frac{d\psi_1^{(a)}}{d\tau},\tag{33}$$

where $\epsilon^0(s)$ denotes $\epsilon^0(k, s)$ at $k>0$. According to (14), ϵ^0 depends only on the sign of k. Further calculations cannot be carried out in general form, and we must use the circumstance that the values of s and τ close to τ_0 are significant near the singularity in the integrals. Using (20), we have

$$\frac{\varepsilon^0(s)}{\varepsilon(k,s)}\approx\frac{s^*}{s^*-rk^2},\qquad r=\frac{a^2}{\mu+i\gamma},\tag{34}$$

so that

$$I=e^{ik[\xi-p(\tau)]}\int_{-\infty}^{\infty}ds\frac{e^{-ikst}}{i(s-\tau+i\delta)}\frac{\varepsilon^0(s)}{\varepsilon(k,s)}\approx\exp[ik(\xi^*-\tau_0t^*$$
$$+\alpha\tau^{*3}+t_0\tau^*)]\int_{-\infty}^{\infty}\frac{ds^*s^*e^{-iks^*t}}{i(s^*-\tau^*+i\delta)(s^*-rk^2)}.$$

Closing the contour of integration with respect to s* in the lower half-plane, we obtain

$$I=-2\pi\frac{\exp[ik(\xi^*-\tau_0t^*-t^*\tau^*+\alpha\tau^{*3})]}{\tau^*-rk^2}$$
$$\times\{\tau^*-rk^2\exp[-ik(rk^2-\tau^*)(t_0+t^*)]\}.\tag{35}$$

The second term in the curly brackets in (35) depends explicitly on t_0—the spill-over time, reckoned from the instant of time at which the initial conditions (25) are specified. For large t_0 this term is small and can be omitted. Indeed, by virtue of the proposed stability, Im r < 0 and the second term decreases exponentially with t_0. Substituting I in (33), we get the final answer:

$$\frac{\partial\psi_1}{\partial\xi}=\operatorname{Re}\frac{1}{\pi}\frac{d\psi_a}{d\tau}(\tau_0)\int_{-\infty}^{\infty}d\tau^*\int_0^\infty dk\frac{\tau^*\exp[ik(\zeta-t^*\tau^*+\alpha\tau^{*3})]}{\tau^*-rk^2},$$
$$\zeta=\xi^*-\tau_0t^*.\tag{36}$$

[8)]Indeed, as explained in Sec. 1, the choice of the roots in accordance with (1) means that S = ∫fdx is the area bounded by the curve f(x) (even in the region where the function f is multiply valued). But the same area can be represented in the form S = ∫xdf, which is equivalent to integration by parts.

We have replaced, with the required accuracy, $d\psi_a(\tau)/d\tau$ by $d\psi_a(\tau_0)/d\tau$ in (36). Formula (36), derived for the case $\gamma > 0$, is meaningful also in the case of an unstable system $\gamma < 0$. The integral with respect to τ^* in (36) should be taken in any case along a contour circuiting the pole $\tau^* = rk^2$ from above. Physically, the answer in this case corresponds to an experimental setup in which the perturbations are so small that the instability does not have time to develop before the instant $t = t_0$. The singularity at the point $t = t_0$ is, however, a strong perturbation that leads to an intense growth of the oscillations at $t > t_0$.

The integral in (36), generally speaking, must be calculated numerically. It is easily seen that the term rk^2 in the denominator leads to a smearing of the singularity at $t^* = 0$, $\xi^* = 0$ and to the appearance of oscillations. When $t^* > 0$ these oscillations grow in the unstable case and attenuate in the stable one.

