ELECTRON-POSITRON PAIR PRODUCTION BY PHOTONS IN THE FIELD OF AN ELECTROMAGNETIC WAVE IN A HOMOGENEOUS MAGNETIC FIELD

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It is shown that due to the appearance of new bands in the quasienergy spectrum of an electron interacting with a classical electromagnetic wave and a homogeneous magnetic field, the production of an electron-positron pair by a photon in the field is possible for $\omega' \ll 2m$ (ω' is the frequency of the photon producing the pair and m is the electron mass) in the absence of real absorption of external-field quanta. Realignment of the quasienergy spectrum of an electron in the external field is analogous to the Lamb level shift in the hydrogen atom. It is a consequence of virtual multiquantum processes involving absorption of a certain number of external-field quanta by the electron and simultaneous return of the same number of quanta to the field. The pair-production probability is calculated for two extreme cases corresponding to two different mechanisms of pair production. In one case pairs are formed as a result of true absorption of a number of quanta from the electromagnetic wave sufficient for overcoming the energy barrier between the electron and positron states. In the second case pair production occurs as a result of the appearance of new bands in the electron quasienergy spectrum

1. INTRODUCTION

WE consider in this paper the generation of an electron-positron pair by a photon in a homogeneous magnetic field of intensity **H** and in the classical field of a monochromatic planar-polarized electromagnetic wave propagating along **H** (these fields will henceforth be called briefly the external field). The interaction of the particle with the external field is investigated with the aid of the exact solution of the Dirac equation[1]; while the interaction with the quantized electromagnetic field is taken into account only in first order of perturbation theory.

The external field does not produce real pair production without an additional interaction, since the action of such a field on the vacuum reduces effectively to the action of only the homogeneous magnetic field, which, as is well known[2], does not produce pairs. Therefore, on the basis of the energy and momentum conservation laws one should expect the production of a pair by a photon with energy $\omega' < 2m$ (m—electron mass) to be possible only as a result of real absorption of a number of quanta of the external electromagnetic field sufficient to overcome the energy barrier between the negative- and positive-frequency states of the particle. An investigation shows, however, that such a conclusion is in error, namely: at $\omega' < 2m$ pairs can also be produced in the absence of processes of real absorption of quanta of the external field.

This effect, which was predicted by us in[3], is explained by the fact that when the particle interacts with the external field, new bands appear in the quasienergy spectrum of the particle[4] (see Fig. 1). These bands lie inside the forbidden band that separates the elecronic and positronic states in the absence of interaction. The electron and positron produced when $\omega' < 2m$ in the absence of real multiquantum absorption processes are in states belonging to the indicated new bands.

The appearance of new quasienergy bands in the spectrum of the system is connected with virtual processes of absorption and emission of quanta of the external field, as a result of which the particle absorbs from the field a certain number of quanta and returns simultaneously the same number of quanta to the field. In other words, the nature of the radical realignment of the quasienergy spectrum of the electron in the external field is the same as the nature of the Lamb level shift in the hydrogen atom—the only difference being that the virtual processes are connected here with the external field and not with spontaneous emission.

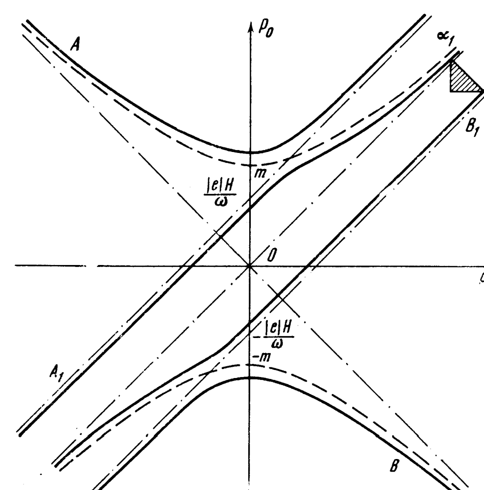

FIG. 1. Quasienergy spectrum of an electron in the field of an electromagnetic wave and a homogeneous magnetic field at fixed values of n and σ(n—number of Landau level, σ—spin variable). A, A_1 and B, B_1 are respectively the electronic and positronic branches of the spectrum. The dashed lines show the lower edge of the electronic energy band and the upper edge of the positronic energy band of the electron in the homogeneous magnetic field in the absence of the wave field, while the dash-dot lines are asymptotes to the branches of the spectrum.

A detailed investigation of the quasienergy spectrum of the electron, in the case when the external electromagnetic field is an arbitrary superposition of electromagnetic waves with different frequencies but with a fixed propagation direction, is contained in[5].

A calculation of the pair-production probability, performed by us in several particular cases, shows that in the absence of real multiquantum absorption, the process begins to have a noticeable probability only in a sufficiently intense electromagnetic field, when $\zeta = e|\mathbf{E}|/2m \gg 1$ ($\mathbf{E}$—amplitude of the intensity of the electric component of the external field). In a weak field ($\zeta \ll 1$), the generation of a pair by a low-energy photon occurs mainly as a result of multiquantum absorption.

2. DERIVATION OF FORMULA FOR THE MATRIX ELEMENT

The probability of production of an electron-positron pair by a photon in a classical external field $A = (A_0, \mathbf{A})$ is determined by the matrix element[1)]

$$M = -ie\int d^4z\,\overline{\Psi}_f(z|A)\hat{A}_{k'\lambda}(z)\Psi_i(z|A), \qquad (1)$$
$$\hat{A}_{k'\lambda}(z) = \hat{e}_\lambda e^{-ik'z}/\sqrt{2\omega' V},$$

where $\Psi_i(z|A)$ and $\Psi_f(z|A)$ are the wave functions of the electron and positron in the external field A, $A_{k'\lambda}(z)$ is the wave function of the pair-producing photon with four-momentum $k' = (\omega', \mathbf{k}')$ and polarization four-vector e_λ, $z = (t, \mathbf{r})$, and V is the normalization volume (we henceforth put V = 1). As the external-field potential we choose the four-vector ($A_y(\tau)$ is an arbitrary function of $\tau = t - z$)

$$A = (0, -Hy, A_y(\tau), 0), \qquad (2)$$

which describes the homogeneous field with intensity H, directed along the z axis, and an arbitrary electromagnetic field propagating along the magnetic field H. The exact solutions of the Dirac equation in the field (2) are given by the formulas (see[1,3])

$$\Psi^{(\pm)}_{p_xp_zn\sigma}(z) = C_{pn\sigma}\exp\{-ip_0t + ip_xx + ip_zz\}\exp\{iyN_p\}$$
$$\times\exp\left\{-\frac{i}{2\bar{n}p}\int_{\tau_0}^{\tau}d\tau' R_p(\tau')\right\}\exp\left\{-\frac{\xi_p^2}{2}\right\}u_{pn\sigma}\Big|_{p_0=\pm p_{0\sigma}}$$
$$(n = 0, 1, \ldots;\ \sigma = \pm 1),$$
$$u_{pn1} = \begin{pmatrix} (m+\bar{n}p)H_n \\ [g_p - i(N_p - eA_y)]H_n - 2nH_{n-1}\sqrt{\beta} \\ (m-\bar{n}p)H_n \\ -[g_p - i(N_p - eA_y)]H_n + 2nH_{n-1}\sqrt{\beta} \end{pmatrix},$$
$$u_{pn,-1} = \begin{pmatrix} -[g_p + i(N_p - eA_y)]H_n + H_{n+1}\sqrt{\beta} \\ (m+\bar{n}p)H_n \\ -[g_p + i(N_p - eA_y)]H_n + H_{n+1}\sqrt{\beta} \\ (-m+\bar{n}p)H_n \end{pmatrix}. \qquad (3)$$

