EXACT THEORY OF TWO-DIMENSIONAL SELF-FOCUSING AND ONE-DIMENSIONAL SELF-MODULATION OF WAVES IN NONLINEAR MEDIA

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It is demonstrated that the equation $i\partial\psi/\partial t + \psi_{xx} + \kappa|\psi|^2\psi = 0$, which describes plane self-focusing and one-dimensional self-modulation can be solved exactly by reducing it to the inverse scattering problem for a certain linear differential operator. In this case exact solutions can be obtained which describe the interaction of solitary wave packets--solitons. It is shown that the amplitude and velocity of interacting solitons do not change, whereas the phase has a discontinuity. It is also demonstrated that only paired collisions of solitons occur. The results obtained are used for assessing the nonlinear stage of development of self-modulated instability.

A plane stationary light beam in a medium with nonlinear refractive index is described by the equation[1,2]

$$2ik\frac{\partial E}{\partial z}+\frac{\partial^2 E}{\partial x^2}=-k^2\frac{\delta n_{nl}}{n_0}|E|^2E. \tag{1}$$

Here E is the complex envelope of the electric field; it is assumed that the refractive index is given by the formula $n = n_0 + \delta n_{nl}|E|^2$. The same equation expressed in terms of suitable variables can be used to describe plane wave beams in other nonlinear media.

A similar equation (see [3,4])

$$i\left(\frac{\partial\psi}{\partial t}+\omega_k'\frac{\partial\psi}{\partial x}\right)+\frac{1}{2}\omega_k''\frac{\partial^2\psi}{\partial x^2}=q|\psi|^2\psi \tag{2}$$

also holds for the complex amplitude ψ of a quasimonochromatic one-dimensional wave in a medium with dispersion and inertialess nonlinearity. In (2), ω_k is the wave dispersion law and $q|\psi|^2$ is the nonlinear correction to the frequency of the wave with amplitude ψ.

Equations (1) and (2) can be reduced to a standard dimensionless form

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\varkappa|u|^2u=0. \tag{3}$$

It is convenient to assign the variable t the meaning of time.

In the present paper we shall investigate Eq. (3) with $\kappa > 0$. As applied to Eq. (1), this means $\delta n_{nl} > 0$. Under this condition, Eq. (1) describes stationary two-dimensional self-focusing and the associated transverse instability of a plane monochromatic wave.[5] For Eq. (2), the condition $\kappa > 0$ is equivalent to the requirement $q\omega_k'' < 0$, which, when satisfied, produces in the nonlinear medium longitudinal instability of the monochromatic wave—self-modulation.[3,4,6]

Equation (3) can be solved exactly by the inverse-problem method. This method is applicable to equations of the type

$$\partial u/\partial t=\hat{S}[u]$$

where $\hat{S}$, generally speaking, is a nonlinear operator differential in x), which can be represented in the form (see [7])

$$\partial\hat{L}/\partial t=i[\hat{L},\hat{A}]. \tag{4}$$

Here $\hat{L}$ and $\hat{A}$ are linear differential operators containing the sought function u(x, t) in the form of a coefficient. If the condition (4) is satisfied, then the spectrum of the operator L does not depend on the time, and the asymptotic characteristics of its eigenfunctions can easily be calculated at any instant of time from their initial values. The reconstruction of the function u(x, t) at an arbitrary instant of time is realized by solving the inverse scattering problem for the operator $\hat{L}$.

As can easily be verified, Eq. (3) can be written in the form (4), with the operators $\hat{L}$ and $\hat{A}$ taking the form

$$\hat{L}=i\begin{bmatrix}1+p & 0\\ 0 & 1-p\end{bmatrix}\frac{\partial}{\partial x}+\begin{bmatrix}0 & u^*\\ u & 0\end{bmatrix},\quad \varkappa=\frac{2}{1-p^2},$$
$$\hat{A}=-p\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{\partial^2}{\partial x^2}+\begin{bmatrix}|u|^2/(1+p) & iu_x^*\\ -iu_x & -|u|^2/(1-p)\end{bmatrix} \tag{5}$$

Without loss of generality, we can assume that $\kappa > 2$ and $p^2 > 0$.

The inverse-problem method was discovered by Kruskal, Green, Gardner, and Miura[8] and was applied by them to the well-known Korteweg-de Vries (KDV) equation. In the case considered by them, the role of the operator $\hat{L}$ was played by the one-dimensional Schrödinger operator. They revealed the fundamental role played by the particular solutions of the KDV equation—solitons, which are directly connected with the discrete spectrum of the operator $\hat{L}$, namely, it was established that the asymptotic state as $t \to \pm\infty$ of any initial condition is a finite set of solitons. In our problem, an analogous role is played by the particular solutions of Eq. (3):

$$u(x,t)=\sqrt{2\varkappa}\,\eta\frac{\exp\{-4i(\xi^2-\eta^2)t-2i\xi x+i\varphi\}}{\operatorname{ch}(2\eta(x-x_0)+8\eta\xi t)}, \tag{6}$$

which we shall also call solitons. A soliton, as will be shown by us, is a stable formation.

