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THE MODEL OF "MIXMASTER UNIVERSE" WITH ARBITRARILY MOVING MATTER

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The evolution of a cosmological model of a "mixmaster universe" filled with arbitrarily moving matter is considered. The analysis is carried out in a system of coordinates in which the matrix of the reference components of the three-dimensional metric tensor is reduced to the principal axes. The motion of the matter results in the rotation of this system of coordinates. The rotation, however, slows down on approach to a singularity and the total angle of rotation for the whole time of evolution of the model is finite. It is shown that light does not have time to circle the universe even once during the strong anisotropy stage when the motion of the matter is taken into account. Mixing therefore does not occur. Some necessary conditions for isotropization of the model at the late stages of the expansion are obtained.

1. INTRODUCTION

THE homogeneous cosmological model of the type IX according to the Bianchi classification has been widely discussed of late. This model has been considered in papers by Lifshitz, Khalatnikov and Belinskiĭ[1,2] in connection with the singularity problem in the general solution of the equations of gravitation, and also by Misner[3] in connection with the attempt to construct an example of the cosmological model in which an effective smoothing out of inhomogeneities (the model of a "mixmaster universe") may occur at the early stage of expansion. The different aspects of this problem are discussed in[4-11]. It has been noted in Misner's papers that a natural explanation of the observed isotropy of the relict radiation may be that the smoothing out of the density and temperature inhomogeneities and the subsequent isotropization of the cosmological expansion occur at the early stages of the expansion.

For the smoothing out of the inhomogeneities to be effective, light must have time at the very early stages of the expansion to traverse distances which are large compared with the radius of curvature. For the type-IX model, this presupposes the possibility of a multiple circumnavigation by light of the entire closed space, the circumnavigation being possible in all the directions of the space.

It was shown in[7,8] that the above-mentioned possibility is practically not realized in the framework of the type-IX homogeneous model with diagonal reference components of the metric. However, in the article by Matzner et al.[9] it is pointed out that in the general case of the model filled with arbitrarily moving matter (which leads to the appearance of off-diagonal reference components of the metric and, connected with them, a rotation of the matter), the conditions for the smoothing out of an inhomogeneity are improved.

In the present paper we also consider the general case of a homogeneous model with arbitrarily moving matter. A comparative analysis is given of the properties of the empty model and the model filled with matter under various assumptions about the character of the motion of the matter. The question of the effect of matter on the dynamical properties of the model is discussed. It is shown that the inclusion of an arbitrarily moving matter cannot lead to the appearance of the conditions for the circumnavigation of the universe by light under any initial conditions, i.e., to the smoothing out of the inhomogeneities at the early stages of the expansion.

2. BASIC EQUATIONS

The metric of the homogeneous model of the Bianchi type-IX may be represented in the form[12-14] (the velocity of light and the Einstein gravitational constant are assumed equal to unity)

$$ds^2 = dt^2 - \gamma_{ab} e_\mu^a e_\nu^b dx^\mu dx^\nu, \tag{1}$$

where $e_\mu^a = e_\mu^a(x^\nu)$, $\gamma_{ab} = \gamma_{ab}(t)$ and the indices a, b, μ and ν assume the values 1, 2 and 3. The matrix γ_{ab} determines the dependence of the metric on time and has six independent components. The evolution of the model with the metric (1) is considered below under the assumption that the universe is filled with an ideal liquid with the equation of state $p = \epsilon/3$[1].

It is convenient to carry out the investigation of the properties of the model in the system of coordinates in which the matrix γ_{ab} is reduced to the principal axes. The transformation to the new system of coordinates is made with the aid of the orthogonal rotation matrix S which depends on the three Euler angles φ_1, φ_2 and φ_3[15]:

$$S^T S = 1, \quad S = \begin{Vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{Vmatrix}, \quad S^T \gamma S = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix} \tag{2}$$

Thus, instead of six components of the matrix γ_{ab}, we have introduced three functions $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and three time-dependent Euler angles $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$ determining the location of the coordinate system whose orientation relative to the initial coordinate system (1) is given by the principal values of the matrix γ_{ab}. The introduced system of coordinates is even more preferable from the point of view of geometry, since the three-dimensional Ricci tensor is reduced to the principal axes simultaneously with the matrix γ_{ab}[2].