The asymptotic behavior of $\partial\psi_1/\partial\xi$ at large t^* turns out to be quite complicated in the general case of complex r. In the next section we shall investigate this asymptotic behavior for the particular case when the imaginary part of r is small. We confine ourselves here to estimating the field intensity at $\xi^* = t^* = 0$. From (36) we find readily that in this case

$$\frac{\partial\psi_1}{\partial\xi} \sim \frac{1}{|r|^{2/7}} \sim \frac{1}{r_D^{4/7}},$$

where r_D is the Debye radius in the plasma. In the quasineutral limit $r_D \to 0$ the intensity $\partial\psi_1/\partial\xi$ tends, as it should, to infinity at this point.

4. ASYMPTOTIC BEHAVIOR OF THE POTENTIAL

In the present section we investigate the asymptotic behavior of the integral (36) at large t_0. As already mentioned, we shall assume here that the imaginary part of r is small:

$$|\operatorname{Im} r| \ll |r|. \tag{37}$$

This can mean either that the breaking of the front occurs near a point where $(\partial^2 g_a/\partial u^2)_{u=\tau_0} = 0$ or that when plasma flows into a plasma the concentration of the particles on one end is much smaller than on the other, $g_I \gg g_{II}$[9] Assumption (37) is used by us only to simplify the formulas; in principle it is possible to carry out a sufficiently complete investigation also without this assumption.

Under the condition (37), r can be taken in the sense of

$$r - i\delta, \quad \delta \to +0, \tag{38}$$

where r is a real number. For concreteness, we assume that $r > 0$, as is usually the case. We shall show first of all that for real r the integral (36) can be reduced to a single integral. To this end we turn the contour of integration with respect to k about the imaginary axis, i.e., we put

$$k = \pm i\varkappa,$$

With the sign coinciding with that of the expression

$$S = \zeta - t^*\tau^* + \alpha\tau^{*3}.$$

[9]It should be borne in mind, however, that this assumption cannot be used for the problem in which plasma escapes into vacuum (see the next section).

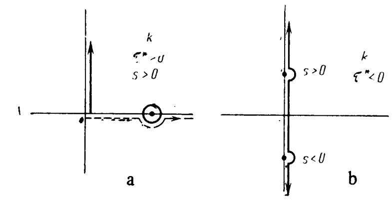

FIG. 4

When turning the contour, we must pay attention to the positions of the poles in k, and in particular when $s > 0$ and $\tau^* > 0$ it is necessary to add an integral along an infinitesimally small contour. The position of the poles and the new integration contour in the k plane are shown in Fig. 4. Contributions are then made only by the residue of the pole at $\tau^* > 0$ and the circuits around the pole at $\tau^* < 0$. The integral along the imaginary axis is, on the other hand, pure imaginary, and therefore makes no contribution. As a result

$$E = \partial\psi_1/\partial\xi = E_1 + E_2, \tag{39}$$

$$E_1 = \frac{E_a}{2}\int_{-\infty}^{0}\left(\frac{|\tau^*|}{r}\right)^{1/2}\exp\left[-\left(\frac{|\tau^*|}{r}\right)^{1/2}|S|\right]d\tau^*,$$

$$E_2 = -E_a \operatorname{Re} i \int_{\substack{0 \\ S(\tau^*)>0}} \left(\frac{\tau^*}{r}\right)^{1/2}\exp\left[i\left(\frac{\tau^*}{r}\right)^{1/2}S\right]d\tau^*, \tag{40}$$

$$S = \zeta - t^*\tau^* + \alpha\tau^{*3},$$

$$E_a = \left(\frac{d\psi_a}{d\tau}\right)_{\tau=\tau_0} = \frac{1}{n_a(\tau_0)}\left(\frac{dn_a}{d\tau}\right)_{\tau=\tau_0}.$$

We introduce the new variables

$$|\tau^*| = \left(\frac{t^*}{\alpha}\right)^{1/2}y^2, \quad z = \frac{\zeta\alpha^{1/2}}{t^{*3/2}} = \frac{(\xi^* - \tau_0 t^*)\alpha^{1/2}}{t^{*3/2}}, \quad l = t^{*7/4}\alpha^{-3/4}r^{-1/2}. \tag{41}$$