The plus and minus signs in (3) correspond to the electronic and positronic states, respectively; p_x and p_z are the momentum components; n and σ are the number of the Landau level and the spin index; τ_0 is an arbitrary constant; $H_n \equiv H_n(\xi_p)$ is the Hermite polynomial. The remaining notation is as follows:

$$\bar{n} = (1, 0, 0, 1),\quad \bar{n}p = p_0 - p_z,\quad \beta = |e|H,$$
$$p_{0\sigma} = \sqrt{p_z^2 + m^2 + \beta(2n+1-\sigma)},\quad \xi_p = [\beta y + p_x + g_p]/\sqrt{\beta},$$
$$C_{pn\sigma} = {}^1\!/_2(\beta/\pi)^{1/4}(2^n n! p_0\bar{n}p)^{-1/2},\qquad R_p(\tau) = (N_p - eA_y)^2 - g_p^2 - 2p_xg_p.$$

The real functions $g_p \equiv g_p(\tau)$ and $N_p \equiv N_p(\tau)$ are defined by the equation

$$g_p(\tau) + iN_p(\tau) = -\frac{\beta}{2\bar{n}p}\int_{-\infty}^{\infty}d\tau'\,eA_y(\tau')\exp\left\{i\frac{\beta}{\bar{n}p}(\tau-\tau')\right\}$$
$$+\frac{\beta}{\bar{n}p}\int_{-\infty}^{\tau}d\tau'\,eA_y(\tau')\exp\left\{i\frac{\beta}{\bar{n}n}(\tau-\tau')\right\}. \qquad (4)$$

The wave functions (3) satisfy the following orthogonality and normalization condition (at $-\infty < t < +\infty$):

$$\int d\mathbf{r}\,[\Psi^{(j')}_{p_x'p_z'n'\sigma'}(\mathbf{r},t)]^+\Psi^{(j)}_{p_xp_zn\sigma}(\mathbf{r},t) = (2\pi)^2\delta(p_x - p_x')\cdot$$
$$\cdot\delta(p_z - p_z')\delta_{nn'}\delta_{\sigma\sigma'}\delta_{jj'}$$
$$(j, j' = \pm;\quad \delta_{+-} = \delta_{-+} = 0;\quad \delta_{++} = \delta_{--} = 1). \qquad (5)$$

Substitution of the wave functions (3) into the expression for the matrix element and subsequent integration with respect to the coordinates lead to the formula (we choose for the wave function of the positron, as usual, the electron function $\Psi^{(+)}_{p_xp_zn\sigma}(\mathbf{r}, t)$, in which we make the substitutions $p_{0\sigma} \to -p_{0\sigma}$; $p_x \to -p_x$; $p_z \to -p_z$)

$p_{0\sigma} \to -p_{0\sigma};\ p_x \to -p_x;\ p_z \to -p_z)$

$$M = -ie(2\pi)^2\frac{C_{pn\sigma}C_{p'n'\sigma'}}{\sqrt{2\omega'\beta}}\delta(p_x + p_x' - k_x')\delta(\bar{n}p + \bar{n}p' - \bar{n}k')$$
$$\times\int_{-\infty}^{\infty}d\tau\exp\{i\tau(p_z + p_z' - k_z')\}\exp\left\{\frac{i}{2\beta}\Phi_{p'p}(\tau) - |d_{p'p}|^2\right\}L_{\sigma'\sigma}[Y_{n'n}(p'p)]; \qquad (6)$$

here

$$\Phi_{p'p}(\tau) = \int_{\tau_0}^{\tau}d\tau' eA_y(\tau')\left(\frac{dg_{p'}(\tau')}{d\tau'} - \frac{dg_{-p}(\tau')}{d\tau'}\right) + g_{-p}N_{p'}$$
$$- g_{p'}N_{-p} - k_x'(N_{p'} + N_{-p}) - k_y'(g_{p'} + g_{-p}),$$
$$d_{p'p} \equiv d_{p'p}(\tau) = [(g_{p'} - g_{-p} + k_x') - i(N_{p'} - N_{-p} - k_y')]/2\sqrt{\beta}.$$

In formula (6), the indices p′, n′, and σ′ pertain to the electron, while p, n, and σ pertain to the positron, with

$$p_{0\sigma} = +\sqrt{p_z^2 + m^2 + \beta(2n+1-\sigma)},\quad p_{0\sigma'}' = +\sqrt{p_z'^2 + m^2 + \beta(2n'+1-\sigma')};$$

$L_{\sigma'\sigma}[Y_{n'n}(p'p)$ is a linear combination of the functions $Y_{n'n}(p'p)$, defined by the relation

$$\sqrt{\beta}\int_{-\infty}^{\infty}dy\exp\{-iy(N_{p'} - N_{-p} - k_y')\}\exp\{-\xi_{p'}^2/2\}\exp\{-\xi_{-p}^2/2\}$$
$$\times\bar{u}_{p'n'\sigma'}\hat{e}u_{-pn\sigma} = \exp\left\{\frac{i}{2\beta}(N_{p'} - N_{-p} - k_y')(k_x' - 2p_x + g_{p'} + g_{-p})\right.$$
$$\left.-|d_{p'p}|^2\right\}L_{\sigma'\sigma}[Y_{n'n}(p'p)]. \qquad (7)$$

The function $Y_{n'n}(p'p)$ is defined as follows:

$$Y_{n'n}(p'p) = \int_{-\infty}^{\infty}d\eta\,e^{-\eta^2}H_{n'}(\eta + d_{p'p})H_n(\eta - d_{p'p}). \qquad (8)$$

In explicit form, the expressions for $L_{\sigma'\sigma}[Y_{n'n}(p'p)]$ are given in Appendix I. We note that in formula (6) we have omitted a constant phase factor which arises when the integrals are taken in the initial matrix element.

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1) We use in this paper a four-dimensional notation: $ab = a_0b_0 - \mathbf{a}\cdot\mathbf{b}$ is the scalar product of the four-vectors $a = (a_0, \mathbf{a})$ and $b = (b_0, \mathbf{b})$; $\hat{a} = \gamma_0a_0 - \boldsymbol{\gamma}\mathbf{a}$; γ_0 and $\boldsymbol{\gamma}$ are Dirac matrices ($\gamma^+ = -\gamma$). We use a system of units in which $c = \hbar = 1$.

We confine ourselves to the case of a monochromatic plane-polarized electromagnetic wave of frequency ω:

$$A_y(\tau) = a\cos\omega\tau, \quad (a = \text{const}). \tag{9}$$

The functions $g_p(\tau)$ and $N_p(\tau)$ defined in (4) are written in this case in the form[2)]

$$g_p(\tau) = -ea\frac{\beta\omega\bar{n}p}{\beta^2-(\omega\bar{n}p)^2}\sin\omega\tau, \quad N_p(\tau) = ea\frac{\beta^2}{\beta^2-(\omega\bar{n}p)^2}\cos\omega\tau. \tag{10}$$