The Einstein equations split into a system of equations which do not contain the Euler angles:

$$(\ln \lambda_1)^{\cdot\cdot} + \lambda_1^2 - (\lambda_2 - \lambda_3)^2 + 4L^3 \left[\frac{\lambda_1 \lambda_3 (\lambda_1 + \lambda_3)}{(\lambda_3 - \lambda_1)^3} v_2^2 - \frac{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^3} v_3^2 \right] = 2\lambda \left[w^{1/2} \frac{v_1^2}{\lambda_1} + \frac{\varepsilon - p}{2} \right], \tag{3}$$

$$\dot{v}_1 + \frac{v_2 v_3}{v_0} \sqrt{\frac{\lambda_1}{\lambda_2 \lambda_3}} (\lambda_2 - \lambda_3) \left[1 + 2 w^{1/2} v_0^2 \lambda \frac{\lambda_1^2 - \lambda_2 \lambda_3}{(\lambda_1 - \lambda_2)^2 (\lambda_3 - \lambda_1)^2} \right] = 0 \tag{4}$$

and four other equations obtained from the above equations by a cyclic permutation of the indices, three equations determining the angles (see (6)), and

$$v_0^2 \lambda w = L^3, \quad v_1^2 + v_2^2 + v_3^2 = \delta L.$$

The dot denotes differentiation with respect to τ which is related to the time t through the condition $dt = \sqrt{\lambda}\, d\tau = \sqrt{\lambda_1 \lambda_2 \lambda_3}\, d\tau$; $v_\mu = u_\mu w^{1/4}$; (u_μ, u_0) is the 4-velocity in the rotating coordinate system whose orientation is defined by the principal values of the tensor γ_{ab}; p is the pressure, ϵ the energy density of the matter, and $w = \epsilon + p$ the enthalpy density. The constant L has the dimensions of length and δ is a dimensionless constant. If at the later stages of the expansion the cosmological model being considered is close to the closed Friedmann model, then $u_0 \approx 1$, $w \approx L^2/\lambda^{2/3}$ and in the vicinity of the point of maximum expansion, $\lambda \approx L^6$ and $w \approx L^{-2}$. Thus, in this case L determines the maximum radius of curvature of the model. The condition $u_0 \approx 1$ leads to the requirement $\delta \ll 1$.

The equations of gravitation have still another solution which should be considered as a limitation on the initial values of λ_μ, $\dot{\lambda}_\mu$ and v_μ:

$$(\ln \lambda_1)^{\cdot} (\ln \lambda_2)^{\cdot} + (\ln \lambda_2)^{\cdot} (\ln \lambda_3)^{\cdot} + (\ln \lambda_3)^{\cdot} (\ln \lambda_1)^{\cdot} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1 + 4L^3 \left[\frac{v_1^2 \lambda_2 \lambda_3}{(\lambda_2 - \lambda_3)^2} + \frac{v_2^2 \lambda_1 \lambda_3}{(\lambda_1 - \lambda_3)^2} + \frac{v_3^2 \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \right] + 4\lambda [w^{1/2} v_0^2 - p]. \tag{5}$$

The system of equations (3)–(5) does not contain the Euler angles and permits us to find the dependence on time of the principal values of the metric tensor $\lambda_\mu(t)$ and of the quantities u_μ and ϵ which determine the behavior of the matter. The time dependence of the Euler angles φ_μ, giving the position of the rotating system of coordinates relative to the fixed system (1) in terms of the known quantities v_μ, λ_μ and ϵ, is determined by the equations

$$\alpha_\mu = -2L^{3/2} \frac{\lambda_\nu \lambda_\sigma}{(\lambda_\nu - \lambda_\sigma)^2} v_\mu, \quad \mu \neq \nu \neq \sigma, \tag{6}$$

where

$$\alpha_1 = \dot{m}_1 n_1 + \dot{m}_2 n_2 + \dot{m}_3 n_3, \quad \alpha_2 = \dot{n}_1 l_1 + \dot{n}_2 l_2 + \dot{n}_3 l_3, \quad \alpha_3 = \dot{l}_1 m_1 + \dot{l}_2 m_2 + \dot{l}_3 m_3.$$

The solution of the system (3)–(6), which completely describes the evolution of the model, depends on twelve arbitrary constants, but only seven of them are important. The five unimportant constants are connected with the choice of initial values for the Euler angles and the time, and with the scale transformation of the matrix γ_{ab} and the time t (or τ).

Notice that any homogeneous cosmological model can depend on not more than seven arbitrary constants, whereas the general solution of the Einstein equations with a hydrodynamic energy-momentum tensor $T_{\mu\nu}$ should depend on eight arbitrary functions of the spatial coordinates. The number of arbitrary constants determining the homogeneous model is less by one than the number of arbitrary functions in the general solution precisely because of the possibility of a conformal scale transformation of the matrix γ_{ab} and the time t. This is connected with the fact that no conformal transformation of the metric exists which does not change the form of the Einstein equations with the exception of the case when the conformal factor is a constant.

Before proceeding to the analysis of the dynamics of the general homogeneous model with moving matter, let us recall the main properties of the model filled with quiescent matter[6,7].

