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THE PROBLEM OF COLLAPSE TO THE CENTER IN QUANTUM MECHANICS

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The quantum-mechanical problem of the motion of a particle in a singular attractive field is investigated with a non-Hermitean boundary condition at zero, corresponding to the absorption of particles at the coordinate origin (collapse to the center). The eigenvalue spectrum and the set of eigenfunctions are found for particular potentials. The completeness of the set of eigenfunctions is investigated.

1. INTRODUCTION

EXAMINATION of the radial Schrödinger equation set up for a singular attractive potential

$$U(r) = -\beta / r^n, \quad \beta > 0, \quad n \geqslant 2,$$

shows[1-4] that, unlike the case of the usual (nonsingular) attractive potential or a singular repulsive potential both the linearly independent solutions are square-integrable and have the same singularity as $r \to 0$. In fact, for $n > 2$ (the case $n = 2$ will be examined in detail below) for $r \to 0$, the semi-classical approximation[2] is valid and we can write

$$u(r) \infty r^{n/4} \exp\left(\pm \frac{i}{\hbar} \int \sqrt{\frac{2m\beta}{r^n}}\, dr\right), \quad r \to 0, \tag{1}$$

i.e., the two solutions oscillate with the same square-integrable amplitude. Thus, we have no criterion for choosing the necessary linear combination of the solutions (1).

If we reject fixing the solution for $r \to 0$, it is not difficult to show that any value of E (including also complex values) will be an eigenvalue, the eigenfunctions will be nonorthogonal, and the set of eigenfunctions will be over-complete. From a mathematical point of view, this circumstance is connected with the fact that the operator under consideration

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 - \frac{\beta}{r^n}, \quad \beta > 0, \quad n \geqslant 2$$

is only symmetric (and not self-adjoint) and its departure from self-adjointenss can be characterized by the so-called deficiency indices (1, 1)[5,6]. As was first shown by Case[7], the self-adjointness of $\hat{H}$ can be restored if we impose on the solutions of the problem under consideration some condition fixing the phase of the oscillating part of the solution:

$$u(r) \infty r^{n/4} \cos\left\{\frac{2}{n-2} \sqrt{\frac{2m\beta}{\hbar^2 r_0^n}} \left(\frac{r_0}{r}\right)^{(n-2)/2} + \varphi\right\}, \quad r \to 0. \tag{2}$$

The presence in this condition of the fixed phase φ (some real parameter) is in accordance with the general mathematical technique for the so-called Hermitean extension of a symmetric operator[5,6]. The eigenfunctions of the operator $\hat{H}$ that satisfy condition (2) turn out to be orthogonal and form a complete set, while the eigenvalues E are real. However, the essential dependence of both the eigenvalues and the eigenfunctions on the arbitrary parameter (phase) φ is undoubtedly a negative feature of this procedure.

But another approach to the problem under study is also possible, in which a non-Hermitean (quasi-self-adjoint[6]) extension of the operator $\hat{H}$ is considered and some non-Hermitean condition is imposed on the function u(r) as $r \to 0$. As this condition, we shall require that, for $r \to 0$, there exist only one wave converging toward the coordinate origin

$$u(r) \infty r^{n/4} \exp\left(-\frac{i}{\hbar} \int^r \sqrt{\frac{2m\beta}{r^n}}\, dr\right), \quad r \to 0. \tag{3}$$

This condition leads to the collapse (in the full sense of the word) of particles into the attracting center. Usually the "collapse" is perceived in the fact (cf.[2]) of the existence in this problem of arbitrarily large (in absolute magnitude) negative energy values, to which correspond eigenfunctions concentrated arbitrarily close to the coordinate origin.

What reasons are there for introducing the non-Hermitean condition (3)? This condition is natural from a physical point of view, since it corresponds to the classical limiting case, in which, as is well-known[8], the collapse of particles to the center corresponds to the case of a singular attractive potential. We note also the following. In the case of a singular repulsive potential

$$U(r) = -\beta / r^n, \quad \beta < 0,$$

it follows from formula (1) that the solution that falls away as $r \to 0$ has the following behavior:

$$u(r) \infty r^{n/4} \exp\left(\frac{1}{\hbar} \int^r \sqrt{-\frac{2m\beta}{r^n}}\, dr\right). \tag{4}$$

Comparison of formulas (3) and (4) shows that the solution (3) is an analytic continuation, in the force constant β, of the solution (4). Thus, the condition (3) not only removes the arbitrariness in the choice of the phase φ in the relation (2), but also distinguishes a solution which is the natural (analytic) continuation of the decreasing solution corresponding to a repulsive potential and which, as is well-known, is unique.