Then

$$E_1 = E_a\frac{t^{*3/4}}{\alpha^{3/4}r^{1/2}}\int_0^\infty y^2\exp[-l|Q(y)|]\,dy,$$

$$Q(y) = zy + y^3 - y^7,$$

$$E_2 = -2E_a\frac{t^{*3/4}}{\alpha^{3/4}r^{1/2}}\operatorname{Re} i\int_{\substack{0 \\ P(y)>0}}^{\infty} y^2\exp[ilP(y)]\,dy,$$

$$P(y) = zy - y^3 + y^7,$$

Let us investigate the asymptotic behavior of E at large positive values of t^*, or more accurately under the condition $l \gg 1$, i.e., $t^* \gg \alpha^{3/7}r^{2/7}$.

The main contribution to E_1 and E_2 is then made by the singular points, at which P and Q, respectively, vanish, and the saddle points, in which the derivative $\partial P/\partial y$ or $\partial Q/\partial y$ vanishes. The contribution of the roots of P and Q can be readily obtained. Expanding the exponential about these points and integrating, we readily reduce E to the form

$$\frac{\partial\psi_1}{\partial\xi} \approx -\frac{d\psi_a}{d\tau}\sum_i\left|\frac{\partial\tau_i^*}{\partial\xi}\right|, \tag{42}$$

where τ_i^* are the roots of the equation

$$\zeta = \tau^*t^* - \alpha\tau^{*3},$$

which coincides, with the required degree of accuracy, with the quasineutral solution (29b). The latter is shown in Fig. 5. Curve 1 gives the course of the potential ψ, and curve 2 the field intensity $E(-E_a/t^*)^{-1}$.

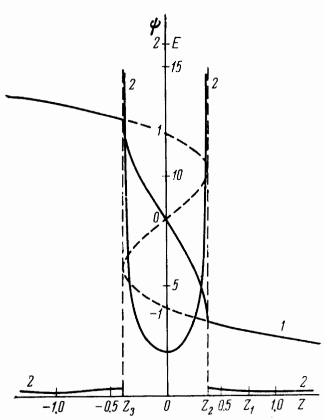

FIG. 5

With increasing t^* (i.e., with increasing l), the field intensity decreases like $1/t^* \sim 1/l^{4/7}$.

In the region "behind" the spill-over, with $z < 0$, E_2 has no saddle points. The contribution of these points to E_1, generally speaking, is exponentially small. A singular situation takes place near the point $z_3 = -\frac{2}{3}\sqrt{3}$. Here two roots of $Q(y)$ coalesce, corresponding, as seen from Fig. 5, to the limit of the region where the solution is multiply valued. The quasineutral formula (42) ceases to hold here, since $\partial\tau_i/\partial\xi \to \infty$ as $z \to z_3$. Expanding the exponent in (41) with accuracy to terms of second order and integrating, we obtain

$$E_1 \approx \frac{-E_a}{3^{3/8}t^{1/8}r^{1/4}\alpha^{3/8}} R_1\left(\sqrt{\frac{|u|l}{3^{1/4}}}\right),$$

$$R_1(x) = \begin{cases} e^{x^2}\int\limits_x e^{-v^2}\,dv + e^{-x^2}\int\limits_0 e^{v^2}\,dv, & u > 0 \\ \frac{1}{2}\sqrt{\pi}e^{-x^2}, & u < 0 \end{cases} \tag{43}$$

$$u = z - z_3.$$

Thus, near the point $z = z_3$ there is a maximum of the field intensity. Its height is finite and not infinite as in the quasineutral theory, but decreases quite slowly, like $t^{*-1/8}$. The width of the maximum is $\Delta\zeta \sim \alpha^{1/4}r^{1/2}t^{*-1/4}$.