3. THE HOMOGENEOUS COSMOLOGICAL MODEL FILLED WITH QUIESCENT MATTER (MODEL I)

If in the synchronous reference frame (1) the matter is stationary, i.e., the three velocity components u_μ are equal to zero, then the system of equations (3)–(6) are very much simplified. First, from (6) follows the important relation

$$\varphi_\mu = \text{const} \tag{7}$$

and without loss of generality we may set $\varphi_\mu = 0$. This means that if the matter is at rest then the tensor γ_{ab}, reduced to the diagonal form at the initial moment of time, remains always diagonal.

[1] According to the paper by Belinskiĭ, Lifshitz and Khalatnikov [10], the assumptions about the properties of the medium filling the universe do not affect the basic laws of evolution of the metric, under the condition that the terms involving matter in the six "tensor" equations of gravitation can be neglected.

[2] A different approach to this model has been developed by Belinskiĭ, Lifshitz and Khalatnikov [10].

The system of equations for λ_μ, (3), determines the evolution of the diagonal metric. The dynamics of a cosmological model with a diagonal metric has been studied in detail in[6,7]. Consequently, we shall only briefly recall the main properties of such a model which will be helpful for the understanding of what follows.

If we do not consider the late stages of the evolution, when the anisotropy is mild and the model is close to the isotropic Friedmann model, then the gravitation of the matter is insignificant. The solution has the form schematically shown in Fig. 1. Two functions oscillate in some interval of time τ while the third function monotonically decreases (the growth of τ corresponds to approach towards a singularity). During the following long period one of the functions, which were oscillating, monotonically decreases while the two others oscillate, and so on. An important feature of the evolution of the model is the growth of the amplitude of the oscillations on approach to a singularity; hence oscillations with large amplitudes are the most typical.

Let us put, at the maximum points of the function λ_1, $Q = \lambda_1/\lambda_2$; $P = \lambda_1/\lambda_3$. Then for $P \gg 1$, $Q \gg 1$ the solution may be written in the form

$$a^2 = \frac{qu}{\operatorname{ch}\eta}, \quad b^2 = \frac{qu}{Q}\operatorname{ch}\eta e^{-\eta/u}, \quad c^2 = \frac{qu}{P}\operatorname{ch}\eta e^{-u\eta}, \tag{8}$$

where $qu(\tau - \tau_0) = \eta$; and q, u and τ_0 are constants.

When $\lambda_1 \gg \lambda_2$, the equalities $a^2 = \lambda_1$, $b^2 = \lambda_2$ and $c^2 = \lambda_3$ are valid.

When $\lambda_1 \ll \lambda_2$ we must set $\lambda_1 = b^2$, $\lambda_2 = a^2$ and $\lambda_3 = c^2$, and "match" the solutions at the point $\lambda_1 = \lambda_2$[3]. It should be noted that when $|\eta| \gg 1$, the curvature of the model is insignificant at points far from the extremum points of $a^2(b^2)$ and the solution coincides with the well-known Kasner solution[16]

$$a^2 = a_0^2 t^{2p_1}, \quad b^2 = b_0^2 t^{2p_2}, \quad c^2 = c_0^2 t^{2p_3},$$

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2} \tag{9}$$

where a_0, b_0 and c_0 are constants.

In between neighboring "Kasner" sections u changes by unity (when $u > 1$; if $u < 1$, an alternation of long periods takes place and $u_{s+1} = 1/(u_s - \lfloor u_s \rfloor)$, where u_s is the parameter u at the beginning of the long period s, and $\lfloor u \rfloor$ is the integer part of u). The condition $P \gg 1$ and $Q \gg 1$ ensures emergence on the "Kasner" asymptote near the points of intersection of the functions λ_1 and λ_2 and a good accuracy in the "matching."

Notice that at this stage of the evolution, the amplitude of the oscillations (Q and P) increases during the motion towards the singularity. It is shown in[6,7] that for a random assignment of the initial conditions the probability of decrease of the amplitude (during the motion towards the singularity) and of the appearance of an evolution with a small amplitude is close to zero[4]. In[8] a conclusion is drawn to the effect that for $Q \gg 1$ and $P \gg 1$ the light, with a probability close to unity (for a random prescription of the initial conditions), traverses a distance which is small compared with the radius of curvature of the model, and does not have time to go around the universe even once in any direction right up to the actual singularity. This is connected with the fact that (in spite of the sharp decrease of λ_3) during the motion towards the singularity, the period of oscillations of the functions λ_1 and λ_2 (with respect to t) decreases so rapidly that light during that time is not able to circumnavigate the universe even once along the smallest axis (λ_3)[5]. Since the probability of a repeated appearance of the situation with the small amplitude $Q \approx 1$ is, practically, equal to zero, then, there is in the present model, as in the Friedmann model, an optical horizon for virtually all initial conditions.