The non-Hermitean condition (3) was first introduced by Vogt and Wannier[9] in connection with a discussion of the potential

$$U(r) = -\beta / r^4, \quad \beta > 0.$$

In the recent work of Perelomov and Popov[10], a more general condition at r = 0, the generalization of Case's condition (2) to the case of complex values of φ, is considered. The limiting value $\varphi = i\infty$ corresponds to the condition for complete absorption (3).

The present paper is devoted to the detailed investigation of an approach based on condition (3). It is necessary to ascertain the internal consistency of this approach. In fact, abandoning Hermiticity means abandoning the reality of the energy eigenvalues E and the orthogonality (in the usual sense) of the eigenfunctions. But one property of the Hermitean problem must be preserved, namely the completeness of the set of eigenfunctions.

In the paper, the completeness of the set of eigenfunctions is proved in two of the four examples considered. In examining the particular cases, we confine ourselves to those potentials for which the eigenvalues and eigenfunctions have more or less simple analytic expressions. The case of the potential $U = -\beta/r^2$ is treated in most detail. Using this example, we illustrate the spectrum, the set of eigenfunctions, the completeness of the set, the inelastic scattering problem, and the solution of the time problem.

2. MOTION IN THE FIELD $U(r) = -\beta/r^2$

Since the radial dependence of the potential

$$U(r) = -\beta/r^2, \quad \beta > 0 \tag{5}$$

is the same as for a centrifugal potential, it is clear that the problem of the motion of a particle with well-defined orbital angular momentum l in the field (5) formally coincides with the problem of the motion of a free particle with "angular momentum" l', where

$$l' + \frac{1}{2} = \sqrt{\left(l+\frac{1}{2}\right)^2 - \gamma}, \quad \gamma = \frac{2m\beta}{\hbar^2}. \tag{6}$$

Therefore, the wavefunction for our problem can be obtained from the wavefunction for free motion[2]

$$\sqrt{kr}J_{l+1/2}(kr) \tag{7}$$

by replacing l by l'. If $\gamma < (l + 1/2)^2$, no difficulties arise (in expression (6), the positive value of the square root must be taken). But let $\gamma > (l + 1/2)^2$. Then it is not clear which of the two complex conjugate values of l' from relation (6) we should take, since in this case both solutions have the same singularity as $r \to 0$

$$\sqrt{kr}\, J_{\pm i\nu}(kr) \propto \sqrt{kr}\exp(\pm i\nu \ln kr).$$

Since the function $J_{-i\nu}$ corresponds to a wave converging toward the coordinate origin, the solution satisfying condition (3) has the form

$$u_{k,-i\nu}(r) \equiv \sqrt{kr}J_{-i\nu}(kr), \quad \nu = \sqrt{\gamma - (l+1/2)^2} > 0. \tag{8}$$

We shall consider the problem of the scattering of particles in the field (5). It is not difficult to reason out that the phase shifts have the form

$$\delta_l = 1/2\pi(l - l'), \tag{9}$$

whence, taking account of (6) and (8), we have

$$\delta_l = \begin{cases} 1/2\pi(l + 1/2 - \sqrt{(l+1/2)^2 - \gamma}) & \text{for } l > l_{max} \\ 1/2\pi(l+1/2) + 1/2 i\pi\sqrt{\gamma - (l+1/2)^2} & \text{for } l < l_{max} \end{cases}, \tag{10}$$

where l_{max} is the whole-number part of $\sqrt{\gamma} - 1/2$.