When $z > 0$, of course, a contribution is made to E by the roots of the polynomial P in accordance with (42). But besides this "smooth" contribution there is also a contribution from the saddle points of the exponential in E_2 in (41). This contribution oscillates like a function of t^* and ξ^*, and must be taken into account separately. The saddle points of the integral E_2 are determined by the conditions

$$\frac{\partial P}{\partial y} = 0, \quad y > 0, \quad P > 0. \tag{44}$$

A simple investigation shows that these conditions are compatible in the region of values of z

$$0 \leqslant z \leqslant z_1, \quad z_1 = 2/\sqrt{7}. \tag{45}$$

In this entire region of values of z, the exponent of P has a maximum at a certain value $y = y_M(z)$. Expanding $P(y)$ near the maximum and calculating the integral, we find the corresponding contribution to E_2,

$$E_M = -E_a \frac{2\sqrt{2\pi}y_M^2}{\alpha^{3/8}r^{1/4}t^{1/8}|P_M''|^{1/2}} \cos\{lP_M + \varphi_M\}, \tag{46}$$

where

$$P_M = P[z, y_M(z)], \quad P_M'' = \left(\frac{\partial^2 P}{\partial y^2}\right)_{y_M}, \quad \varphi_M = \frac{\pi}{4}.$$

In the range of values of z

$$z_2 \leqslant z \leqslant z_1, \quad z_1 = 2/\sqrt{7}, \quad z_2 = 2/3\sqrt{3}. \tag{47}$$

$P(y)$ also has a minimum at $y = y_m(z)$. (In the remaining part of the region (45) the minimum lies at negative values of $P(y)$.) The contribution of this minimum to E differs from (46) in that M is replaced by m, with $\varphi_m = 3\pi/4$.

The derived formulas cease to hold when $z \to z_1$, where the maximum approaches the minimum. In this case E_2 can be expressed in terms of the Airy function. We present this formula without derivation:

$$E_2 \approx \frac{2^{4/3}\pi^{1/2}}{7^{1/2}3^{1/3}} \sin\left(\frac{8l}{7^{7/4}}\right) \cdot \Phi\left(\frac{l^{2/3}u}{3^{1/3}4^{1/3}}\right) \frac{E_a t^{*1/6}}{\alpha^{1/2}r^{1/3}}, \tag{48}$$

where $\Phi(x)$ is the Airy function defined in accordance with[10] (formula (c, 3)). Finally, when $z \approx z_2$ $(z < z_2)$ there is a contribution connected with the fact that the roots and the minimum of P coincide. The corresponding formula is

$$E_2 \approx -\frac{2E_a}{3^{3/8}t^{*1/8}r^{1/4}\alpha^{3/8}} R_2\left(\sqrt{\frac{|u|l}{3^{1/4}}}\right), \quad R_2(x) = \operatorname{Re} ie^{-ix^2}\int\limits_x^\infty e^{iv^2}\,dv.$$

Thus, after the breaking of the front, oscillations are produced in the plasma and move forward from the spill-over point $z = 0$ $(\zeta = 0)$. The amplitude of the oscillations increases with increasing ζ. It is maximal in the vicinity of the point $z = z_1$. There are no oscillations beyond this point. The dimension of the region occupied by the oscillations is $\Delta\zeta \sim t^{*3/2}$. The amplitude of the oscillations in the region $0 < z < z_1$ decreases slowly with time, like $t^{*-1/8}$, and is the larger, the smaller the Debye radius $(\sim r_D^{-1/4})$. The order of magnitude of the wave vector in this region is

$$k \sim t^{*1/4} / \alpha^{1/4}r^{1/2},$$

i.e., it increases with increasing t^*. The oscillations constitute a sum of two deformed sinusoids with variable amplitudes and wave vectors.

In a small vicinity near the singular point $z = z_1$ (with width $\Delta\zeta \sim t^{*1/3}$), the amplitude of the oscillations even increases with the time, like $t^{*1/6}$. This is seen from formula (48). It is interesting to note that in the quasineutral solution there is nothing to distinguish the point $z = z_1$. In essence, this is the caustic point for the resultant oscillations, beyond which they cannot pass. We note also that in accordance with (48) the maximum amplitude itself oscillates rapidly with time, as a result of beats between the aforementioned sinusoids[10)].