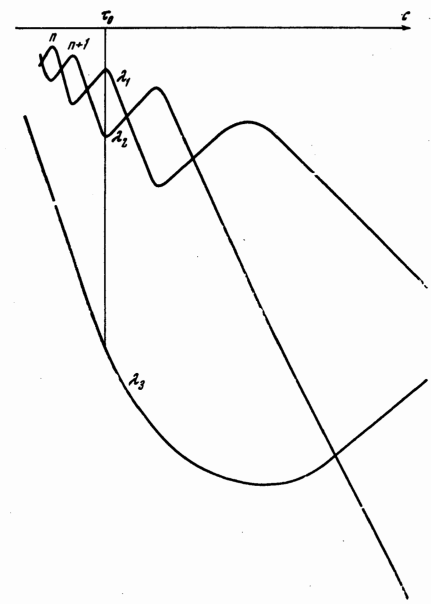

FIG. 1. The evolution of the model of a "mixmaster universe" filled with quiescent matter.

4. THE HOMOGENEOUS COSMOLOGICAL MODEL WITH ARBITRARILY MOVING MATTER (MODEL II)

The general pattern of evolution of the principal values λ_1, λ_2 and λ_3 of the tensor γ_{ab} in the model with arbitrarily moving matter resembles the model with quiescent matter (model I) considered above. As in model I, one of the functions monotonically decreases during movement towards a singularity while the two others oscillate, although the manner in which the oscillations take place is somewhat different from the manner in model I (Fig. 2). The amplitude of the oscillations increases during the motion towards the singular point and therefore the situation with a large amplitude of oscillation is the most typical (like model I). This case is considered below[6]. The main feature of the model with an arbitrarily moving matter is the

[3]Such a "matching" procedure is carried out for $u > 1$; for $u < 1$, it is necessary to "interchange" b^2 and c^2. The "matching" point $\lambda_1 = \lambda_2$ $(u > 1)$ is, in fact, unimportant, and the solutions are "matched" in the "Kasner" regions (see [7]).

[4]We shall not discuss here the so-called dangerous cases, the probability for which is negligibly small (see below).

[5]On the contrary, when $Q \approx 1$, the principal role is played by the decrease of λ_3 (the period varies slowly) and light is able to go around the universe many times along the axis λ_3.

[6]The evolution of the model with a small amplitude of oscillation is considered in [2,10].

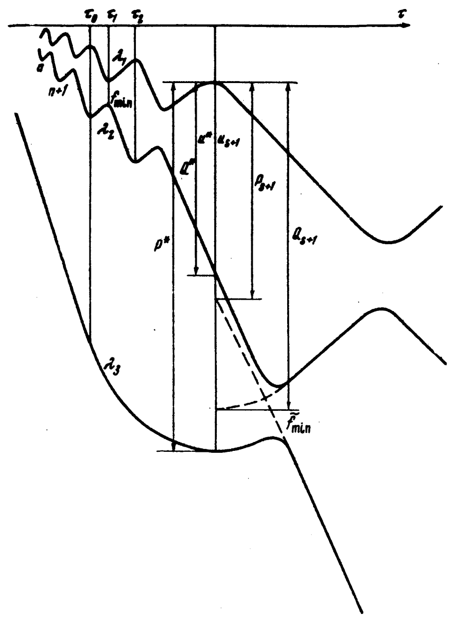

FIG. 2. The evolution of the principal values of γ_{ab} in the model of a "mixmaster universe" with arbitrarily moving matter.

nonintersection, as a rule, of the functions $\lambda_1(\tau)$, $\lambda_2(\tau)$ and $\lambda_3(\tau)$[7)]. Therefore, if we assume that in some arbitrary cycle $\lambda_1 > \lambda_2 > \lambda_3$, then, generally speaking, this relation will subsequently be valid right up to a singularity.

As in model I, it is necessary, for a complete description of the entire evolution, to consider two typical situations:

1) $\lambda_1 > \lambda_2 \gg \lambda_3$—a prolonged period of oscillations;
2) $\lambda_2 \gg \lambda_2 > \lambda_3$—an alternation of long periods.

Let us begin with the analysis of the prolonged period. In this case ($\lambda_1 > \lambda_2 \gg \lambda_3$) the equations of hydrodynamics reduce to the form

$$\dot v_1 + v_2 v_3\left[\frac{\sqrt{\lambda_1\lambda_2}}{v_0\sqrt{\lambda_3}} + 2L^{3/2}\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\right]=0,\quad \dot v_2 - v_1 v_3\left[\frac{\sqrt{\lambda_1\lambda_2}}{v_0\sqrt{\lambda_3}} + 2L^{3/2}\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\right]=0,\quad \dot v_3 + v_1 v_2\sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}(\lambda_1-\lambda_2)\cdot\left[\frac{1}{v_0} - 2L^{3/2}\sqrt{\frac{\lambda_3}{\lambda_1\lambda_2}}\right]=0. \quad (10)$$