The cross-section for the inelastic scattering due to collapse of particles into the center has the form[2]

$$\sigma_{\text{inel}} \equiv \sigma_{\text{coll}} = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1 - |\eta_l|^2)$$

$$= \frac{\pi}{k^2}\sum_{l=0}^{l_{max}}(2l+1)\left\{1 - \exp\left(-2\pi\sqrt{\gamma - \left(l+\frac{1}{2}\right)^2}\right)\right\}, \tag{11}$$

where we have taken into account that

$$|\eta_l|^2 = |\exp(2i\delta_l)|^2 = \exp(-2\pi\nu).$$

We shall consider the semi-classical limiting case $\gamma \gg 1$. Then the summation over l in relation (11) can be replaced by integration, and we obtain

$$\sigma_{\text{coll}} \approx \frac{2\pi}{k^2}\int_0^{\sqrt{\gamma}} \lambda\{1 - \exp(-2\pi\sqrt{\gamma - \lambda^2})\}d\lambda$$

$$= \frac{2\pi}{k^2}\int_0^{\sqrt{\gamma}} \mu\{1 - \exp(-2\pi\mu)\}d\mu \approx \frac{\pi\gamma}{k^2} = \pi\rho^2_{max}; \tag{12}$$

here, ρ_{max} is the limiting value of the classical impact parameter corresponding to the collapse of a classical particle to the center in the field (5). It is obvious that (12) coincides with the corresponding classical cross-section[8].

We shall now examine the question of the spectrum of eigenvalues E and the completeness of the set of eigenfunctions. It is found expedient to go over from the functions (8) to the functions

$$u_{k,p}(r) \equiv \sqrt{kr}J_p(kr), \tag{13}$$

which describe in a unified manner the following three problems:

1) $p = l + 1/2$—the free particle;

2) $p = [(l + 1/2)^2 - \gamma]^{1/2}$—a particle in the field (5) with $l > l_{max}$ (the "normal case");

3) $p = -i[\gamma - (l + 1/2)^2]^{1/2}$—a particle in the field (5) with $l < l_{max}$ (the case of collapse to the center).

In investigating the energy spectrum, we shall start from the functions (13). It is obvious that, for all three particular cases of p, the permissible values of E fill the positive semi-axis. In fact, for any other value of E the parameter k is essentially complex and as $r \to \infty$ the function (13) tends to infinity, since the asymptotic expression for the Bessel function (with any value of the index p) essentially contains both exponentials. Thus, for the potential (5), there is no discrete spectrum for any value of the force constant β (if, of course, the non-Hermitean condition (3) is imposed on the solution of the problem).

We turn now to investigate the question of the normalization and of the orthogonality and completeness relations for the solutions (13). For a free particle (half-integer p), it is known[12] that the solutions (13) are normalized to a δ-function:

$$\int_0^{\infty} u_{k',p}(r)u_{k,p}(r)\,dr = \delta(k' - k), \quad p = 1/2,\ 3/2,\ 5/2,\ .. \tag{14}$$

We shall show that (14) also holds for the other values of p of interest to us.

From the radial Schrödinger equation written for the two functions $u_{k'}$ and u_k (we temporarily omit the fixed index p), the relation

$$\frac{d}{dr}\left(u_{k'}\frac{du_k}{dr} - u_k\frac{du_{k'}}{dr}\right) + (k^2 - k'^2)u_{k'}u_k = 0. \tag{15}$$

follows directly. Integrating this equality over r from zero to some large value R, we obtain

$$\int_0^R u_{k'}u_k\,dr=\frac{1}{k'^2-k^2}\left(u_{k'}\frac{du_k}{dr}-u_k\frac{du_{k'}}{dr}\right)\Big|_0^R. \qquad (16)$$

For r → 0 and r → ∞, we have from the asymptotic expressions for the Bessel functions[11]:

$$u_{k,p}(r)\approx\begin{cases}\dfrac{(kr)^{p+1/2}}{2^p\Gamma(p+1)}, & kr\to 0\\[2mm] \sqrt{\dfrac{2}{\pi}}\sin\left(kr+\dfrac{\pi}{4}-\dfrac{\pi p}{2}\right), & kr\to\infty\end{cases}. \qquad (17)$$