The results of a numerical tabulation of formulas (41) are shown in Figs. 6a–e. We see that up to the breaking point $(t^* < 0)$ the intensity of the electric field is a smooth function of ζ and increases as this point is approached. Beyond the point $t^* = 0$, in the region $\zeta > 0$, an oscillatory structure develops gradually. We see that the amplitude of the oscillations

[10)]In order for the employed approximations to be valid, it is necessary that t^* be small, $t^* \ll \alpha$. It turns out here that $kr^{1/2} \ll 1$, as was assumed in the derivation of (34).

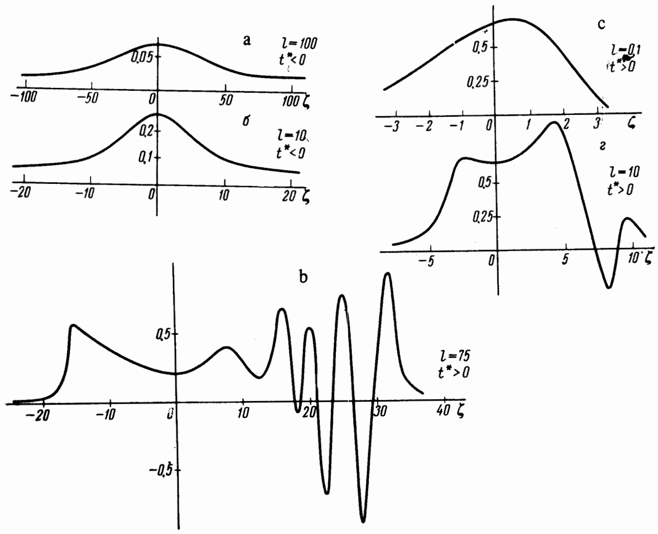

FIG. 6

is maximal near the point z_1 (Fig. 6e). The oscillations terminate abruptly beyond this point.

In the derivation of the obtained formulas we have assumed r to be real. It can be shown, however, that they remain valid in the main also if r has an imaginary part. In this case l likewise acquires an imaginary part, so that the oscillations attenuate or grow exponentially, depending on the sign of γ.

5. NONLINEAR EFFECTS. THE KORTEWEG-DE VRIES EQUATION FOR EXPANSION IN VACUUM

So far we have solved the problem in the linear approximation, i.e., we took into account only the influence of small spatial dispersion on the rate of propagation of the perturbations. We shall now take into account the influence of the nonlinear effects. This influence is particularly significant for a problem such as expansion in vacuum, where the effect is equal to zero in the linear approximation.

The problem is to solve the system (2)—(17) with accuracy up to terms of second order in the amplitude. From (17) we obtain at this accuracy

$$\frac{n_1}{n_a^0} = \psi_1 + \frac{\psi_1^2}{2} - a^2\frac{\partial^2\psi_1}{\partial\xi^2}. \tag{49}$$

We rewrite Eq. (2) and separate explicitly the terms of first and second order:

$$\frac{\partial g_1}{\partial t} + u\frac{\partial g_1}{\partial\xi} - \frac{1}{2}\frac{\partial g_a^0}{\partial u}\frac{\partial\psi_1}{\partial\xi} = \frac{1}{2}\frac{\partial g_1}{\partial u}\frac{\partial\psi_1}{\partial\xi} \equiv A. \tag{50}$$

We apply the Fourier transformation

$$i(ku-\omega)g_{\omega k} - \frac{1}{2}\frac{\partial g_a^0}{\partial u}ik\psi_{\omega k} = A_{\omega k} + \bar{g}_k,$$

$$A_{\omega k} = \frac{1}{2}\left(\frac{\partial g_1}{\partial u}\frac{\partial\psi_1}{\partial\xi}\right)_{\omega k}.$$