In the self-focusing problem, the soliton has the meaning of a homogeneous waveguide channel inclined to the z axis at an angle $\theta = -\tan^{-1}4\xi$. In the self-modulation problem, the soliton is a single wave packet propagating without distortion of its envelope and with a velocity $v = -4\xi$. The soliton is characterized by four constants: η, ξ, x_0, and φ. Unlike the KDV soliton, the constant η characterizing the amplitude and the constant ξ which determines the velocity of the soliton are independent and can be chosen arbitrarily.

The soliton (6) is the simplest representative of an extensive family of exact solutions of Eq. (3), which can be expressed in explicit form. In the general case such a solution—we shall call this an N-soliton solution—depends on 4N arbitrary constants: η_j, ξ_j, x_{0j}, and φ_j; for non-coinciding ξ_j this solution breaks up into individual solitons if $t \to \pm\infty$. The N-soliton solution thus describes the process of scattering of N solitons by one another. In this scattering, the amplitudes and velocities of the solitons remain unchanged, and only their center coordinates x_0 and phases φ are altered. Just as in the case of KDV solitons,[9] only paired collisions of solitons contribute to this change. What is principally new compared with the KDV case is the possibility of formation of a bound state of a finite number of solitons having identical velocities. In the simplest case of two solitons, the bound state is a periodic-in-time solution of Eq. (3), and in the case of N solitons it is an arbitrarily-periodic solution characterized by N periods. As applied to the self-focusing problem, the N-soliton solution describes the intersection of N homogeneous channels, and the bound state describes an oscillating "complicated" channel.

1. THE DIRECT SCATTERING PROBLEM

Let us assume that u(x, t) decreases sufficiently rapidly as $|x| \to \infty$, and let us examine the scattering problem for the operator $\hat{L}$. To this end, we consider the system of equations

$$\hat{L}\psi = \lambda\psi, \quad \psi = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} \tag{7}$$

and make the change of variables

$$\psi_1 = \sqrt{1-p}\exp\left\{-i\frac{\lambda}{1-p^2}x\right\}v_2, \quad \psi_2 = \sqrt{1+p}\exp\left\{-i\frac{\lambda}{1-p^2}x\right\}v_1.$$

Equation (7) can be rewritten in the form of the following system:

$$\begin{gathered} v_1' + i\zeta v_1 = qv_2, \quad v_2' - i\zeta v_2 = -q^* v_1, \\ q = iu/\sqrt{1-p^2}, \quad \zeta = \lambda p/(1-p^2). \end{gathered} \tag{8}$$

In spite of the fact that this system is not self-adjoint, the problem of scattering for this system is analogous in many respects to the problem of scattering for the one-dimensional Schrödinger equation.

Let v and w be solutions of the system (8) at $\zeta = \zeta_1$ and ζ_2, respectively. Then

$$\frac{d}{dx}(v_1w_2 - w_1v_2) + i(\zeta_1 - \zeta_2)(v_1w_2 + v_2w_1) = 0. \tag{9}$$

In addition, if v is a solution of the system (8) at $\zeta_1 = \xi_1 + i\eta_1$, then

$$\bar{v} = \begin{Bmatrix} v_2^* \\ -v_1^* \end{Bmatrix}$$

satisfies the system (8) at $\zeta_2 = \zeta_1^* = \xi_1 - i\eta_1$.

We define, for real $\zeta = \xi$, the Jost functions φ and ψ as solutions of Eq. (8) with asymptotic values

$$\varphi \to \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} e^{-i\xi x} \quad \text{as } x \to -\infty,$$

$$\psi \to \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} e^{i\xi x} \quad \text{as } x \to +\infty.$$

The pair of solutions ψ and $\bar{\psi}$ forms a complete system of solutions, and therefore

$$\varphi = a(\xi)\bar{\psi} + b(\xi)\psi. \tag{10}$$

Applying relation (9) to the pair of solutions φ and $\bar{\varphi}$ of Eq. (8), we obtain

$$|a(\xi)|^2 + |b(\xi)|^2 = 1. \tag{11}$$

The Jost functions φ and ψ admit of analytic continuation into the upper half-plane Im $\zeta > 0$. By virtue of (9) we have

$$a(\xi) = (\varphi_1\psi_2 - \varphi_2\psi_1)(x, \xi),$$

and therefore $a(\xi)$ also admits of an analytic continuation. It is clear that

$$a(\zeta) \to 1 \quad \text{as } |\zeta| \to \infty, \quad \operatorname{Im}\zeta \geqslant 0.$$

The points of the upper half-plane Im $\zeta > 0$, $\zeta = \zeta_j$, $j = 1, \ldots, N$, at which $a(\zeta) = 0$, correspond to the eigenvalues of the problem (8). Here

$$\varphi(x, \zeta_j) = c_j\psi(x, \zeta_j), \quad j = 1, \ldots, N.$$

We note also that for real q(x) we have the equalities

$$\varphi(x, -\xi) = \varphi^*(x, \xi), \quad \psi(x, -\xi) = \psi^*(x, \xi)$$

and consequently

$$a(\xi) = a^*(-\xi).$$

Continuing the last equality into the upper half-plane, we obtain

$$a(\zeta) = a^*(-\zeta^*).$$

It is clear that in this case the zeroes of $a(\zeta)$ lie on the imaginary axis.