Taking account of (17) in the relation (16), we obtain

$$\int_0^R u_{k',p}(r)u_{k,p}(r)dr\approx\frac{1}{\pi}\frac{\sin[(k'-k)R]}{k'-k}-\frac{1}{\pi}\frac{\sin[(k'+k)R-\pi(p-1/2)]}{k'+k} \qquad (18)$$

As R tends to infinity, the first term in the right-hand side of (18) tends to a δ-function, while the second is a rapidly-oscillating function having no singularity at $k' = k$. Thus,

$$\int_0^\infty u_{k',p}(r)u_{k,p}(r)dr\equiv\int_0^\infty (u_{k',p^*})^*u_{k,p}\,dr\equiv\langle k',p^*|k,p\rangle=\delta(k'-k), \qquad (19)$$

where p is a real positive number or is pure imaginary. $|k, p\rangle$ in formula (19) denotes the Dirac set corresponding to the function $u_{k,p}$.

Since k and r occur in (13) only in the form of the product kr, from (19) by a simple interchange of k and r we obtain the relation

$$\int_0^\infty u_{k,p}(r')u_{k,p}(r)dk\equiv\int_0^\infty \{u_{k,p^*}(r')\}^*u_{k,p}(r)dk=\delta(r'-r), \qquad (20)$$

which, as can be seen without difficulty, is the completeness condition for the set (13). This assertion becomes yet more obvious if we rewrite (20) in a slightly different form, using Dirac's notation:

$$\int_0^\infty |k,p\rangle\langle k,p^*|dk=1. \qquad (21)$$

Thus, in the case of collapse to the center, apart from the set (8) there arises naturally the complementary (or conjugate) set of functions

$$\tilde{u}_{k,-i\nu}(r)\equiv\{u_{k,-i\nu}(r)\}^*=u_{k,i\nu}\equiv\langle r|k,i\nu\rangle,$$
$$u_{k,-i\nu}\equiv\langle r|k,-i\nu\rangle. \qquad (22)$$

It follows from the relations (19)—(21) that these sets are biorthogonal and complete. It is obvious that the set of functions (22) describes the problem that is the Hermitean conjugate of ours, i.e., one in which, as r → 0, there is only a wave diverging from the coordinate origin. We shall see below that $\tilde{u}$ is associated with the time-reversed motion of the particle.

We shall examine the question of the expansion of an arbitrary function f(r) as an integral over the continuous set (8):

$$f(r)=\int_0^\infty c(k)u_{k,-i\nu}(r)dk\equiv\int_0^\infty c(k)\sqrt{kr}J_{-i\nu}(kr)dk. \qquad (23)$$

For the amplitude c(k), using (19), we have

$$c(k)=\langle k,i\nu|f\rangle\equiv\int_0^\infty f(r)u_{k,-i\nu}(r)dr\equiv\int_0^\infty f(r)\sqrt{kr}\;J_{-i\nu}(kr)dr, \qquad (24)$$

where $|f\rangle$ is the Dirac set corresponding to the function f(r). The relations (23) and (24) when expressed explicitly in terms of Bessel functions are known in mathematics by the name of Hankel transforms[12]. The fact that the formulas for the direct and inverse transformations coincide (this is true for both real and complex p) is characteristic for this transformation. We note also the existence in the mathematical literature of special tables of Hankel transforms[13].

The completeness of the set of eigenfunctions enables us to solve the time problem. Suppose at t = 0 we have v(r, 0) = f; then for t ≠ 0, the solution of the problem is given by the expression

$$v(r,t)=\int_0^\infty c(k)\exp\left(-i\frac{\hbar k^2 t}{2m}\right)u_{k,-i\nu}(r)dk, \qquad (25)$$

where c(k) is given by formula (24). We shall also consider the solution of the conjugate problem (i.e., the problem with creation of particles) with the same initial condition $\tilde{v}(r, 0) = f$ at t = 0. We shall seek the solution of this problem in the form of an integral over the conjugate set (22):

$$\tilde{v}(r,t)=\int_0^\infty \tilde{c}(k)\exp\left(-i\frac{\hbar k^2 t}{2m}\right)u_{k,i\nu}(r)dk. \qquad (26)$$

We find the amplitude $\tilde{c}(k)$ occurring in (26), using (19):

$$\tilde{c}(k)=\langle k,-i\nu|f\rangle\equiv\int_0^\infty f(r)u_{k,i\nu}(r)dr. \qquad (27)$$

We shall now consider the time-reversed motion of our particle. This, evidently, is described by the function v*(r, −t). Performing the complex conjugation in (25) and replacing t by −t, we obtain (26), in which

$$\tilde{c}(k)=c^*(k). \qquad (27')$$

Thus, the time-reversed motion is a solution of the conjugate problem and corresponds to the initial state f*.