Let $v_1 = R\sin\psi$, $v_2 = R\cos\psi$. Then, correct to terms of the order λ_3/λ_2 and λ_3/λ_1, we have

$$R = \text{const},\quad v_3 = \text{const} = (cL)^{1/2}, \quad (11)$$

$$\dot\psi = -\frac{(cL)^{1/2}}{v_0}\sqrt{\frac{\lambda_1\lambda_2}{\lambda_3}} - 2c^{1/2}L^2\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2} \quad (12)$$

The second term in (12) coincides with the expression describing the rotation of the principal axes relative to the initial reference frame (see (17)), and is connected with the transition to the rotating reference frame. The first term in (12) describes the rotation of the velocity vector relative to the stationary reference frame. It is easy to show that the angle of rotation connected with this term is small. Indeed,

$$|\dot\psi_{(1)}| = \left|\frac{v_3}{v_0}\sqrt{\frac{\lambda_1\lambda_2}{\lambda_3}}\right| < \sqrt{\lambda_1\lambda_2},\quad \Delta\psi_1 < \int\sqrt{\lambda_1\lambda_2}\,d\tau \ll 1. \quad (13)$$

The last integral determines the number of times light goes around the three-dimensional space of the model in the direction of the λ_3-axis and, as has been established in[7,8] for model I, it is small compared with unity[8)]. Allowance for the motion of the matter does not, as will be shown below, alter this conclusion.

During one oscillation the magnitude of the velocity changes by not more than

$$\Delta v_3 \approx \delta L^{1/2}(Q/P^2)^{1/3} \ll 1.$$

Therefore the formula (11) is meaningful only when $v_3 \gg \Delta v_3$. If this condition is not fulfilled and the velocity v_3 is small, then the effect of the velocity v_3 is negligible and can be considered as a small perturbation of the model I. In that case $u_0 \approx 1$ and $w \backsim \lambda^{-1/3}$.

Let us first consider the evolution under the assumption that v_3 = const. In the equations for λ_μ, (3), we may neglect the right hand sides as well as the terms containing λ_3. Then

$$(\ln\lambda_1\lambda_2)^{\cdot\cdot} = 0,\quad \left(\ln\frac{\lambda_1}{\lambda_2}\right)^{\cdot\cdot} + 2(\lambda_1^2 - \lambda_2^2) = 8cL^4\frac{\lambda_1\lambda_2(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)^3},\quad (\ln\lambda_3)^{\cdot\cdot} - (\lambda_1-\lambda_2)^2 = 0. \quad (14)$$

Thus, when the condition v_3 = const is met, the evolution of the model with an arbitrary velocity, being considered, is similar to the evolution of the model with only one velocity component (v_3). The general solution of (14) can be written out separately in the vicinity of the maximum and minimum points of λ_1.

In the vicinity of the maximum point of λ_1 the solution of (14) coincides with the "single-rotation" solution (8), where

$$a^2 = \lambda_1,\quad b^2 = \lambda_2,\quad c^2 = \lambda_3.$$

In the vicinity of a minimum point of λ_1

$$\lambda_1\lambda_2 = \frac{q^2u^2}{Q}e^{-q(\tau-\tau_0)},\quad \operatorname{ch} f = \operatorname{ch} f_{min}\operatorname{ch}\frac{2u-1}{2}q(\tau-\tau_1), \quad (15)$$

where

$$f = \frac{1}{2}\ln\frac{\lambda_1}{\lambda_2},\quad \operatorname{sh} f_{min} = \frac{2c^{1/2}L^2}{q(2u-1)},\quad q(\tau_1-\tau_0) = \frac{2}{2u-1}\ln\frac{\sqrt{4Q}}{\operatorname{ch} f_{min}}.$$

The condition of applicability of the obtained solution is $Q/\cosh^2 f_{min} \gg 1$. The solutions (8) and (15) may then be "matched" in the "Kasner" regions. By successively "matching" these solutions, we may construct the complete pattern of the evolution of the model in the prolonged period ($\lambda_3 \ll \lambda_2 < \lambda_1$).

For comparison with the model I we give the formulas connecting the parameters of the model in the

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7)The intersection is possible only in exceptional cases; in these cases the corresponding velocity component vanishes at the point of intersection.

8)As is well known, the Einstein equations include the equations of motion of matter, i.e. in this case, the equations of hydrodynamics. In particular, the system of equations (4) can be obtained from the off-diagonal Einstein equations. It can then be shown that the neglect of the first term in the square brackets of Eqs. (4) and (10) is equivalent to the neglect of the right hand side in the six Einstein "tensor" equations.