It follows from (4) that the eigenfunctions of the operator $\hat{L}$ obey the equation

$$i\partial\psi/\partial t = A\psi. \tag{12}$$

More accurately speaking, if at $t = 0$ the function ψ_0, regarded as the initial condition for Eq. (12), satisfies the system (7), then the corresponding solution of (12) at an arbitrary instant of time also satisfies the system (7) with unaltered value of λ.

From Eq. (12) for the eigenfunctions it follows that as $|x| \to \infty$, the solution $V(x, \zeta, t)$ of the system (8) satisfies the equation

$$i\frac{\partial v}{\partial t} = \frac{1}{p}\zeta^2 v + 2i\zeta\frac{\partial v}{\partial x} - p\frac{\partial^2 v}{\partial x^2},$$

from which we get that $a(\zeta)$ does not depend on the time, and

$$b(\xi, t) = b(\xi, 0)e^{4i\xi^2 t}, \quad c_j(t) = c_j(0)e^{4i\zeta_j^2 t}. \tag{13}$$

2. THE INVERSE SCATTERING PROBLEM

Let us consider the problem of reconstructing u(x, t) from the scattering data $a(\xi)$, $b(\xi, t)$, $-\infty < \xi$

$< +\infty$; $c_j(t)$, $j = 1, \ldots, N$. The values of these quantities at $t = 0$ are calculated from the initial data for Eq. (3), and their variation with t is indicated in formula (13). In the inverse scattering problem, the time t plays the role of the parameter It suffices therefore to consider the question of the reconstruction of the coefficient $q(x)$ of Eqs. (8) from $a(\xi)$, $b(\xi)$, and c_j.

We introduce the function

$$\Phi(\zeta) \equiv \Phi(\zeta, x) = \begin{cases} a^{-1}(\zeta)\varphi(x,\zeta)e^{i\zeta x}, & \operatorname{Im}\zeta > 0 \\ \begin{Bmatrix} \psi_2^*(x,\zeta^*) \\ -\psi_1^*(x,\zeta^*) \end{Bmatrix} e^{i\zeta x}, & \operatorname{Im}\zeta < 0 \end{cases}$$

and denote $\phi(\xi)$ the discontinuity of this function on crossing the real axis:

$$\phi(\xi) = \Phi(\xi + i0) - \Phi(\xi - i0).$$

Assuming the zeroes $\zeta_1, \ldots, \zeta_N$ of the function $a(\zeta)$ to be simple, we have a formula that reconstructs the piecewise-analytic function $\Phi(\zeta)$ from the discontinuity $\phi(\xi)$ and the residues at the poles ζ_j:

$$\Phi(\zeta) = \begin{Bmatrix}1\\0\end{Bmatrix} + \sum_{k=1}^{N} \frac{e^{i\zeta_k x}}{\zeta - \zeta_k}\frac{\varphi(x\ \zeta_k)}{a'(\zeta_k)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi(\xi)}{\xi - \zeta}d\xi,$$

or

$$\Phi = \Phi^{(1)} + \Phi^{(2)},\quad \Phi^{(1)}(\zeta) = \begin{Bmatrix}1\\0\end{Bmatrix} + \sum_{k=1}^{N}\frac{e^{i\zeta_k x}}{\zeta - \zeta_k}\tilde{c}_k\psi(x,\zeta_k),$$

$$\tilde{c}_k = \frac{c_k}{a'(\zeta_k)},\quad \Phi^{(2)}(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi(\xi)}{\xi - \zeta}d\xi. \tag{14}$$

The tilde over the c_k will henceforth be omitted. From (10) we obtain the following expression for the discontinuity:

$$\phi(\xi) = \frac{b(\xi)}{a(\xi)}e^{i\xi x}\psi(x,\xi). \tag{15}$$

The system of equations of the inverse scattering-theory problem for the function $\phi(\xi)$, $\xi < +\infty$ and for the parameters $\psi(x, \zeta_j)$ (x is fixed), $j = 1, \ldots, N$, is obtained by putting in (14) $\zeta = \zeta_j^*$, $j = 1, \ldots, N$. At $\zeta = \xi - i0$, formula (14) yields

$$\begin{Bmatrix}\psi_2^*(x,\xi)\\ -\psi_1^*(x,\xi)\end{Bmatrix} e^{i\xi x} = -\frac{1}{2}(1 - J)\phi(\xi) + \Phi^{(1)}(\xi).$$

Here J is the Hilbert transform

$$(J\phi)(\xi) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\phi(\xi')}{\xi' - \xi}d\xi',\quad (J\phi)^* = -J\phi^*.$$