Introducing further the symbol

$$F_{p'p} = -\frac{1}{8}(ea)^2\omega\frac{\bar{n}p[\beta^2+(\omega\bar{n}p')^2]+\bar{n}p'[\beta^2+(\omega\bar{n}p)^2]}{[\beta^2-(\omega\bar{n}p)^2][\beta^2-(\omega\bar{n}p')^2]}$$

and using the Fourier expansion

$$\exp\left\{-iF_{p'p}\sin 2\omega\tau - \frac{i}{2\beta}[k_x'(N_{p'}+N_{-p})+k_y'[g_{p'}+g_{-p})] - |d_{p'p}|^2\right\}$$

$$\times L_{\sigma'\sigma}[Y_{n'n}(p'p)]\frac{1}{\sqrt{2^n n!\,2^{n'}n'!}} = \sum_{s=-\infty} e^{-is\omega\tau}B^{(\sigma'\sigma)}_{s,\,n'n}, \tag{11}$$

we obtain for the square of the matrix element the expression (L_x and L_z are the dimensions of the principal region in the directions of the axes x and z; $\alpha = e^2/4\pi$)

$$\frac{|M|^2}{TL_xL_z} = \alpha\frac{(2\pi)^3}{8\omega'}\frac{1}{p_0p_0'(\bar{n}p)(\bar{n}p')}\delta(p_x'+p_x-k_x')$$

$$\times\delta(\bar{n}p'+\bar{n}p-\bar{n}k')\sum_{s=-\infty}^{\infty}|B^{(\sigma'\sigma)}_{s,\,n'n}|^2\delta\Big(p_z'+p_z-s\omega-k_z'-\frac{(ea)^2}{4}\omega^2$$

$$\times\left(\frac{\bar{n}p}{\beta^2-(\omega\bar{n}p)^2}+\frac{\bar{n}p'}{\beta^2-(\omega\bar{n}p')^2}\right)\Big) \tag{12}$$

The probability (per unit time) of production of an electron-positron pair in a state with quantum numbers $n'\sigma'$ and $n\sigma$, summed over the momenta of the final states, is written in final form as follows:

$$W_{n'\sigma';\,n\sigma} = \frac{\alpha}{16\pi}\frac{\beta}{\omega'}\int_0^{\bar{n}k'}d\eta\frac{1}{\eta^2(\bar{n}k'-\eta)^2}\sum_{s=-\infty}^{\infty}|B^{(\sigma'\sigma)}_{s,\,n'n}|^2\Big|_{\bar{n}p=\eta,\,\bar{n}p'=\bar{n}k'-\eta}$$

$$\times\delta\Big(\frac{m^2+\beta(2n+1-\sigma)-\eta^2}{2\eta}+\frac{m^2+\beta(2n'+1-\sigma')-(\bar{n}k'-\eta)^2}{2(\bar{n}k'-\eta)}$$

$$-s\omega-k_z'-\frac{1}{4}(ea)^2\omega^2\left(\frac{\eta}{\beta^2-(\omega\eta)^2}+\frac{\bar{n}k'-\eta}{\beta^2-\omega^2(\bar{n}k'-\eta)^2}\right)\Big). \tag{13}$$

The separate terms in the sum over s in (13) describe the processes of absorption and emission of quanta of the external field $A_y(\tau)$, with the terms with $s > 0$ corresponding to the absorption of s quanta, and those with $s < 0$ to the emission of $|s|$ quanta.

3. CONSERVATION LAWS AND QUASIENERGY SPECTRUM

Before we proceed to calculate the probability of pair production by a photon in accord with the formula derived in the preceding section, let us describe in greater detail the consequences ensuing from the energy and momentum (more accurately—quasienergy and quasimomentum) conservation laws, which are contained in formulas (12) and (13). The conservation laws can be written in the following symmetrical form:

$$p_0'+p_0-\omega'-s\omega-\Delta=0, \tag{14}$$

$$p_z'+p_z-k_z'-s\omega-\Delta=0, \tag{15}$$

where

$$\Delta=\frac{1}{4}(ea)^2\omega\left(\frac{kp}{\beta^2-(kp)^2}+\frac{kp'}{\beta^2-(kp')^2}\right), \quad k=\omega\bar{n}.$$

We shall be interested henceforth exclusively in the case when the frequencies of the external field and of the pair-producing photon satisfy the conditions

$$\omega \ll m, \quad \omega' \ll m, \tag{16}$$

and the intensity of the homogeneous magnetic field and the frequency of the external field are such that

$$\omega_H \ll \omega \quad (\omega_H = |e|H/m). \tag{17}$$

Taking into account the relation

$$\bar{n}p+\bar{n}p'=\bar{n}k', \tag{18}$$

which follows from the conservation laws (14) and (15) and applying the second inequality of (16), we find that pair production by a photon is possible only if $\bar{n}p' \ll 2m$ and $\bar{n}p \ll 2m$. The last two inequalities mean that the produced electrons and positrons have ultrarelativistic momenta ($p_z, p_z' \gg m$) in the direction of wave propagation, with $p_\perp \ll p_z$ ($p_\perp^2 = \beta(2n+1-\sigma)$ is the square of the transverse momentum of the particle). At small values of the transverse momentum ($p_\perp < m$), the longitudinal momentum of the particle is of the order of $m^2/\bar{n}k'$ (at $\bar{n}p \approx \bar{n}p'$). The electron and positron generated by the photon thus move within a narrow cone with its axis along the direction of propagation of the wave, and with a vertex angle $\theta \approx 2p_\perp/p_z \ll 1$.

Equations (14) and (15) contain a characteristic term Δ proportional to the intensity of the electromagnetic wave and having a resonant character: as $\beta \to kp$ (or $\beta \to kp'$), this term assumes arbitrarily large values. The equality $\beta = kp$, which can be written in the more usual form

$$\omega(1-v_z)=|e|H/\epsilon_p$$

($\epsilon_p = p_0$, $v_z = p_z/\epsilon_p$, $|e|H/\epsilon_p$ is the cyclotron frequency of the electron), is the condition for cyclotron resonance of the electron in the homogeneous magnetic field and in the field of the electromagnetic wave[3)]. We note that the energy and momentum conservation laws for the processes occurring in the field of the monochromatic electromagnetic wave in the absence of the homogeneous magnetic field also contain a term that depends on the wave intensity. The latter, however, does not have a resonant form, in contrast to the present problem, and reduces simply to a replacement of the electron mass m by the effective mass $m^* = \sqrt{m^2+\frac{1}{2}(ea)^2}$. This means, obviously, an increase in the energy gap separating the electronic and positronic states of the particle (see the dispersion Equation (21) with H = 0).

The conservation laws (14) and (15) admit of the following two main mechanisms of pair production by a low-energy photon:

a) A multiquantum process, wherein the number s

[2)] Fractions of the form $(\beta^2-(\omega\bar{n}p)^2)^{-1}$ must be understood throughout in the sense of the principal value.

[3)] Strictly speaking, this inequality is the condition for cyclotron resonance of a particle in a state with $p_\perp = 0$.

of quanta absorbed from the external field suffices to overcome the energy barrier between the electronic and positronic states; in this case $s \sim (p_z + p_z')/\omega \gg 1$.

b) Allowance for the term Δ in the conservation laws; this term can be positive and sufficiently large in magnitude to satisfy the conservation laws at $\omega' \ll m$. It is easily seen that when $|ea|/m \ll 1$, such a possibility is realized in the region of cyclotron resonance, when at least one of the denominators in the expression for Δ becomes sufficiently small. The second mechanism, as will become clear later on, is also essentially connected with multiquantum processes, but with processes of a different kind--virtual processes of absorption and emission of external-field quanta, as a result of which the entire energy spectrum of the electron-photon system is considerably realigned.