We shall consider the scalar product

$$\langle\tilde{v}(t)|v(t)\rangle\equiv\int_0^\infty \tilde{v}^*(r,t)v(r,t)dr.$$

Using (25), (26) and (19), we find

$$\langle\tilde{v}(t)|v(t)\rangle=\int_0^\infty\int_0^\infty \tilde{c}^*(k')c(k)\exp\left(i\frac{\hbar(k'^2-k^2)t}{2m}\right)$$
$$\times\langle k',i\nu|k,-i\nu\rangle dk'dk=\int_0^\infty \tilde{c}^*(k)c(k)dk=\langle f|f\rangle=1, \qquad (28)$$

in which the last equality assumes that the initial state $|f\rangle$ is normalized to unity:

$$\langle f|f\rangle=\int_0^\infty |f(r)|^2dr=1. \qquad (29)$$

Thus, the equality (28) is a constant of the motion for our non-Hermitean problem, whereas the "normalization" integrals

$$W(t)\equiv\int_0^\infty |v(r,t)|^2dr=\langle v(t)|v(t)\rangle, \qquad (30)$$

$$\tilde{W}(t)\equiv\int_0^\infty |\tilde{v}(r,t)|^2dr=\langle\tilde{v}(t)|\tilde{v}(t)\rangle \qquad (31)$$

essentially depend on the time, with (30) decreasing and (31) increasing, but in such a way that (28) remains constant.

We shall now study the question of the time dependence of expression (30). It is evident that (30) is the probability of finding the particle in a finite part of space (i.e., the probability that the particle has not fallen into the center by the moment of time t). We shall find the value of the scalar product $\langle k', -i\nu | k, -i\nu\rangle$. Since $u^*_{k'}$ satisfies the same equation as $u_{k'}$, we can start from the relation (16), having replaced $u_{k'}$ by $u^*_{k'}$ in it. Then, using (17) we obtain the expression

$$\int_0^R u^*_{k',-i\nu}(r)u_{k,-i\nu}(r)\,dr \approx \operatorname{ch}(\pi\nu)\frac{\sin[(k'-k)R]}{\pi(k'-k)} \quad (32)$$

$$-\frac{i}{\pi}\operatorname{sh}(\pi\nu)\frac{\cos[(k'-k)R]}{k'-k}+\frac{2i}{\pi}\operatorname{sh}(\pi\nu)\frac{\sqrt{k'k}(k'/k)^{i\nu}}{k'^2-k^2}+$$

+ rapidly oscillating terms having no singularity at $k' = k$.

Making R go to infinity, we find

$$\langle k', -i\nu | k, -i\nu\rangle = \operatorname{ch}(\pi\nu)\delta(k'-k) + \frac{2i}{\pi}\operatorname{sh}(\pi\nu)\sqrt{k'k}\left(\frac{k'}{k}\right)^{i\nu}\mathrm{P}\frac{1}{k'^2-k^2}, \quad (33)$$

where P denotes the principle value. Now, using (33), we can represent the expression (30) in the form

$$W(t) = \operatorname{ch}(\pi\nu)\int_0^\infty |c(k)|^2\,dk + \frac{2i}{\pi}\operatorname{sh}(\pi\nu) \quad (34)$$

$$\times \mathrm{P}\iint_0^\infty dk'\,dk\frac{c^*(k')c(k)\sqrt{k'k}}{k'^2-k^2}\left(\frac{k'}{k}\right)^{i\nu}\exp\left(i\frac{\hbar(k'^2-k^2)t}{2m}\right).$$