(n + 1)-st cycle with the parameters of the model in the n-th cycle:

$$q_{n+1} = q_n, \quad u_{n+1} = u_n - 1, \quad (\operatorname{sh} f_{min})_{n+1} = \frac{2u_n - 1}{2u_n - 3}(\operatorname{sh} f_{min})_n,$$
$$\frac{Q_{n+1}}{Q_n} = \left(\frac{2u_n - 1}{u_n}\right)^2 \left[\frac{4Q_n}{(\operatorname{ch}^2 f_{min})_n} \frac{u_n - 1}{u_n}\right]^{1/(u_n - 1)},$$
$$\frac{P_{n+1}}{P_n} = \frac{u_n - 1}{u_n}\left[\frac{4Q_n}{(\operatorname{ch}^2 f_{min})_n} \frac{u_n - 1}{u_n}\right]^{u_n}. \tag{16}$$

These formulas formally go over, as $f_{min} \to 0$, into the analogous formulas for the model I. Consequently, all the conclusions about the evolution laws for model parameters which pertain to model I are automatically carried over to the model under consideration (it is only necessary to substitute $Q_n/(\cosh^2 f_{min})_n$ for Q_n). For example, the overall change in the amplitude of λ_3 during the prolonged period can be written, in analogy with model I, as:

$$P_\Sigma \approx [4Q_0/(\operatorname{ch}^2 f_{min})_0]^{u_0^2};$$

and the total time of the prolonged cycle, as:

$$\tau_\Sigma \approx \frac{u_0}{q}\ln\frac{4Q_0}{(\operatorname{ch}^2 f_{min})_0}.$$

The obtained results permit us to elucidate the dependence on time of the Euler angles which describe the mutual position of the initial reference frame and the reference frame defined by the principal values.

Let $\varphi_1 = \varphi_2 = \varphi_3 = 0$ when $\tau = \tau_0$—at the moment when λ_1 attains a maximum value. Then it can be shown that the change in the angles φ_1 and φ_2 is small. For the angle φ_3 (the angle of rotation in the plane $\lambda_1\lambda_2$), we have

$$2\varphi_3 = \arcsin(\operatorname{th} f_{min}) - \arcsin\left(\frac{\operatorname{th} f_{min}}{\operatorname{th} f}\right)$$
$$\text{for } \tau_0 \leqslant \tau \leqslant \tau_1, \quad f < 0; \tag{17a}$$
$$2\varphi_3 = -\pi + \arcsin(\operatorname{th} f_{min}) + \arcsin\left(\frac{\operatorname{th} f_{min}}{\operatorname{th} f}\right)$$
$$\text{for } \tau_1 \leqslant \tau \leqslant \tau_2, \quad f > 0. \tag{17b}$$

It is not difficult to see that the following inequalities

$$\gamma_{a3} \sim \lambda_a^2\varphi_1^2 \ll \gamma_{aa}\gamma_{33} \sim \lambda_a\lambda_3, \quad a = 1, 2,$$

are valid. Thus, the off-diagonal components γ_{13} and γ_{23} do not, in the first approximation, affect the dynamics of the model[5].

The quantity

$$G = \gamma_{12}/\sqrt{\gamma_{11}\gamma_{22} - \gamma_{12}^2}, \tag{18}$$

may serve as a measure of the nondiagonality of the matrix γ_{ab}. Substituting (17) in (18), we find that in the interval of time $(\tau_2 - \tau_0)$ between two maxima the quantity G changes from zero to $G_1 \approx \sqrt{Q}\sinh f_{min}/\cosh^2 f_{min}$. Therefore the appearance of even a small misalignment of the frame of the principal axes and the initial stationary reference frame can lead to the appearance of a strong nondiagonality of γ_{ab}. It is only under the condition $f_{min} \ll 1/\sqrt{Q}$, that we have $G_1 \ll 1$ and the model under consideration close to the model I in the whole interval $\tau_0 \le \tau \le \tau_2$ (as $f_{min} \to 0$, $\Delta\varphi_3 \to \pi/2$ and this amounts simply to a transformation of the axes in comparison with the model I). The effect of matter on the metric may then be considered in the framework of perturbation theory. In particular, the formulas for "matching" in this case are the same as in the model I. Notice that in this case matter can be inserted in the metric of model I without a back effect when $u_0 \approx 1$ as well as when $u_0 \approx u_3/\sqrt{\lambda_3}$. The case $v_3 \lesssim \Delta v_3 \ll 1$ $(u_0 \approx 1)$ noted above, is a special case as compared to the case $G_1 \ll 1$. Because of the growth of the quantity Q, the condition $G_1 \ll 1$ can be fulfilled only in a limited interval of time τ. During motion towards a singularity the condition $G_1 \ll 1$ is quickly violated and it is necessary to use the general formulas (16). Subsequently, the probability of a chance appearance of the situation $\sqrt{Q}\, f_{min} \ll 1$ becomes negligibly small.

The formulas obtained above are valid for $u > 1$. When, however, u becomes less than unity, the function λ_3 begins to grow and a change of long periods of oscillations takes place. The functions λ_2 and λ_3 approach each other. However, if $v_1 \neq 0$, then unlike the model I, an intersection of λ_2 and λ_3 does not occur; following the approach, a period of oscillations of the functions λ_1 and λ_2 and monotonical decrease of λ_3 begins again (Fig. 2).