The situation with pair production by a photon of low energy, without allowance for real multiquantum absorption processes, can be understood in terms of the quasienergy dispersion curves. According to the definitions of quasienergy[4] and quasimomentum, an electron in a state with a wave function (3), when the vector potential of the external field is chosen in the form (9), has a quasienergy

$$P_0 = p_0 - \frac{e^2a^2}{4\bar{n}p}\frac{(kp)^2}{\beta^2-(kp)^2} \tag{19}$$

and a z-component of the quasimomentum

$$P_z = p_z - \frac{e^2a^2}{4\bar{n}p}\frac{(kp)^2}{\beta^2-(kp)^2}; \tag{20}$$

here

$$p_0 = \sqrt{p_z^2 + m^2 + \beta(2n+1-\sigma)}.$$

In terms of the new notation, the conservation laws (14) and (15) take the following sample form:

$$P_0' + P_0 - \omega' - s\omega = 0, \tag{14a}$$

$$P_z' + P_z - k_z' - s\omega = 0, \tag{15a}$$

which is analogous to the usual energy and momentum conservation laws in processes in which $k' = (\omega', k_z')$, $sk = (s\omega, s\omega)$ and $P' = (P_0', P_z')$; $P = (P_0, P_z)$ are quantum numbers pertaining to the initial and final states of the system, respectively. It is important here that the dispersion law (the dependence of P_0 on P_z) on the particles in the final state differ quite considerably from the dispersion law of a particle in a homogeneous magnetic field. In fact, eliminating p_z from (19) and (20), we obtain the following dispersion equation[3]

$$P_0^2 - P_z^2 - m^2 - \beta(2n+1-\sigma) + \frac{e^2a^2}{2}\frac{(kP)^2}{\beta^2-(kP)^2} = 0. \tag{21}$$

At fixed values of the quantum numbers n and σ, this equation determines four quasienergy branches--two electronic (branches A and A_1 in Fig. 1) and two positronic (branches B and B_1). Branches with different values of n and σ are grouped into bands of allowed values of quasienergy, and for these bands we retain the earlier notation (bands A, A_1, etc.). Bands A_1 and B_1 lie inside the forbidden band that separates the electronic and positronic states in the absence of a wave field. It is the appearance of new quasienergy bands which causes the already mentioned second mechanism of pair production by a low-energy photon, namely, the produced electron and positron are in states belonging to just these bands.

We note that the separation of the quasienergy spectrum into four bands is Lorentz-invariant. The electronic bands A and A_1 correspond to positive values of $\bar{n}P$, and the positronic B and B_1 to negative values; the new bands A_1 and B_1 correspond to $(kP)^2 < \beta^2$, and therefore in the figure these bands lie between the two straight lines parallel to the bisector of the first coordinate angle and intersecting on the ordinate axis the segment $(-|e|H/\omega)$ (see Fig. 1).

The physical meaning of the quantities P_0 and P_z is explained by the relation

$$\overline{v_{n\sigma}^{(z)}} = \frac{dP_0}{dP_z}\ \Big(v_{n\sigma}^{(z)} \int\limits_{(L_xL_z)} dxdz \int\limits_{-\infty}^{\infty} dy\Psi_{pn\sigma}^+\Psi_{pn\sigma}$$

$$= \int\limits_{(L_xL_z)} dxdz \int\limits_{-\infty}^{\infty} dy\Psi_{pn\sigma}^+\hat{v}_z\Psi_{pn\sigma}\Big),$$

where $\hat{v}_z$ is the operator of the z-component of the electron velocity and the bar denotes averaging over the time. The tangent to the dispersion curve thus determines the electron flux density along the z axis.

We emphasize that formulas (19) and (20), as well as Eq. (21), are valid in any vicinity of the cyclotron-resonance point, with the exception of the point $kp = |e|H$ itself. This point, however, makes no contribution to the pair-production probability since, first, at $kp = |e|H$ (or $kp' = |e|H$) the conservation laws (14) and (15) are not satisfied simultaneously, and second, the coefficients that determine the probability $B_{s,n'n} \to 0$ as $kp \to |e|H$.

Taking into account the complete analogy between the usual energy and momentum conservation laws and the quasienergy and quasimomentum conservation laws (14a) and (15), we can readily understand that it is possible to operate with the quasienergy and quasimomentum just as well as with the energy and momentum. Let us show, for example, how it is possible to calculate with the aid of the quasienergy dispersion curves the photon energy ω_{min} necessary to produce a pair by the second mechanism.

It is obvious that to calculate ω_{min} it is necessary to consider those regions of the quasimomentum P_z, in which the branches A_1 and B_1 come closest together, i.e., the regions $|P_z| \gg m$. Let us consider for concreteness the region of positive quasimomenta. It is easy to establish that the branches of band A_1 approach asymptotically the bisector of the first quadrant, and the branches of band B_1 approach the straight line $P_0 = P_z - |e|H/\omega$. Let the produced electron and positron correspond to points α_1 and α_2 on Fig. 1. From the conservation laws (14a) and (15a) it follows that if a pair is produced by a photon, then $\Delta P_0 \geq \Delta P_z$, where $\Delta P_0(\Delta P_z)$ is the difference of the quasienergies (the z-components of the quasimomenta) of the electron and positron, equal to the energy (z-component of the momentum) of the pair-producing photon. The minimum energy of the photon is obtained at $\Delta P_0 = \Delta P_z$. Recognizing that the length of the segment $\alpha_1\alpha_2$ is $|e|H/\sqrt{2}\omega$, we obtain from the shaded triangle

$$2(\Delta P_0)^2 = (\alpha_1\alpha_2)^2 = \frac{1}{2}\left(\frac{|e|H}{\omega}\right)^2.$$

Hence

$$\omega_{min} = \frac{1}{2}\frac{\omega_H}{\omega} m.$$

This formula, like the schematic diagram (Fig. 1), is valid only when $|ea|/m \ll 1$. When $|ea|/m \gg 1$, the sections of the branches A_1 and B_1 having the largest curvature approach the origin and as a result ω_{min} decreases with increasing $|ea|/m$, with $\omega_{min} \to 0$ as $|ea|/m \to \infty$.

The realignment of the energy spectrum of the electron in the external field is a consequence of definite features of the dynamics of the interaction between the particle and the external field. The character of these features becomes clear from a perturbation-theory calculation of the shift of the electron energy in the field of the electromagnetic wave (9).

The correction of second order in eA_y to the Green's function of the electron in a homogeneous magnetic field can be calculated in the usual manner (see, for example,[6]). The Feynman diagrams corresponding to this correction (Fig. 2) describe the exchange of a quantum between the electron and the wave—such a process will be called virtual absorption and emission. As a result of the exchange process, the electron does not return to the initial state, but goes over into a state with changed energy. Omitting the details of the calculations, we present the final expression for the correction $\Delta\epsilon_p$ to the electron energy

$$\Delta\varepsilon_p = \frac{(ea)^2}{4p_0}\frac{(kp)^2}{\beta^2-(kp)^2} \qquad (p_0 = \sqrt{p_z^2+m^2+\beta(2n+1-\sigma)}). \quad (22)$$

It can be shown that summation of all the terms of the perturbation-theory series for the Green's function corresponding to the diagrams of the self-energy of the electron reduces to a replacement of the expression

$$x \equiv [p_0^2 - p_z^2 - m^2 - \beta(2n+1-\sigma) + i\varepsilon]^{-1} \qquad (\varepsilon > 0),$$

which determines the Fourier transform of the Green's function in a homogeneous magnetic field, by an infinite series of the form

$$x\left[1 - x\frac{(ea)^2}{2}\frac{(kp)^2}{\beta^2-(kp)^2} + \left(x\frac{(ea)^2}{2}\frac{(kp)^2}{\beta^2-(kp)^2}\right)^2 - \ldots\right].$$

If the denominator of this geometric progression is smaller than unity, then the sum of the series can be represented in the form of the fraction

$$\left[p_0^2 - p_z^2 - m^2 - \beta(2n+1-\sigma) + \frac{(ea)^2}{2}\frac{(kp)^2}{\beta^2-(kp)^2} + i\varepsilon\right]^{-1},$$

the denominator of which gives the dispersion equation (21) that is obtained from an exact solution of the Dirac equation.