We rewrite the obtained relation in the form

$$\psi_2^*(x,\xi)e^{i\xi x} + {}^1/_2(1 - J)\phi_1 = \Phi_1^{(1)}(\xi),$$
$$-\psi_1(x,\xi)e^{-i\xi x} + {}^1/_2(1 + J)\phi_2^* = \Phi_2^{(1)*}(\xi),$$

and multiply the first of these equations by $c^*(x, \xi)$ and the second by $c(x, \xi)$, where

$$c(x,\xi) = \frac{b(\xi)}{a(\xi)}e^{2i\xi x}.$$

Taking (15) into account, we obtain

$$\phi_1 - c(x,\xi)\frac{1+J}{2}\phi_2^*$$
$$= -c(x,\xi)\sum_{k=1}^{N}\frac{\exp(-i\zeta_k^* x)}{\xi - \zeta^*}c_k^*\psi_2^*(x,\zeta_k),$$

$$c^*(x,\xi)\frac{1-J}{2}\phi_1 + \phi_2^*$$
$$= c^*(x,\xi) + c^*(x,\xi)\sum_{k=1}\frac{\exp(i\zeta_k x)}{\xi - \zeta_k}c_k\psi_1(x,\zeta_k). \tag{16}$$

In addition to Eqs. (16), we obtain 2N equations for $\psi_1(x, \zeta_j)$ and $\psi_2^*(x, \zeta_j)$, putting in (14) $\zeta = \zeta_j^*$, $j = 1, \ldots, N$:

$$\psi_1(x,\zeta_j)\exp(-i\zeta_j x) + \sum_{k=1}^{N}\frac{\exp(-i\zeta_k^* x)}{\zeta_j - \zeta_k^*}c_k^*\psi_2^*(x,\zeta_k)$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi_2^*(\xi)}{\xi - \zeta_j}d\xi,$$

$$-\sum_{k=1}^{N}\frac{e^{i\zeta_k x}}{\zeta_j^* - \zeta_k}c_k\psi_1(x,\zeta_k) + \psi_2^*(x,\zeta_j)\exp(i\zeta_j^* x)$$
$$= 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi_1(\xi)}{\xi - \zeta_j^*}d\xi. \tag{17}$$

The system of equations (16) and (17) relative to

$$\phi(\xi) = \phi(x,\xi),\quad \psi(x,\zeta_j)$$

makes it possible to obtain these quantities from the scattering data. A formula for the reconstruction of $q(x)$ from $\phi(\xi)$ and $\psi(x, \zeta_j)$ can be obtained readily by comparing the asymptotic behavior of $\psi(x, \zeta)$ as $\zeta \to \infty$, obtained from the formula (14):

$$\begin{Bmatrix}\psi_2(x,\zeta)\\ -\psi_1(x,\zeta)\end{Bmatrix}e^{-i\zeta x} = \begin{Bmatrix}1\\0\end{Bmatrix} + \frac{1}{\zeta}\Big[\sum_{k=1}^{N}c_k^*\exp(-i\zeta_k^* x)\psi^*(x,\zeta_k)$$
$$+ \frac{1}{2\pi i}\int\phi^*(\xi)d\xi\Big] + O\left(\frac{1}{\zeta^2}\right)$$

and directly from the differential equation (8):

$$\psi(x,\zeta)e^{-i\zeta x} = \begin{Bmatrix}0\\1\end{Bmatrix} + \frac{1}{2i\zeta}\begin{Bmatrix} q(x) \\ \int_x^\infty |q|^2(s)\,ds\end{Bmatrix} + O\left(\frac{1}{\zeta^2}\right).$$

From these two relations we get

$$q(x) = -2i\sum c_k^*\exp(-i\zeta_k^* x)\psi_2^*(x,\zeta_k) - \frac{1}{\pi}\int\phi_2^*(\xi)d\xi,$$

$$\int_x^\infty |q(s)|^2 ds = -2i\sum c_k\exp(i\zeta_k x)\psi_1(x,\zeta_k) + \frac{1}{\pi}\int\phi_1(\xi)d\xi. \tag{18}$$

In concluding this section, we present the equations of the inverse problem of scattering theory, written in the form of an equation of the Marchenko type (see [10, 11]). These equations are obtained from (16) and (17) as a result of the Fourier transformation with respect to ξ.