During the time of alternation of the prolonged periods of oscillations the velocities v_2 and v_3 rotate in accordance with the rotation of the reference frame whose orientation relative to the initial reference frame is given by the principal values of the tensor γ_{ab} (the rotation takes place about the axis λ_1). The quantity v_1 changes by not more than

$$\Delta v_1 \leqslant \sqrt{\delta L}\int\sqrt{\lambda_1\lambda_2}\,d\tau \ll 1$$

and therefore we may assume $v_1 = \text{const} = a^{1/2}L^{1/2}$.

The time dependence of $\lambda_\mu(\tau)$ is determined by formulas similar to (15). The input parameters of the (s + 1)-st prlonged cycle of oscillations are determined in terms of the values of the parameters at the end of the previous s-th cycle by the formulas

$$q_{s+1} = \frac{q_s}{u_{s+1}^2}, \quad u_{s+1} = \frac{1}{u^*}, \quad \operatorname{sh}\tilde{f}_{min} = \frac{2a^{1/2}L^2}{(1 - u^{*2})q_s},$$
$$Q_{s+1} = \frac{P^*}{\operatorname{ch}^2\tilde{f}_{min}}, \quad P_{s+1} = Q^*\operatorname{ch}^2\tilde{f}_{min}, \quad \tilde{f} = \frac{1}{2}\ln\frac{\lambda_2}{\lambda_3} \tag{19}$$

(the asterisks indicate quantities pertaining to the end of the s-th cycle). As $\tilde{f}_{min} \to 0$ these formulas go over into the analogous formulas for model I.

As in model I, if $u^* < (\cosh^2 f_{min})^*/4Q^*$, then the matching formulas (19) are not valid and, in principle, the possibility of a sharp decrease of the amplitude of the oscillations is not to be excluded. However, such an event is extremely improbable[6,7].

The angle of rotation of the principal axes relative to the initial reference frame is described by formulas similar to (17), where

$$\operatorname{sh}\tilde{f}_{min} = 2a^{1/2}L^2/(1 - u^{*2})q_s,$$

while the quantity $\tilde{f}$ varies within the limits $\tilde{f}_{min} \le \tilde{f} \le \frac{1}{2}\ln(P_{s+1}/Q_{s+1})$. Therefore, if at the end of the s-th cycle the matrix γ_{ab} is diagonal, then the component γ_{23} appears at the beginning of the (s + 1)-st cycle, where

$$G_2 = \frac{\gamma_{23}}{\sqrt{\gamma_{22}\gamma_{33}}} \approx \sqrt{\frac{P_{s+1}}{Q_{s+1}}} \frac{\operatorname{sh} f_{min}}{\operatorname{ch}^2 f_{min}}.$$

As a rule, $G_2 \gtrsim 1$ due to the fact that the quantity $P_{s+1}/Q_{s+1} \gg 1$ and rapidly increases during the motion towards a singularity.

Asymptotically near a singularity, not only P and Q, but also f_{min} and $\tilde{f}_{min}$ grow. Hence $\lambda_1 \gg \lambda_2 \gg \lambda_3$ near a singular point. This leads to a strong slowing down of the rotation of the reference frame whose orientation relative to the initial stationary reference frame is given by the principal values of the tensor γ_{ab}, and the total angle of rotation of one frame relative to the other tends to a finite limit[9)]. We recall, however, that even a small change in the Euler angles will lead to the appearance of large off-diagonal terms in the matrix γ_{ab}.

As an example, let us find the total angle of rotation in the plane $\lambda_1\lambda_2$, beginning from some moment of time $\tau = \tau_0$ right up to a singularity under the condition $\lambda_1 \gg \lambda_2 \gg \lambda_3$. During one oscillation the angle φ_3 changes by the amount (17):

$$\Delta\varphi_3 \approx -\operatorname{arcctg} \operatorname{sh} f_{min} \approx -1/\operatorname{sh} f_{min}.$$

Substituting the definition $\sinh f_{min} = 2c^{1/2}L^2/q(2u-1)$ and summing within the limits $1 \le u \le u_s$, we find that during the s-th long period the angle φ_3 changes by the amount

$$\Delta\varphi_s \approx -q_{s-1}/2c^{1/2}L^2.$$

The quantity q_s quickly decreases with growth of the number of s. Using the probability distribution for u obtained in[8,10] and the law $q_s = q_{s-1}/u_s^2$, we obtain

$$\bar{q}_{s-1} = q_0/3^{s-1}, \quad \Delta\varphi_\Sigma = -{}^3/_2\Delta\varphi_0 \ll 1, \tag{20}$$

where q_0 and $\Delta\varphi_0$ are the values of the parameters q and $\Delta\varphi_3$ at $\tau = \tau_0$.