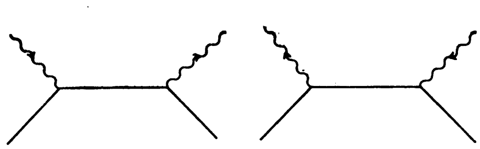

FIG. 2. The external straight lines and the internal lines represent respectively the wave and Green's functions of the electron in a homogeneous magnetic field; the external wavy lines represent the scattering of the electron by the potential A, which describes the field of the electromagnetic wave (a downward arrow on a wavy line means absorption of a quantum of the field A, and an upward arrow the emission of a quantum of this field).

The foregoing analysis reveals the cause of the radical realignment of the energy spectrum of this system, which was discussed above. This cause lies in a characteristic feature of the dynamics of the system—the existence of virtual processes, in which the electron borrows from the field of the electromagnetic wave a definite number of quanta, which it simultaneously returns to the wave field. The nature of the splitting of the energy bands in this problem is the same as the nature of the Lamb shift of the levels in the hydrogen atom, the only difference being that here the virtual processes are connected not with the spontaneous emission but with the external electromagnetic field.

Such an appreciable realignment of the energy spectrum of the system, caused by virtual multiquantum processes, leads to an appreciable redistribution of the charges in the vacuum. The results obtained above concerning pair production by a low-energy photon demonstrate that in the presence of an external field not all the levels with negative quasienergy are filled completely, and not all levels with positive quasienergy are free. Namely, all the states corresponding to the positron bands B and B_1 are completely filled, and the remaining states are free. This statement can be proved also formally, for example with the aid of the electron Green's function, by assuming the Dirac hypothesis that all the particle states with negative energy are filled in the absence of the electromagnetic wave field and by turning on the interaction with the wave adiabatically.

As follows from our reasoning, in spite of the deep analogy between the concepts of energy and quasienergy, there is also a strong difference between them, manifest, in particular, in the fact that the ground state of the quasistationary system need not be a state with a minimum quasienergy, as is the case in the present problem.

In concluding this section we emphasize that the energy necessary for the production of a pair with the aid of a second mechanism is drawn from the external field. The energy of the external field goes to redistribution of the vacuum charges, accompanied by realignment of the energy spectrum, and it is borrowed from the field not in the form of an integer number of quanta, but continuously—in proportion to the intensity of the field (see formula (22)). Attention should be called to the fact that $\Delta\epsilon_p$ can have either sign, but the effect of pair production occurs at positive values of Δ (see (14) and (15), or, what is the same, at positive values of $\Delta\epsilon_p$.

4. CALCULATION OF PAIR-PRODUCTION PROBABILITY

We confine ourselves to the case when the wave vector $\mathbf{k}'$ of the photon producing the electron-positron pair is directed opposite to the external electromagnetic wave, i.e., $\mathbf{k}' = (0, 0, -\omega')$, $\bar{n}\mathbf{k}' = 2\omega'$, and the photon polarization vector is $\mathbf{e} = (0, 1, 0)$.

According to (13), calculation of the pair-production probability reduces to a calculation of the coefficients

$B^{(\sigma'\sigma)}_{s,n',n}$, which are determined by the Fourier expansion (11). We denote by $\bar{B}_{s,n'n}$ the functions obtained from $B^{(\sigma'\sigma)}_{s,n'n}$ by means of the substitution $L_{\sigma'\sigma}[Y_{n'n}] \to Y_{n'n}$. The coefficients $B^{(\sigma'\sigma)}_{s,n'n}$ can be expressed in terms of $\bar{B}_{s,n'n}$ with the aid of formulas given in Appendix I.

The functions $\bar{B}_{s,n'n}$ are determined by the formula

$$\bar{B}_{s,n'n} = (-1)^{n'-n}\frac{1}{2\sqrt{\pi}}\int_{-\pi}^{\pi} d\varphi\, u_{n'n} I_{n'n}(\rho)\exp i(s\varphi + c_0 \sin 2\varphi), \quad (23)$$

where ($L_n^{(\alpha)}(\rho)$ is a Laguerre polynomial)

$$u_{n'n} = [\operatorname{sign}(a_0 \sin\varphi)]^{n'-n}\exp\left\{i(n'-n)\operatorname{arctg}\left(\frac{b_0}{a_0}\operatorname{ctg}\varphi\right)\right\},$$

$$I_{n'n}(\rho) = \sqrt{\frac{n!}{n'!}}\rho^{(n'-n)/2}e^{-\rho/2}L_n^{(n'-n)}(\rho), \quad I_{n'n}(\rho) = (-1)^{n'-n}I_{nn'}(\rho);$$

$$\rho \equiv \rho(\varphi) = 2(a_0^2\sin^2\varphi + b_0^2\cos^2\varphi),$$

$$a_0 = \beta y[\beta^2 - \omega^2\eta(\bar{n}k' - \eta)], \qquad b_0 = \beta^2\omega y(\bar{n}k' - 2\eta),$$

$$c_0 = \frac{ea}{4}y\sqrt{\beta}[\beta^2 + \omega^2\eta(\bar{n}k' - \eta)],$$

$$y = \frac{ea\omega\bar{n}k'}{[\beta^2 - (\omega\eta)^2][\beta^2 - \omega^2(\bar{n}k' - \eta)^2]2\sqrt{\beta}}. \quad (24)$$

We write out the conservation law contained in (13), and apply it to the present case:

$$F(\eta) \equiv \frac{m^2 + \beta(2n + 1 - \sigma)}{2\omega\eta} + \frac{m^2 + \beta(2n' + 1 - \sigma')}{2\omega(\bar{n}k' - \eta)} - s - \frac{ea}{2\sqrt{\beta}}a_0 = 0. \quad (25)$$

It can be shown that at small values of the quantum numbers n and n′ the coefficients $\bar{B}_{s,n'n}$ are exponentially small (we recall that satisfaction of the condition (16) is assumed). It is therefore necessary to use the asymptotic form of the Laguerre polynomials at large values of the indices (and of the argument). We note also that in accordance with the known asymptotic form of the Laguerre polynomial $L_n^{(\alpha)}(x)$[7] in the region of $\nu \gg 1$, $x > \nu$ ($\nu = 4n + 2\alpha + 2$) where the polynomials are monotonic, the function $e^{-x/2}L_n^{(\alpha)}(x)$ is exponentially small. Those regions of values of φ in the integral (23) which correspond to the transition point ($x \approx \nu$) or the oscillation region ($1 \ll x < \nu$) likewise make no noticeable contribution to the integral, for in this case the stationary points of the argument of the exponential in the integrand of (23) are complex. For this reason, it is of interest to consider only those values of the parameters that determine expression (23) for which the argument ρ of the function $I_{n'n}$ can be small. Such a possibility is realized when $b_0/a_0 \ll 1$ and $\sin^2\varphi \ll 1$. The second possibility ($a_0/b_0 \ll 1$ and $\cos^2\varphi \ll 1$) will not be considered, since it leads to an exponentially small contribution to the integral.