Let

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{b(\xi)}{a(\xi)}e^{i\xi x}d\xi + \sum_{k=1}^{N}c_k e^{i\zeta_k x}. \tag{19}$$

Then, after simple derivations, we get from (16) and (17)

$$K_1(x,y) = F^*(x+y) + \int_x^\infty K_2^*(x,s)F^*(s+y)\,ds,$$

$$K_2^*(x,y) = -\int_x^\infty K_1(x,s)F(s+y)\,ds, \tag{20}$$

where the kernel $K(x, y)$ is connected with $\psi(x, \xi)$ by the relation

$$\psi(x,\zeta)=e^{i\zeta x}+\int_x^\infty K(x,s)e^{i\zeta s}\,ds,\quad \operatorname{Im}\zeta\geqslant 0.$$

We note also that in terms of this notation, formulas (18) take the form

$$q(x)=-2K_1(x,x),\quad \int_x^\infty |q(s)|^2\,ds=-2K_2(x,x).$$

3. N-SOLITON SOLUTIONS (EXPLICIT FORMULA)

We consider the inverse scattering problem in the case when $b(\xi, t) \equiv 0$. Then $\phi(\xi) \equiv 0$, and the solution of the inverse problem reduces to a solution of the finite system (17) of linear algebraic equations. We rewrite this system in a more symmetrical form:

$$\psi_{1j}+\sum_{k=1}^N\frac{\lambda_j\lambda_k^*}{\zeta_j-\zeta_k^*}\psi_{2k}^*=0,$$

$$-\sum_{k=1}^N\frac{\lambda_k\lambda_j^*}{\zeta_j^*-\zeta_k}\psi_{1k}+\psi_{2j}^*=\lambda_j^*; \tag{17'}$$

here

$$\psi_j=\begin{Bmatrix}\psi_{1j}\\ \psi_{2j}\end{Bmatrix}=\sqrt{c_j}\psi(x,\zeta_j),\quad \lambda_j=\sqrt{c_j}e^{i\zeta_j x}.$$

Formulas (18) also assume a simpler form in this case:

$$q(x)=-2i\sum_{k=1}^N\lambda_k^*\psi_{2k}^*,\quad \int_x^\infty|q(s)|^2\,ds=-2i\sum_{k=1}^N\lambda_k\psi_{1k}. \tag{18'}$$

If N = 1 and $a(\zeta)$ has only one zero in the upper half-plane, then the system (17′) takes the form

$$\psi_1+\frac{|\lambda|^2}{2i\eta}\psi_2^*=0,\quad \frac{|\lambda|^2}{2i\eta}\psi_1+\psi_2^*=\lambda^*. \tag{21}$$

It is easy to verify that the system (21) describes the soliton (6) with parameters

$$x_0=(2\eta)^{-1}\ln\frac{|\lambda(0)|^2}{2\eta},\quad \varphi=-2\arg\lambda(0).$$

In the general case the system (17′) describes an N-soliton solution. This system, as will be shown in the Appendix, is not degenerate, and in addition there takes place the formula

$$-\int_x^\infty|q(s)|^2\,ds=i\left(\sum\lambda_k\psi_{1k}-\sum\lambda_k^*\psi_{1k}^*\right)=\frac{d}{dx}\ln\det\|A\|, \tag{22}$$

where $\|A\|$ is the matrix of the system (17′). For the one-dimensional Schrödinger equation, a similar formula was obtained by Kay and Moses.[12]

To prove formula (22), we note that

$$\frac{d}{dx}\ln\det\|A\|=\frac{1}{\det\|A\|}\sum_{k=1}^{2N}\det\|A_k\|,$$

where the matrix $\|A_k\|$ differs from the matrix $\|A\|$ by the column numbered k, which is the derivative of the corresponding column of the matrix $\|A\|$. For $1 \le k \le N$, the aforementioned column of the matrix $\|A_k\|$ is

$$i\lambda_k\begin{Vmatrix}0\\ \vdots\\ 0\\ \lambda_1^*\\ \vdots\\ \lambda_N^*\end{Vmatrix}.$$

By virtue of the Cramer formula, we have

$$i\lambda_k\psi_{1k}=\det\|A_k\|\,/\,\det\|A\|.$$

Thus,

$$i\sum_{k=1}^N\lambda_k\psi_{1k}=\frac{1}{\det\|A\|}\sum_{k=1}^N\det\|A_k\|.$$

To prove formula (22), it remains to verify that

$$-i\sum\lambda_k^*\psi_{1k}^*=\frac{1}{\det\|A\|}\sum_{k=N+1}^{2N}\det\|A_k\|. \tag{23}$$

To this end we rewrite the system (17′) in the form

$$\psi_{2j}+\sum_{k=1}^N\frac{\lambda_j\lambda_k^*}{\zeta_j-\zeta_k^*}(-\psi_{1k}^*)=\lambda_j,$$

$$-\sum_{k=1}^N\frac{\lambda_j^*\lambda_k}{\zeta_j^*-\zeta_k}\psi_{2k}+(-\psi_{1j}^*)=0.$$

The matrix of this system relative to $\{\psi_{21}, \ldots, \psi_{2N}, -\psi_{11}^*, \ldots, -\psi_{1N}^*\}$ coincides with the matrix $\|A\|$, from which follow formulas (23) and (22). For the N-soliton solutions we obtain finally the explicit formula

$$|u(x,t)|^2=\sqrt{2\varkappa}\frac{d^2}{dx^2}\ln\det\|A\|=\sqrt{2\varkappa}\frac{d^2}{dx^2}\ln\det\|BB^*+1\|,$$

where

$$B_{jk}=\frac{\sqrt{c_jc_k^*}}{\zeta_j-\zeta_k^*}\exp\{i(\zeta_j-\zeta_k^*)x\}.$$

The dependence of the quantity c_j on the time is given by formula (13).