We divide by $i(ku-\omega)$ and integrate with respect to $du/\sqrt{\pi}$. We obtain

$$n_{\omega k} - \int\frac{k}{i(ku-\omega)}\frac{\partial g_a^0}{\partial u}\psi_{\omega k}\frac{du}{\sqrt{\pi}} = \int\frac{A_{\omega k}+\bar{g}_k}{i(ku-\omega)}\frac{du}{\sqrt{\pi}}. \tag{51}$$

We shall henceforth deal with problems of the type of expansion in vacuum, where the distribution function is equal to zero in a finite vicinity of the point $u = \tau$. It is easy to show that in this case it is necessary to set equal to zero the term containing the initial distribution function

$$n^0 = \int\frac{\bar{g}_k}{i(ku-\omega)}\frac{du}{\sqrt{\pi}}.$$

Indeed, this term is simply the Fourier component of the concentrations of the particles that would have a distribution function $\bar{g}$ at $t = 0$, and move freely from then on. The distribution function of such particles satisfies the equation

$$(u-\tau)\,\partial g^{(0)}/\partial\tau = 0$$

or

$$\partial g^{(0)}/\partial\tau = B(u)\,\delta(u-\tau)$$

(see[11], formula (27)). Since the function $g^{(0)}$ should vanish at $u \approx \tau$, we see therefore that $\partial g^{(0)}/\partial\tau = 0$, $dn^{(0)}/d\tau = 0$. Taking this circumstance into account and substituting (49) in (51), we have

$$\varepsilon\psi_{\omega k} = \int\frac{A_{\omega k}}{i(ku-\omega)}\frac{du}{\sqrt{\pi}} - \frac{1}{2}(\psi_1^2)_{\omega k}. \tag{52}$$

We now recognize that we are considering perturbations moving with a velocity close to τ_0, and can therefore regard $s^* = s - \tau$ as a small quantity, on a par with the amplitude. We transform the term $A_{\omega k}$. In the zeroth approximation we have

$$g_{\omega k} = \frac{ik}{2i(ku-\omega)}\frac{\partial g_a}{\partial u}\psi_{\omega k} \approx \frac{1}{2(u-\tau_0)}\frac{\partial g_a}{\partial u}\psi_{\omega k},$$

so that

$$g_1 \approx \frac{1}{2(u-\tau_0)}\frac{\partial g_a}{\partial u}\psi_1.$$

Here

$$A \approx \frac{1}{8}\frac{\partial\psi_1^2}{\partial\xi}\frac{\partial}{\partial u}\left(\frac{\partial g_a/\partial u}{u-\tau_0}\right),$$

and Eq. (52) takes the form

$$\varepsilon(k,s)\psi_{\omega k} = \frac{\nu-1}{2}(\psi^2)_{\omega k}, \tag{53}$$

$$\nu = \frac{3}{4\sqrt{\pi}\,n_a}\int\frac{g_a}{(u-\tau_0)^4}\,du. \tag{54}$$

We now use the approximate expression (20) for ϵ and change over to the coordinate system moving with velocity τ_0. In this coordinate system we have

$$-\frac{\partial}{\partial t} \to i\omega^* = i(\omega - k\tau_0), \qquad \frac{\partial}{\partial\zeta^*} \to ik,$$

so that $s^* = \omega^*/k$. We obtain

$$\left[(ak)^2 - \mu\frac{\omega^*}{k}\right]\psi_{\omega k} = \frac{\nu-1}{2}(\psi^2)_{\omega k}$$

or, taking the inverse Fourier transforms,

$$\mu\frac{\partial\psi_1}{\partial t^*} - a^2\frac{\partial^3\psi_1}{\partial\zeta^{*3}} = [\nu-1]\psi_1\frac{\partial\psi_1}{\partial\zeta^*}. \tag{55}$$

The stationary solutions of this equation, of the type of solitary waves $\psi_1 = \psi_1(\zeta^* - u_0t^*)$, naturally coincide with those considered in Sec. 2.