In the calculation of the coefficients $\bar{B}_{s,n'n}$ it is convenient to use the Tricomi relation[7]

$$e^{-\rho/2}L_n^{(\alpha)}(\rho) = \frac{(n+\alpha)!}{n!}\left(\frac{\nu\rho}{4}\right)^{-\alpha/2}\sum_{m=0}^{\infty}A_m\left(\frac{\rho}{\nu}\right)^{m/2}J_{\alpha+m}(\sqrt{\nu\rho}), \quad (26)$$

where A_m are constants ($A_0 = 1$, $A_1 = 0, \ldots$) and $\nu = 4n + 2\alpha + 2$. It is easy to verify that when the inequalities $\alpha\rho/\nu \ll 1$ and $\rho^3/\nu \ll 1$ are satisfied, we can confine ourselves to the first term of the series in (26). Using the Tricomi relation, we obtain the formula ($J_{n'-n}(x)$ is a Bessel function)

$$I_{n'n}(\rho) \approx J_{n'-n}(\sqrt{\nu\rho}), \quad (27)$$

which is valid if

$$\frac{(n'-n)\rho}{\nu} \ll 1, \quad \frac{\rho^3}{\nu} \ll 1, \quad \frac{(n'-n)^3}{\nu^2} \ll 1, \quad \frac{n'-n}{\nu} \ll 1. \quad (28)$$

Now the function $\bar{B}_{s,n'n}$ can be represented in the form (we put $a_0 > 0$)

$$\bar{B}_{s,n'n} \approx (-1)^{n'-n}\frac{1}{2\sqrt{\pi}}\int_0^{\pi} d\varphi\, J_{n'-n}(\sqrt{\nu\rho})[e^{ih(\varphi)} + (-1)^{n'-n}e^{-ih(\varphi)}],$$

$$h(\varphi) = s\varphi + c_0\sin 2\varphi + (n'-n)\operatorname{arctg}\left(\frac{b_0}{a_0}\operatorname{ctg}\varphi\right). \quad (29)$$

We calculate the integral (29) by the stationary-phase method, assuming $c_0 \gg 1$ and $n' - n \gg 1$. The function $h(\varphi)$ has two stationary points, only one of which leads to values $\rho(\varphi)$ satisfying the conditions (28). This point (φ_0) is determined by formula (A.2) of the Appendix; it is real when

$$(n'-n)\left[s + 2c_0\frac{a_0^2 + b_0^2}{a_0^2 - b_0^2}\right]^{-1}(1 + \varepsilon) \geqslant \frac{b_0}{a_0} \qquad (b_0 > 0). \quad (30)$$

A simple derivation shows that $n' - n \geq \sqrt{\nu\rho_0}$ ($\rho_0 = \rho(\varphi_0)$). Taking this inequality into account, we can readily establish with the aid of the asymptotic form of the Bessel function $J_{n'-n}(\sqrt{\nu\rho_0})$ for $n' - n \geq \sqrt{\nu\rho_0} \gg 1$ that the largest value of the integral of (29) is obtained with the equals sign in (30), i.e., $\varphi_0 = 0$. Using now formula (A.4) of the Appendix, in the derivation of which it was assumed that $\varphi_0 = 0$, we get

$$|\bar{B}_{s,n'n}| \approx \frac{\Gamma(1/3)}{2\sqrt{\pi}\,3^{1/6}}\frac{(n'-n)^{2/3}}{|s + 2c_0|}|J_{n'-n}(2b_0\sqrt{n'+n})|. \quad (31)$$

The coefficients $\bar{B}_{s,n'n}$ assume the largest value at $n' - n \approx 2b_0\sqrt{n' + n}$. We shall henceforth use the notation

$$\frac{2b_0\sqrt{n'+n}}{n'-n} = 1 - \alpha_0 \quad (0 < \alpha_0 \ll 1), \quad \delta = \frac{b_0}{a_0} \quad (\delta \ll 1). \quad (32)$$

According to the results of the Appendix (see (A.5)) expression (31) is valid subject to satisfaction of the conditions

$$\begin{aligned}&(n'-n)^{2/3} \gg 1, && \text{if} \quad {}^1/_3(n'-n)(2\alpha_0)^{3/2} \ll 1;\\ &(2\alpha_0)^{3/2}(n'-n)^{1/3} \ll 1, && \text{if} \quad {}^1/_3(n'-n)(2\alpha_0)^{3/2} \gg 1,\end{aligned} \quad (33)$$

which follow from the inequalities (A.6) and

$$a_0\sqrt{n'+n}/|s + 2c_0| \lesssim 1,$$

which is satisfied for all the cases considered below.

The root of Eq. (25) satisfying the condition $\delta \ll 1$, is determined by the approximate equality $\eta_0 \approx \bar{n}k'/2 \approx \omega'$. It therefore follows from (25) that

$$n' + n \approx \gamma s + \frac{ea}{2\sqrt{\beta}}\gamma a_0(\eta_0) - \frac{m^2}{\beta}; \quad (25a)$$

$$\gamma = \frac{\omega\eta_0}{\beta} \quad (a_0(\eta_0) \equiv a_0|_{\eta=\eta_0}).$$

With the aid of the formulas of Appendix I we get

$$B^{(11)}_{s,n'n} \approx -B^{(-1,-1)}_{s,n'n} \approx 4i\eta_0\sqrt{2n\beta}\,\bar{B}_{s,n'n},$$

$$B^{(-1,1)}_{s,n'n} = B^{(1,-1)}_{s,n'n} \approx -4i\eta_0 m\bar{B}_{s,n'n}. \quad (34)$$

In the cases considered by us $|B^{(-1,1)}_{s,n'n}| \ll |B^{(11)}_{s,n'n}|$, and therefore we shall neglect the quantities $B^{(-1,1)}_{s,n'n}$ and $B^{(1,-1)}_{s,n'n}$.

Taking formulas (31) and (34) into account, we obtain finally the following expression for the pair production probability in the case of an s-quantum process:

$$W^{(s)}_{nn'} \equiv \sum_{\sigma,\sigma'=\pm 1} W^{(s)}_{n'\sigma';\,n\sigma} \approx C_1 f_s(\gamma)\,\omega\,\frac{(n'-n)^{1/3}}{n} J^2_{n'-n}(2b_0\sqrt{n'+n}), \quad (35)$$

where

$$C_1 = \frac{\alpha}{\pi^2}\frac{1}{3^{1/3}}\Gamma^2\left(\frac{1}{3}\right),$$

$$f_s(\gamma) = \frac{n^2}{(s+2c_0)^2}\frac{1}{\gamma}\left[1+\frac{a_0^2}{|s+2c_0|}\gamma\frac{3+\gamma^2}{2}\right]^{-1}.$$

In the derivation of (35) we took into account the relation (see (30); we neglect the quantity $\epsilon \sim (n'-n)\delta/(s+2c_0) \ll 1$)

$$n'-n = \frac{b_0}{a_0}\left(s+2c_0\frac{a_0^2+b_0^2}{a_0^2-b_0^2}\right) \approx \frac{b_0}{a_0}(s+2c_0). \quad (36)$$

The quantity $W^{-1}(\omega')$, where

$$W(\omega') = \sum_{n,n'=0}^{\infty}\sum_{s=-\infty}^{\infty} W^{(s)}_{nn'},$$

is the lifetime of the photon in an external field with respect to pair production[8] and $K(\omega') = W(\omega')/c$ is the photon absorption coefficient (here c is the speed of light). The number of electron-positron pairs produced per second at a distance x by a photon beam with cross section S is given by the formula

$$N = \sum_{n,n'=0}^{\infty}\sum_{s=-1}^{\infty} N^{(s)}_{n'n}; \quad N^{(s)}_{n'n} = Sx\int d\omega'\,\rho(\omega')\frac{W^{(s)}_{nn'}}{c}, \quad (37)$$

where $\rho(\omega)\Delta\omega$ is the number of photons having an energy in the interval $(\omega, \omega+\Delta\omega)$ and passing each second through a cross section of 1 cm^2. This formula is valid when $|xK(\omega')| \ll 1$.

We shall consider expression (35) in two limiting cases corresponding to two different mechanisms of pair production by a low-energy photon (see Section 3).