4. N-SOLITON SOLUTIONS (ASYMPTOTIC FORM AS $t \to \pm\infty$)

Let us study the behavior of the N-soliton solution at large $|t|$. We confine ourselves here to the case when all the ξ_j are different, i.e., there are no two solitons having the same velocity. In this case the N-soliton solution breaks up as $t \to \pm\infty$ into diverging solitons. To verify this, we arrange the ξ_j in decreasing order: $\xi_1 > \xi_2 > \ldots > \xi_N$.

From (13) we have

$$\lambda_j(t,x)=\lambda_j(0)\exp\{-\eta_j(x+4\xi_jt)+i[\xi_jx+2(\xi_j^2-\eta_j^2)t]\},$$
$$|\lambda_j(x,t)|=|\lambda_j(0)|e^{-\eta_jy_j},\quad y_j=x+4\xi_jt.$$

Let us consider the asymptotic form of the N-soliton solution on the straight line y_m = const as $t \to \infty$. Then

$$y_j\to+\infty,\quad |\lambda_j|\to 0\quad\text{for } j<m,$$
$$y_j\to-\infty,\quad |\lambda_j|\to\infty\quad\text{for } j>m.$$

It follows from the system (17′) that $\psi_{1j}, \psi_{2j} \to 0$ when $j < m$. In the limit we have a reduced system of equations relative to $2(N - m - 1)$ functions of y_m

$$\psi_{1m},\ \psi_{2m}^*,\quad \varphi_{1k}=\psi_{1k}\lambda_k,\quad \varphi_{2k}^*=\psi_{2k}^*\lambda_k^*,\quad k>m,$$

namely

$$\psi_{1m} + \frac{|\lambda_m|^2}{2i\eta_m}\psi_{2m}^* = -\lambda_m \sum_{k=m+1}^{N} \frac{1}{\zeta_m - \zeta_k^*}\phi_{2k}^*,$$

$$\frac{|\lambda_m|^2}{2i\eta_m}\psi_{1m} + \psi_{2m}^* = \lambda_m^* + \lambda_m^* \sum_{k=m+1}^{N} \frac{1}{\zeta_m^* - \zeta_k}\phi_{1k} \tag{24}$$

and

$$\sum_{k=m+1}^{N} \frac{1}{\zeta_m - \zeta_k^*}\phi_{2k}^* = -\frac{\lambda_m^*}{\zeta_m - \zeta_m^*}\psi_{2m}^*,$$

$$\sum_{k=m+1}^{N} \frac{1}{\zeta_m^* - \zeta_k}\phi_{1k} = -1 - \frac{\lambda_m}{\zeta_m^* - \zeta_m}\psi_{1m}. \tag{25}$$

Solving the last system with respect to ϕ_{1k} and ϕ_{2k}^*, we obtain (see formulas (A.2) and (A.4) of the Appendix) the following simple formulas:

$$\phi_{1k} = a_k + \frac{2i\eta_m}{a_m}\frac{a_k}{\zeta_k - \zeta_m}\psi_{1m}\lambda_m,$$

$$\phi_{2k}^* = -\frac{2i\eta_m}{a_m^*}\frac{a_k^*}{\zeta_k^* - \zeta_m^*}\psi_{2m}^*\lambda_m^*.$$

Here

$$a_k = \prod_{p=m+1}^{N}(\zeta_k - \zeta_p^*) \Big/ \prod_{m<p\neq k}(\zeta_k - \zeta_p),$$

$$a_m = 2i\eta_m \prod_{p=m+1}^{N}\frac{\zeta_m - \zeta_p^*}{\zeta_m - \zeta_p}.$$

Substituting the expressions for ϕ_{1k} and ϕ_{2k}^* in (24) we obtain, using formulas (A.4), (A.5), and (A.6) of the Appendix:

$$\psi_{1m} + \frac{|\lambda_m^+|^2}{2i\eta_m}\psi_{2m}^* = 0,$$

$$\frac{|\lambda_m^+|^2}{2i\eta_m}\psi_{1m} + \psi_{2m}^* = (\lambda_m^+)^*, \quad \lambda_m^+ = \lambda_m \prod_{p=m+1}^{N}\frac{\zeta_m - \zeta_p}{\zeta_m - \zeta_p^*}. \tag{26}$$

The system (26) coincides with the system (21) and describes a soliton with a displaced position of the center x_0^+ and phase φ^+:

$$x_{0m}^+ - x_{0m} = \frac{1}{\eta_m}\sum_{p=m+1}^{N}\ln\left|\frac{\zeta_m - \zeta_p}{\zeta_m - \zeta_p^*}\right| < 0,$$

$$\varphi_m^+ - \varphi_m = -2\sum_{p=m+1}^{N}\arg\left(\frac{\zeta_m - \zeta_p}{\zeta_m - \zeta_p^*}\right).$$

Analogously, for $t \to -\infty$ we obtain the system (21) with

$$\lambda = \lambda_m^- = \lambda_m \prod_{p=1}^{m-1}\frac{\zeta_m - \zeta_p}{\zeta_m - \zeta_p^*}.$$

On the straight lines $y = x + \xi t$, where ξ does not coincide with any ξ_m as $t \to \pm\infty$, the reduced system becomes asymptotically homogeneous, and the solution tends to zero at an asymptotic rate, thus proving the asymptotic breakdown of the N-soliton solution into individual solitons.