We shall show that (34) decreases monotonically with time. We have

$$\frac{dW}{dt} = -\frac{\operatorname{sh}\pi\nu}{\pi}\frac{\hbar}{m}\left|\int_0^\infty c(k,t)k^{1/2-i\nu}\,dk\right|^2, \quad (35)$$

where

$$c(k,t) = c(k)\exp(-i\hbar k^2 t/2m). \quad (36)$$

The relation (35) has a simple physical meaning. It is not difficult to convince oneself that

$$-\frac{\operatorname{sh}\pi\nu}{\pi}\frac{\hbar}{m}\left|\int_0^\infty c(k,t)k^{1/2-i\nu}\,dk\right|^2$$

$$= \lim_{r\to 0}\left(r^2\iint j_r\,d\Omega\right) = \frac{\hbar}{2im}\lim_{r\to 0}\left(u^*\frac{du}{dr}-u\frac{du^*}{dr}\right), \quad (37)$$

i.e., the decrease of W(t) with time is equal to the fraction of particles that have fallen into the center.

We shall find the limit of expression (34) as $t \to \infty$. We have the following formula[14]:

$$\lim_{t\to\infty}\left(\mathrm{P}\frac{e^{i\xi t}}{\xi}\right) = i\pi\delta(\xi). \quad (38)$$

Taking (38) into account, we have

$$W(\infty) = e^{-\pi\nu}\int_0^\infty |c(k)|^2\,dk. \quad (39)$$

It is obvious that $W(\infty)$ cannot exceed unity, and therefore from (39) follows the inequality

$$\int_0^\infty |c(k)|^2\,dk \leqslant e^{\pi\nu}, \quad \langle f|f\rangle = 1. \quad (40)$$

If we had performed the calculations leading to the relations (33)—(39) for the time-reversed motion, we would, clearly, have obtained relations of the form (33)—(39), with ν replaced throughout by $-\nu$ and c(k) by c*(k). Then, allowing for the fact that $\widetilde{W}(\infty) \geq 1$ and also taking (40) into account, we obtain

$$e^{-\pi\nu} \leqslant \int_0^\infty |c(k)|^2\,dk \Big/ \int_0^\infty |f(r)|^2\,dr \leqslant e^{\pi\nu} \quad (41)$$

in place of the equality

$$\int_0^\infty |c(k)|^2\,dk = \int_0^\infty |f(r)|^2\,dr, \quad (42)$$

which held for the "normal" case, where p is real.

We note, finally, that from (41) follows the completeness, which we have already proved, of the set $u_{k,\pm i\nu}$. Indeed, from the fact that $c(k) \equiv 0$, it follows automatically that $f(r) \equiv 0$, i.e., a function orthogonal to all the $u_{k,\pm i\nu}$ ($c(k) \equiv 0$) is identically zero.

3. MOTION IN THE FIELD $U(r) = -\beta/r^2 + m\omega^2 r^2/2$

It is easy to see that the motion of a particle with a well-defined value l of the orbital angular momentum in the field

$$U(r) = -\frac{\beta}{r^2}+\frac{m\omega^2 r^2}{2} \quad (43)$$

formally coincides with the problem of the motion of a particle with "angular momentum" l' (6) in the field

$$U = m\omega^2 r^2/2, \quad (44)$$

i.e., with the problem of a three-dimensional isotropic oscillator, the energy levels and eigenfunctions of which are known[15]:

$$\varepsilon = 2n + l' + 3/2, \quad (45)$$

$$u = Ae^{-\rho^2/2}\rho^{l'+1}F(-n, l'+3/2; \rho^2); \quad (46)$$

here, we have used the notation

$$\varepsilon = \frac{E}{\hbar\omega}, \quad \gamma = \frac{2m\beta}{\hbar^2}, \quad \rho = \sqrt{\frac{m\omega}{\hbar}}r \quad (47)$$

and $F(a, c; \xi)$ is the confluent hypergeometric function, while n is the radial quantum number (from which we drop the index r in order not to complicate the notation) taking the values n = 0, 1, 2, ... Naturally, we are interested in the case of collapse to the center $\gamma > (l + 1/2)^2$, to which corresponds a value of l' equal to

$$l' = -1/2 - i\nu, \quad \nu = \sqrt{\gamma - (l+1/2)^2}. \quad (48)$$

With allowance for (48), the expressions (45) and (46) assume the form

$$\varepsilon_{n,-i\nu} = 2n + 1 - i\nu, \quad (49)$$

$$u_{n,-i\nu} = A\exp(-\rho^2/2)\rho^{1/2-i\nu}F(-n, 1-i\nu; \rho^2). \quad (50)$$