We first study the pair-production mechanisms connected with virtual many-quantum processes, leading to the occurrence of new quasienergy bands. Using (25a) and (36), we obtain the following expression for the ratio of the argument of the Bessel function in (35) to its order at $s = 0$:

$$\left.\frac{2b_0\sqrt{n'+n}}{n'-n}\right|_{s=0} \approx 2\frac{1-\gamma^2}{1+\gamma^2}\frac{1}{\zeta}\left(\frac{2\gamma^2\zeta^2}{1-\gamma^2}-1\right)^{1/2}, \quad \zeta = \frac{|ea|}{2m}. \quad (38)$$

The probability (35) reaches a maximum value in the case when this ratio is equal approximately to unity. In this case

$$\left.\frac{2b_0\sqrt{n'+n}}{n'-n}\right|_{s=0} \to 1 \quad \text{as } \zeta\to\infty, \quad \gamma^2 = \frac{1}{3}.$$

Confining ourselves to the given value of γ^2 we get from (38) at $\zeta \gg 1$

$$2\alpha_0 = \frac{1}{\zeta^2}, \quad f_s(\gamma)|_{s=0;\,\gamma^2=1/3} \approx \frac{\sqrt{3}}{88}. \quad (39)$$

It follows from the formulas written out above that the pair-production probability becomes noticeable only in a sufficiently strong electromagnetic field, when $\zeta \gg 1$. Figure 3 shows a plot of $W^{(0)}_{nn'}$ against ζ for $\omega = 3\times 10^{15}$ sec^{-1}, $H = 10^5$ Oe, and $\delta = 0.5 \times 10^{-7}$ ($\gamma^2 = 1/3$). At the chosen values of ω and H, the frequency of the pair-producing photon is 0.34 $\times 10^{-3}$ m. The dependence of the probability on ζ reaches a maximum at $\zeta \approx 20$. At lower values of ζ, the probability decreases very sharply; thus, at $\zeta = 5$ (this corresponds to an intensity of the electric field of the external electromagnetic wave $|\mathbf{E}| = 0.5 \times 10^{12}$ V/cm)

$$W^{(0)}_{nn'} \approx 8\cdot 10^{-6}\ \text{sec}^{-1}.$$

The condition $\delta \ll 1$ denotes a high degree of resonance between the photon frequency ω' and the frequency $\omega^* = \gamma\omega_H m/\omega$. In fact, the equality $b_0/a_0 = \delta$ can be written in the form ($\gamma^2 = 1/3$)

$$\Delta\omega' \equiv \frac{\omega'-\omega^*}{\omega^*} \approx \frac{\delta}{\sqrt{3}}. \quad (40)$$

With increasing δ (i.e., with deviation from resonance), the pair-production probability decreases very slowly; for example, at $\delta = 2\times 10^{-7}$ and at the remaining parameters given above, the maximum of the probability is 9×10^{-6} sec^{-1} ($\zeta \approx 10^2$).

As $\zeta \to \infty$ (at fixed δ) the probability decreases like $W^{(0)}_{nn'} \sim \zeta^{-8/3}$. With decreasing intensity of the homogeneous magnetic field (δ and ζ are fixed), $W^{(0)}_{nn'}$ decreases exponentially:

$$W^{(0)}_{nn'} \sim \beta^{5/3}\exp\left\{-\frac{4}{\sqrt{3}}\delta\frac{m^2}{\beta}\frac{1}{\zeta}\right\}.$$

This formula describes the dependence of the probability on β at $\beta > \beta_0$, where β_0 is determined by the equality $(2\alpha_0)^{3/2}(n'-n)^{1/3} \approx 1$ (see the second inequality of (33)).

We consider now a pair-production mechanism consisting of real absorption from the external field of a sufficient number of photons for the process to proceed. We put

$$a_0 = \sqrt{s}(1+\varepsilon_1), \quad \gamma = 1-\varepsilon_2, \quad (41)$$

where $|\epsilon_1| \ll 1$ and $0 < \epsilon_2 \le 1$. The ratio of the argument of the Bessel function to its order can be close to unity only at this choice of a_0 and γ. Assuming the inequality

$$|A| \ll 1, \quad A = \frac{m^2}{s\beta}\left(1-\frac{\sqrt{s\beta}}{m}\zeta\right) \quad (42)$$

to be satisfied and retaining the terms of principal order of magnitude, we get

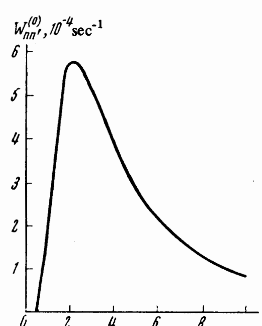

FIG. 3

$$2a_0 \approx A + \varepsilon_2 + \varepsilon_1^2, \qquad f_s(\gamma) \approx {}^1/_{32}. \tag{43}$$

According to expressions (35) and (43), the dependence of $W_{nn'}^{(S)}$ on ζ (at fixed values of γ and δ) has the form of a sharp peak, the height and half-width of which we calculate assuming the following inequalities to be satisfied:

$$\bar{f}(\varepsilon_1) \equiv \frac{n'-n}{3}(2a_0)^{3/2} \gg 1, \qquad \zeta\frac{\sqrt{s\beta}}{m} \ll 1.$$

The function $\bar{f}(\epsilon_1)$ reaches a minimum value f_0 at $\epsilon_1 = \epsilon_1^{(0)} \approx -{}^1/_3(A + \epsilon_2)$, and then the value of the probability at the maximum is

$$\max W_{nn'}^{(s)} = \frac{C_1}{32\pi}\omega\frac{1}{(2\delta)^{2/3}s^{5/3}}\frac{1}{(A+\varepsilon_2)^{1/2}}\exp\{-2\bar{f}_0\},$$
$$\bar{f}_0 \approx {}^2/_3\delta s(1+\varepsilon_1^{(0)})[A+\varepsilon_2+(\varepsilon_1^{(0)})^2]^{3/2}, \tag{44}$$

and the half-width of the maximum is ($\zeta = \zeta_0$ corresponds to the probability maximum)

$$\Delta\zeta \approx \zeta_0 / (2\delta s)^{1/2}(A+\varepsilon_2)^{1/4}. \tag{45}$$

We note that the quantity (44) has a nonzero limit as $\zeta \to 0$, but at the same time $\Delta\zeta \to 0$, and also

$$\Delta\omega' \equiv \frac{\omega'-\omega^*}{\omega^*} \approx \delta\zeta\frac{m}{\sqrt{s\beta}} \to 0 \qquad \left(\omega^* = \gamma\frac{\omega_H}{\omega}m\right). \tag{46}$$

By virtue of the last limiting relation, the number of electron-positron pairs produced by a photon flux described by any smooth distribution function with respect to the frequencies $\rho(\omega')$, vanishes when $\zeta \to 0$ (see (37)).

We present a numerical estimate of the probability of the process at the following values of the frequency, intensity of the electric component of the electromagnetic wave, and intensity of the homogeneous magnetic field: $\omega = 3\times 10^{15}\ \text{sec}^{-1}$, $|E| = 10^8$ V/cm, and $H = 4\times 10^5$ Oe. Assuming $s = 0.5\times 10^{11}$ and $\delta = 10^{-6}$, we obtain with the aid of formulas (44)–(46)

$$\max W_{nn'}^{(s)} \approx 2\cdot 10^{-5}\ \text{sec}^{-1}, \qquad \Delta\zeta/\zeta \approx 1.5\cdot 10^{-2},$$
$$\Delta\omega' = 0.5\cdot 10^{-10}, \qquad \omega' = 2.35\cdot 10^{-3}\ m. \tag{47}$$

With increasing parameter ζ (γ and δ are fixed), the probability of pair production decreases sharply (exponentially), starting with $\zeta \sim m/\sqrt{s\beta}$. With decreasing β, the probability decreases like

$$W_{nn'}^{(s)} \sim s^{-5/3}e^{-cs}, \qquad s \sim 1/\beta, \qquad c = \text{const} > 0.$$