The obtained formulas make it possible to describe the soliton scattering process. As $t \to \infty$, the N-soliton solution breaks up into solitons arranged in such a way that the fastest soliton is in front and the slowest at the rear. At $t \to -\infty$ the arrangement of the solitons is reversed. When the time t varies from $-\infty$ to $+\infty$, the quantity λ_m is changed by a factor λ_m^+/λ_m^-, corresponding to a total change in the coordinate of the soliton center

$$\Delta x_{0m} = x_{0m}^+ - x_{0m}^-$$
$$= \frac{1}{\eta_m}\left(\sum_{k=m+1}^{N}\ln\left|\frac{\zeta_m - \zeta_k}{\zeta_m - \zeta_k^*}\right| - \sum_{k=1}^{m-1}\ln\left|\frac{\zeta_m - \zeta_k}{\zeta_m - \zeta_k^*}\right|\right) \tag{27a}$$

and a total change of phase

$$\Delta\varphi_m = \varphi_m^+ - \varphi_m^- = 2\sum_{k=1}^{m-1}\arg\frac{\zeta_m - \zeta_k}{\zeta_m - \zeta_k^*} - 2\sum_{k=m+1}^{N}\arg\frac{\zeta_m - \zeta_k}{\zeta_m - \zeta_k^*}. \tag{27b}$$

Formulas (27) can be interpreted by assuming that the solitons collide pairwise (and every soliton collides with all others). In each paired collision, the faster soliton moves forward additionally by an amount $\eta_m^{-1} \times \ln|(\zeta_m - \zeta_k^*)/(\zeta_m - \zeta_k)|$, $\xi_m > \xi_k$, and the slower one shifts backwards by an amount $\eta_k^{-1} \times \ln|(\zeta_m - \zeta_k^*)/(\zeta_m - \zeta_k)|$. The total soliton shift is equal to the algebraic sum of its shifts during the paired collisions, so that there is no effect of multiparticle collisions at all. The situation is the same with the phases.

5. BOUND STATES AND MULTIPLE EIGENVALUES

The rate of divergence of a pair of solitons is proportional to the difference between the values of the parameters ξ; at equal values of ξ, the solitons do not diverge, but form a bound state. Let us consider the bound state of N solitons, putting for simplicity $\xi_j = 0$. Then $c_j(t) = c_j(0)\exp(-4i\eta_j^2 t)$. It is seen directly from the general formula (22) that the bound state is an arbitrarily periodic solution of Eq. (3), characterized in the general case by N frequencies $\omega_j = 4\eta_j^2$. Actually the answer contains all the possible frequency differences, and therefore the bound state of two solitons is characterized only by one frequency $\omega = 4(\eta_1^2 - \eta_2^2)$, and constitutes a periodic solution of Eq. (3).

As $\eta_2 \to 0$ we have $\omega \to 4\eta_1^2$, i.e., the period of the oscillations tends to a constant limit. A more detailed analysis shows, however, that the depth of the oscillations tends to zero, and the bound state goes over into a soliton with amplitude η_1. As $\eta_2 \to \eta_1$ the zeroes of $a(\zeta)$ coalesce; the bound state then becomes aperiodic.

The limiting state resulting from the coalescence of the zeroes and formation of a multiple zero of $a(\zeta)$, is best investigated by starting directly from the Marchenko equations (20).

Let us consider the kernel F(x, t) of the Marchenko equation for a system of two solitons with close values of the parameters:

$$F(x,t) = c_1\exp\{i\zeta x + 4i\zeta^2 t\} + c_2\exp\{i(\zeta + \Delta\zeta)x + 4i(\zeta + \Delta\zeta)^2 t\}$$
$$\approx \exp\{i\zeta x + 4i\zeta^2 t\}[c_1 + c_2 + ic_2\Delta\zeta(x + 8\zeta t) + \ldots].$$

Going to the limit as $\Delta\zeta \to 0$, $c_2\Delta\zeta \to \gamma$, $c_1 + c_2 \to a_1$, we obtain

$$F(x,t) = [a_1(t) + a_2(t)x]e^{i\zeta x},$$

where

$$a_1(t) = a_1(0)(1 + 8\gamma\zeta t)e^{4i\zeta^2 t},$$
$$a_2(t) = a_2(0)e^{4i\zeta^2 t}.$$

Solving the Marchenko equations, we obtain after transformations

$$q(x) = -\frac{4\eta\lambda^{*2}a_2^*}{1+|a_2|^2|\lambda|^4}\frac{-2+|\lambda|^4\mu a_2^*}{1+|\mu|^2|\lambda|^4},$$

$$\mu = \frac{2\eta[a_1+2(x+1/2\eta)a_2]}{1+|a_2|^2|\lambda|^4}, \quad \lambda = \frac{e^{i\zeta x}}{2\eta}. \qquad (28)$$

An analysis of this expression shows that as $|t| \to \infty$, it constitutes a superposition of two solitons with amplitude η, the distance between which increases with time like $\ln(4\eta^2 t)$. The solutions resulting from the coalescence of a large number of simple zeroes are of similar form.