We note that the integral μ can be expressed in terms of

$$E_a = \frac{d\psi_a}{d\tau}(\tau = \tau_0).$$

To this end, we differentiate (7) with respect to τ. We have

$$\frac{dn_a}{d\tau}=\frac{1}{2}\int\frac{1}{(u-\tau)^2}\frac{\partial g_a}{\partial u}\frac{du}{\sqrt{\pi}}+\frac{1}{2}\int\frac{1}{(u-\tau)}\frac{\partial^2 g_a}{\partial u\partial\tau}\frac{du}{\sqrt{\pi}},$$

or, expressing $\partial^2 g_a/\partial u\partial\tau$ from (6) and putting $\tau=\tau_0$, we obtain

$$E_a(1-\nu)=\mu,\quad E_a=\left(\frac{1}{n_a}\frac{dn_a}{d\tau}\right)_{\tau=\tau_0}, \tag{56}$$

so that finally the equation for ψ_1 takes the form

$$E_a\frac{\partial\psi_1}{\partial t^*}-\frac{a^2E_a}{\mu}\frac{\partial^3\psi_1}{\partial\zeta^{*3}}+\psi_1\frac{\partial\psi_1}{\partial\zeta^*}=0. \tag{57}$$

This is the Korteweg-de Vries equation. It was investigated in detail in a number of papers (see, for example,[12-15] and the review[16]). As $a^2\to 0$, this equation reduces to

$$E_a\frac{\partial\psi_1}{\partial t^*}+\psi_1\frac{\partial\psi_1}{\partial\zeta^*}=0,$$

which, in particular, has a solution of the type

$$\psi_1=E_a\tau^*,\quad \zeta^*=\tau^*t^*-a\tau^{*3}, \tag{58}$$

which coincides with (8) and (9).

The general problem is to find a solution (57) that goes over into (58) as $\zeta^*\to\pm\infty$, i.e., that behaves like

$$\psi_1=E_a\tau^*,\quad \tau^*\to(\zeta^*/a)^{1/3},\quad \zeta^*\to\pm\infty. \tag{59}$$

No such solution has been obtained as yet. Investigations carried out in[12-15] show that, qualitatively, the solution should have at $\zeta^*<0$ a tail consisting of individual "soliton" pulses, the existence of which is connected essentially with the nonlinearity. Forward, on the other hand, there should exist a region of oscillations that are little influenced by the nonlinearity and whose properties should probably be close to those of the oscillations obtained in Sec. 4.

In conclusion we note that, as seen from (57), the smaller E_a the more important the nonlinearity.

We note also that with decreasing E_a the gap between the line $u=\tau$ and the separatrix also decreases. In this case the soliton can fall in the velocity region where there are particles, and then break in turn. The result is a nonstationary picture similar to that investigated in the numerical calculations of Alikhanov, Sagdeev, and Chebotarev[17] for hydrodynamics ($T_i=0$).

In the case when $\tau_0\gg 1$ we have

$$\mu=2\sqrt{2},\quad \nu=3, \tag{60}$$

and Eq. (55) coincides identically with the equation for a plasma with cold ions ($T_i=0$), described by isothermal hydrodynamics[6,16]. This is as it should be, for when $\tau_0\gg 1$ the self-similar distribution functions of the ions are close to δ functions. It is important that the Korteweg-de Vries equation (55) obtained here does not presuppose in any manner that the distribution functions of the ions are close to δ functions. Therefore the constants ν and μ can differ from (60) by as much as desired (see Fig. 3). In the derivation of (55), the expansion is actually in powers of ψ_1/ψ_k, where $\psi_k=(u_{0k}+\Delta)^2$ is the critical amplitude of the soliton (see Sec. 2)[11)].

[11)] We note that the same equation can also be obtained with other (not self-similar) ion distribution functions. It is only necessary that these functions be cut off, that the speed of sound c fall in a region where $f_i\equiv 0$, and that a finite gap exist between c and $u_k(f_i(u_k)\neq 0)$.

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234