Near a singular point the velocity components $v_\mu = u_\mu w^{1/4}$ become constant quantities. It can then be shown that if the condition $u_0^2 = (1 - v^2)^{-1} \ll \sqrt{\lambda_1\lambda_2}/\lambda_3$ (v^2 is the square of the three-dimensional velocity of the matter) is not fulfilled, then the model can never approximate the Friedmann model.

Thus, the specific effect of the motion of matter is manifested in the fact that the principal values of the components of the metric tensor cannot coincide (with the exception of the degenerate cases, when the corresponding velocity vanishes at the moment of coincidence of two principal values). This conclusion is valid for all metrics possessing the Kasner asymptotic form near a singular point if the velocity of the matter has a rotational component (in the model considered above the velocity is rotational). However, in contrast to the above-considered model, if the curvature has no effect on the dynamics in the vicinity of a singular point, then the Kasner asymptotic form is violated only at that moment when the decreasing and increasing functions approach each other. Subsequently, the motion of the matter can again be neglected. It was therefore perfectly correct for Lifshitz and Khalatnikov in their paper[17] not to take into account the effect of the motion on the metric. A peculiarity of the metric in the type-IX model, however, is the fact that the effect of the curvature is considerable right up to a singularity, as a result of which the cycles in which the decreasing function (λ_1) is larger than the increasing function (λ_2), periodically recur and the motion of the matter influences the dynamics of the model right up to the singularity.

9)This result was also obtained independently by Belinskiĭ and Khalatnikov.

5. PROPAGATION OF LIGHT

For Misner's idea to be realized, it is necessary that light have time to circle the universe many times in all directions. It was shown in[7,8] that when the amplitude of the oscillations is large ($Q \gg 1$) light, with a probability close to unity, does not have time to circle the universe even once in any direction. Let us show that allowance for the motion of the matter does not change this conclusion.

In the model filled with moving matter, the principal axes of the metric tensor do not coincide with the null-geodesics. Therefore, we cannot just as simply as in the model I calculate the distance traversed by the light. However, it is easy to obtain the upper bound of the number of times the light circles the universe. Indeed, during the time between two neighboring minima of the function λ_1, light cannot go around the universe more than N_{max} times:

$$N_{\lambda_3} = \frac{1}{2\pi}\int \frac{k^3}{k_0} dt = \frac{1}{2\pi}\int dt \frac{k_3}{\lambda_3}\left(\frac{k_1^2}{\lambda_1} + \frac{k_2^2}{\lambda_2} + \frac{k_3^2}{\lambda_3}\right)^{-1/2} < N_{max}$$
$$= \frac{1}{2\pi}\int \sqrt{\lambda_1\lambda_2}\, d\tau \approx \frac{2u}{\pi \operatorname{ch} f_{min}}\left(\frac{\sqrt{4Q}}{\operatorname{ch} f_{min}}\right)^{-2u/(2u+1)} \times\left[1 - \left(\frac{\sqrt{4Q}}{\operatorname{ch} f_{min}}\right)^{-4u/(4u^2-1)}\right]; \tag{21}$$

(k_0, $k_a = k_a/\lambda_a$) are the local components of the wave 4-vector, k^3 being the component along the axis corresponding to the smallest scale $\sqrt{\lambda_3}$. For $u \gg 1$, we obtain from this

$$N_{max} \approx \frac{1}{2\pi\sqrt{Q}} \ln \frac{4Q}{\operatorname{ch}^2 f_{min}} \quad \text{for} \quad u \gg \ln\frac{4Q}{\operatorname{ch}^2 f_{min}} \gg 1,$$
$$N_{max} \approx \frac{u}{\pi\sqrt{Q}} \quad \text{for} \quad \ln\frac{4Q}{\operatorname{ch}^2 f_{min}} \gg u \gg 1. \tag{22}$$

We obtain for the number of times light goes around the universe during the period of the cycle s

$$N_s \lesssim u_s/\pi\sqrt{Q_s}. \tag{23}$$

These formulas do not differ from the corresponding formulas for the model I[8]. Therefore, we can make the assertion that, as in the model I, at the stage $Q/\cosh^2 f_{min} \gg 1$, beginning from some arbitrary moment and right up to a singularity, light, with a probability close to unity, does not have time to go around the universe even once in any direction.

In[9] the conclusion has been drawn on the basis of numerical computations that the motion of matter facilitates mixing. As has been shown above, mixing does not occur during the large amplitude oscillations stage. Clearly, the conclusion of the paper[9] pertains to the case of small amplitude oscillations, $Q/\cosh^2 f_{min} \approx 1$, when, in fact, effective mixing in the direction of the smallest principal value of the tensor

γ_{ab} can occur. Thus, if light does go around the universe several times, then this can only be at a comparatively late stage in the course of the isotropization of the model under consideration. Thus, we can expect the model of a "mixmaster universe" to be incapable of explaining the observed homogeneity of the universe, unless the parameters are especially chosen.

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