We rewrite expression (50) in terms of generalized Laguerre polynomials[11]

$$L_n^{(\alpha)}(\xi) = \frac{1}{n!}e^{\xi}\xi^{-\alpha}\frac{d^n}{d\xi^n}(e^{-\xi}\xi^{n+\alpha}) = \binom{n+\alpha}{n}F(-n, \alpha+1; \xi) \quad (51)$$

in the following form:

$$u_{n,-i\nu}(\rho) = \left[\frac{n!2}{\Gamma(n+1-i\nu)}\right]^{1/2}\exp\left(-\frac{\rho^2}{2}\right)\rho^{1/2-i\nu}L_n^{(-i\nu)}(\rho^2). \quad (52)$$

It must be emphasized that we are using the definition of generalized Laguerre polynomials employed in the mathematical literature, which differs from the definition of these same polynomials in the physics literature[2]. The solutions (52) are normalized by the condition

$$\langle n', i\nu | n, -i\nu\rangle = \delta_{n',n}, \quad \langle r | n, \pm i\nu\rangle = u_{n,\pm i\nu}. \tag{53}$$

As in the previous Section, in this problem a conjugate set $u_{n,i\nu}$ has arisen naturally; together with the set $u_{n,-i\nu}$, this set forms a biorthogonal system of functions.

We shall discuss the energy spectrum of our problem. The eigenvalues $E_{n,-i\nu}$ are discrete and have a negative imaginary part, and this entails an exponential time dependence for the probability of finding a particle in a finite part of space. All the "bound" states have the same lifetime:

$$\tau = 1/2\omega\nu, \tag{54}$$

since the imaginary part of $E_{n,-i\nu}$ is independent of n (for fixed l).

We shall now prove the completeness of the set of eigenfunctions (52) of our problem. As is well-known[6], a system of functions is complete (or closed), if there exists no function with an integrable square, apart from that identically equal to zero, that would be orthogonal to all the functions of the given set $\{u_n\}$. We assume the opposite to be true, i.e., let

$$\langle F | n, -i\nu\rangle = 0, \quad n = 0, 1, 2, \ldots \tag{55}$$

Since all the $u_{n,-i\nu}$ contain the common factor $\exp(-\rho^2/2)\rho^{1/2-i\nu}$ and $L_n^{(-i\nu)}(\xi)$ is a polynomial of degree n, the orthogonality conditions (55) are equivalent to the following conditions:

$$\int_0^\infty F^*(\rho)\exp\left(-\frac{\rho^2}{2}\right)\rho^{1/2-i\nu+2n}\,d\rho = 0, \quad n = 0, 1, 2, \ldots \tag{56}$$

Introducing the new independent variable $\xi = \rho^2$ and the new function

$$f(\xi) = \frac{F(\rho)}{\sqrt{2\rho}}, \quad \int_0^\infty |f(\xi)|^2 d\xi = \int_0^\infty |F(\rho)|^2 d\rho,$$

we rewrite (56) in the form

$$\int_0^\infty f^*(\xi)\xi^{-i\nu/2+n}\,d\xi = 0, \quad n = 0, 1, 2, \ldots \tag{56'}$$

Thus, the function $f(\xi)e^{i\nu/2}$ belonging to Hilbert space is orthogonal to all $e^{-\xi/2}\xi^n$, but thereby also to all

$$e^{-\xi/2}L_n(\xi), \tag{57}$$

where $L_n(\xi)$ is the ordinary Laguerre polynomial. But the set of functions (57) is complete[6], i.e.,

$$f(\xi)\xi^{i\nu/2} \equiv 0,$$

and thereby, also,

$$F(\rho) \equiv 0$$

Thus, our set (52) is complete.