5. CONCLUDING REMARKS

It was assumed in this paper that the external field (homogeneous magnetic field and classical electromagnetic wave) acts for an infinitely long time and occupies all of space. In a real experiment, however, the external field has finite dimensions Δr and a finite duration $\Delta\tau$. The influence of these factors on the probability of the process can be assessed, strictly speaking, only on the basis of the results of a solution of the corresponding problem, in the formulation of which the finite character of Δr and $\Delta\tau$ is taken into account from the very outset. Such a problem has not yet been solved, and we can only advance here certain considerations as to the conditions under which the influence of these factors can be disregarded. It is obvious that the results of our paper do not change strongly if $r \ll R$, where (in the usual units) $r = cp_\perp/eH$ is the radius of the electron orbit in the homogeneous magnetic field, $p_\perp$ is the projection of the electron momentum on the xy plane, and R is the radius of the light beam. In the particular cases considered by us we have $p_\perp \approx (2n\hbar\omega_H m)^{1/2}$, and therefore $r \approx (2n\hbar/m\omega_H)^{1/2}$. At the parameter values indicated above (see (47)) we have $r \approx 0.1$ cm. The condition $r \ll R$ can be satisfied at the present time only for electromagnetic fields with intensity E much smaller than taken in (47). For such fields the value of $\max W_{nn'}^{(S)}$ remains approximately the same as before, but $\Delta\omega'$ will be much smaller than in (47), and accordingly the number of pairs produced by the photon decreases.

Besides the effect considered in the present paper, that of pair production by a low-energy photon in the absence of real multiquantum transitions, the appearance of new quasienergy bands in the particle spectrum leads to a number of other effects. For example, emission of a photon by an electron in an external field proceeds via four channels (at fixed quantum numbers of the electron in the initial and final states and at fixed number of external-field quanta participating in the process), and the frequency of the emitted photon depends very strongly on the electric field intensity[5].

I consider it my pleasant duty to thank Ya. B. Zel'dovich for interest in the work, support, and important remarks that have contributed to the clarification of the physical content of the paper. I am also grateful to A. S. Kompaneets for encouraging support, which has greatly stimulated my work on the solution of this problem, to V. M. Buĭmistrov for interest in the work, and to S. I. Pekar for critical remarks.

APPENDIX I

The formulas for $L_{\sigma'\sigma}[Y_{n'n}(p'p)]$ at $e_0 = e_Z = 0$ are of the form

$$L_{11}[Y_{n'n}(p'p)] = 2(e_x - ie_y)\bar{n}p'[(g_{-p} - i(N_{-p} - eA_y))Y_{n'n} - \sqrt{\beta}\cdot 2nY_{n',n-1}] - 2(e_x + ie_y)\bar{n}p[(g_{p'} + i(N_{p'} - eA_y))Y_{n'n} - \sqrt{\beta}\cdot 2n'Y_{n'-1,n}],$$
$$L_{-1,-1}[Y_{n'n}(p'p)] = 2(e_x + ie_y)\bar{n}p'[(g_{-p} + i(N_{-p} - eA_y))Y_{n'n} - \sqrt{\beta}Y_{n',n+1}] - 2(e_x - ie_y)\bar{n}p[(g_{p'} - i(N_{p'} - eA_y))Y_{n'n} - \sqrt{\beta}Y_{n'+1,n}],$$
$$L_{1,-1}[Y_{n'n}(p'p)] = 2m(e_x - ie_y)(\bar{n}p + \bar{n}p')Y_{n'n},$$
$$L_{-1,1}[Y_{n'n}(p'p)] = -2m(e_x + ie_y)(\bar{n}p + \bar{n}p')Y_{n'n}.$$

APPENDIX II

CALCULATION OF THE INTEGRAL BY THE STATIONARY-PHASE METHOD

We consider the integral

$$I = \int_0^\pi d\varphi e^{ic_0\bar{h}(\varphi)}J_{n'-n}(\sqrt{\nu\varphi}).$$
$$c_0\bar{h}(\varphi) \equiv h(\varphi) = s\varphi + c_0\sin 2\varphi + (n'-n)\,\text{arctg}\left(\frac{b_0}{a_0}\text{ctg}\,\varphi\right)$$
$$c_0 \gg 1; \qquad a_0, b_0 > 0. \tag{A.1}$$

The stationary point $\varphi = \varphi_0$ of the function $h(\varphi)$, satisfying the first two conditions of (28), is given by the expression

$$\cos 2\varphi_0 = 1 - \frac{2a_0b_0}{a_0^2 - b_0^2}\left[(n'-n)\left(s + 2c_0\frac{a_0^2+b_0^2}{a_0^2-b_0^2}\right)^{-1}(1+\varepsilon) - \frac{b_0}{a_0}\right],$$
$$\varepsilon \approx \frac{4c_0(a_0^2-b_0^2)(n'-n)a_0b_0}{[2c_0(a_0^2+b_0^2)+s(a_0^2-b_0^2)]^2}. \tag{A.2}$$

We shall assume that the expression in the square brackets in (A.2) vanishes, i.e., $\varphi_0 = 0$.

We calculate the integral I from the formula (see[9], p. 69)

$$I \approx \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{n!} k_0^{(n)} \Gamma\left(\frac{1+n}{3}\right) \exp\left[\frac{i\pi(n+1)}{6}\right] c_0^{-(1+n)/3} e^{ih_0}; \quad (A.3)$$

Here $\Gamma(x)$ is the Euler gamma function, $h_0 = h(\varphi_0)$,

$$k(u) = J_{n'-n}(\sqrt{\nu\rho}) \left(\frac{du}{d\varphi}\right)^{-1}, \quad u^3 = \bar{h}(\varphi) - \bar{h}_0,$$

$$k_0^{(n)} = \frac{d^n}{du^n} k(u) \Big|_{u=0}.$$

In (A.3) we took into account the fact that $d^2h/d\varphi$ $d^2h/d\varphi^2 |_{\varphi=\varphi_0} = 0$. In this formula we retain only the first two nonvanishing terms. The final result is as follows $h_0^{(n)} \equiv d^n h(\varphi)/d\varphi^n |_{\varphi=\varphi_0}$, $\rho_0 = \rho(\varphi_0)$):

$$I \approx \frac{1}{3} i^{n'-n} \left\{ \Gamma\left(\frac{1}{3}\right) e^{i\pi/6} J_{n'-n}(\sqrt{\nu\rho_0}) \left(\frac{6}{h_0'''}\right)^{1/3} + \right. \quad (A.4)$$

$$+ \frac{3}{2} i \left[4J'_{n'-n}(\sqrt{\nu\rho_0}) \sqrt{\frac{\nu}{\rho_0}} \frac{a_0^2 - b_0^2}{h_0'''} + \frac{1}{5} J_{n'-n}(\sqrt{\nu\rho_0}) \right.$$

$$\left. \left. \times \frac{h_0^{(5)}}{(h_0''')^2} \right] \right\}, \quad J_n'(x) \equiv \frac{d}{dx} J_n(x).$$

The condition under which we can confine ourselves in (A.4) to the first term in the curly brackets is the inequality

$$\left| \frac{6}{5} \frac{1}{(n'-n)^{2/3}} + 2\sqrt{n'+n} \frac{(n'-n)^{1/3} a_0}{s+2c_0} \frac{J'_{n'-n}(\sqrt{\nu\rho_0})}{J_{n'-n}(\sqrt{\nu\rho_0})} \right| \ll 1, \quad (A.5)$$

derived under the assumption that

$$\frac{a_0(s+2c_0)}{(n'-n)b_0} \approx \left(\frac{s+2c_0}{n'-n}\right)^2 \gg 1. \quad (A.6)$$

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