4. MOTION IN THE FIELD $U(r) = -Ze^2/4 - \beta/r^2$

We shall introduce atomic units and the dimensionless variable r. Then the field takes the form

$$U(r) = -\frac{Z}{r} - \frac{\gamma}{2r^2}, \quad Z > 0, \quad \gamma > 0. \tag{58}$$

If we introduce l' from formula (48), the problem reduces formally to the hydrogen-atom problem. Therefore, for the energy levels (the discrete part of the spectrum), we have the formula

$$E_{l,n} = -\frac{Z^2}{2(l'+n_r+1)^2} = -\frac{Z^2}{2(n_r+1/2-i\nu)^2} \tag{59}$$

As follows from (59), Im $E_{l,n_r} < 0$. However, in contrast to the "oscillator", in the "hydrogen-atom" case the different energy levels have different lifetimes. As in the case of the "oscillator", it is convenient to normalize the wavefunctions of the discrete spectrum by a condition of the form (53).

Consider the continuous spectrum. It is not difficult to see that it fills the positive semi-axis of E. It is evident that the wavefunctions will be essentially complex. They can be normalized by the condition

$$\langle k', i\nu | k, -i\nu\rangle = \delta(k'-k), \tag{60}$$

analogous to condition (19). It is obvious that the functions $\langle r|k, -i\nu\rangle$ are continuous-spectrum Coulomb wavefunctions, in which l is replaced by $l' = -1/2 - i\nu$.

Consider elastic scattering due to the collapse of particles to the center. We write out the value of $\eta_l = e^{2i\delta_l}$ in our case:

$$\eta_l = e^{i\pi(l-l')}\frac{\Gamma(1+l'-iZ/k)}{\Gamma(1+l'+iZ/k)}. \tag{61}$$

For $l' = l$, this expression coincides with the usual expression for Coulomb phases. Putting $l' = -1/2 - i\nu$ into (61), we have

$$|\eta_l|^2 = e^{-2\pi\nu}\frac{\mathrm{ch}[\pi(\nu - Z/k)]}{\mathrm{ch}[\pi(\nu + Z/k)]}. \tag{62}$$

It is interesting to apply (62) to the case $Z < 0$, i.e., to the case of Coulomb repulsion and examine then the case of small k. Then for the capture coefficient we obtain the following approximate expression

$$1 - |\eta_l|^2 \approx 2\,\mathrm{sh}\,(2\pi\nu)\exp(-2|Z|\pi/k), \tag{63}$$

in which appears the characteristic factor

$$\exp(-2\pi|Z|/k),$$

equal to the penetration factor of the Coulomb barrier. It is worth emphasizing that for $Z < 0$ there is no discrete spectrum.

5. DIRAC ELECTRON IN THE FIELD $-Ze^2/r$ FOR $Z > 137$

We shall consider, finally, the relativistic problem of the motion of an electron in the Coulomb field of a point nucleus with $Z > 137$. As is well-known[16,17], the "collapse to the center" situation also arises in this case. If we take a non-Hermitean condition of the form (3), which in the case of Dirac's equation must be somewhat transformed since the expression for the probability flux density has a different form (in comparison with the nonrelativistic case), omitting the intermediate calculations we obtain for the discrete energy levels the expression that can be obtained by analytic continuation of Sommerfeld's formula

$$E = mc^2\left\{1 + \frac{Z^2\alpha^2}{[n_r + (\varkappa^2 - Z^2\alpha^2)^{1/2}]^2}\right\}^{-1/2} \tag{64}$$

with respect to the parameter Z. For levels with $|\kappa| = 1$, where κ is the Dirac quantum number[16,17], we have

$$E_{n_r,\pm 1} = mc^2\left\{1 + \frac{Z^2\alpha^2}{[n_r - i(Z^2\alpha^2 - 1)^{1/2}]^2}\right\}^{-1/2}. \tag{65}$$

(65) takes an especially simple form for the "ground" state with $n_r = 0$ and $\kappa = -1$:

$$E_{0,-1} = -imc^2\sqrt{Z^2\alpha^2 - 1}. \tag{66}$$

Finally, it is worth mentioning the fact that, as in the "normal" case ($Z\alpha < 1$), the energy levels (65) are degenerate with respect to the sign of the quantum number κ.

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