6. STABILITY OF SOLITONS

We do not consider in this paper the evolution of the initial conditions of general form, in which an important role may be played by the "nonsoliton" part of the solution, which is connected with the quantity $b(\xi, t)$. We consider only the case when this quantity is small, i.e.,

$$|b(\xi,t)/a(\xi)| = |b(\xi,0)/a(\xi)| \ll 1, \qquad (29)$$

and the coefficient $a(\zeta)$ has in the upper half-plane only one zero $\zeta = \zeta_1$. Such a choice of the initial conditions corresponds to posing the problem of the stability of a soliton with parameters $\xi_1 + i\eta_1 = \zeta_1$ when the soliton is perturbed by a continuous spectrum.

Let us consider the system (16) and (17) at $N = 1$ and express the functions $\psi_2^*(x, \zeta_1)$ and $\psi_1(x, \zeta_1)$ from (17) in terms of

$$\Phi^{(2)}(\zeta_1^*) = \frac{1}{2\pi i}\int \frac{\phi(\xi)}{\xi - \zeta_1^*} d\xi.$$

After substituting in (16), we obtain a system only for the quantities ϕ_1 and ϕ_2^*. This system contains as a coefficient the small quantity $c(x, \xi, t) = [b(\xi, t)\exp i\xi x]/a(\xi)$ Accurate to terms quadratic in c, we have

$$\phi_1 = 0, \quad \phi_2 = c(x,\xi,t)\left[1 + \frac{\zeta_1^* - \zeta_1}{\xi - \zeta_1}\left(1 - \frac{1}{1+|\lambda|^4}\right)\right]$$

$$\lambda = \sqrt{c_1(0)}\, e^{2i\zeta_1^2 t} e^{i\zeta_1 x}. \qquad (30)$$

From (18) we now obtain

$$q(x,t) = q_0(x,t) - \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi_2^*(x,\xi,t)\, d\xi,$$

$$q_0(x,t) = (\lambda^*)^2/1 + |\lambda|^4. \qquad (31)$$

Here q_0 is the soliton (6) with parameters $\xi + i\eta = \zeta_1$. Recognizing that

$$c(x,\xi,t) = \frac{b(\xi,0)}{a(\xi)}\exp\{2i\xi x + 4i\xi^2 t\},$$

we find that the integral in (31) decreases like $1/\sqrt{t}$, as $t \to \infty$. This means that the soliton is stable against a perturbation by a continuous spectrum—as $t \to \infty$ the solution goes over asymptotically into a soliton.

A perturbation of general form, other than the production of a continuous spectrum, shifts the position of the zero of $a(\zeta)$ on the complex plane and by the same token perturbs the parameters of the soliton. As $t \to \infty$, the solution tends asymptotically to this perturbed soliton.

7. QUASICLASSICAL APPROXIMATION

Let us consider for Eq. (3) initial conditions such that the quasiclassical approximation can be applied to the operator $\hat{L}$. To this end, we eliminate the function v_2 from the system (8). We have for v_1

$$v_1'' + (|q|^2 + \zeta^2)v_1 - (v_1' + i\zeta v_1)q'/q = 0. \qquad (32)$$

In the quasiclassical approximation, $q'/q \ll 1$, and therefore Eq. (32) can be replaced by the Schrödinger equation

$$v_1'' + (|q|^2 + \zeta^2)v_1 = 0.$$

It follows therefore that in the quasiclassical approximation all the eigenvalues of ζ lie on the imaginary axis and are described by the Bohr quantization rules

$$\int \sqrt{|q|^2 - \eta_n^2}\, dx = 2\pi(n + 1/2), \quad \zeta_n = i\eta_n.$$

Actually, the levels do not lie strictly on the imaginary axis, but have small real increments, which ensure divergence of the individual solitons. A simple estimate shows that the characteristic ratio of the real part to the imaginary one, $\xi/\eta \sim N^{-1/2}$, where N is the total number of solitons, is a large quantity in the quasiclassical limit.

We note (see Sec. 1) that if the initial condition is pure real, then all the ξ_n are exactly equal to zero. In this case we can neglect the "nonsoliton" part of the solution, owing to the exponential smallness of the reflection coefficient, and assume that we are dealing with a bound state of a very large number of solitons. Such a state is characterized by a large number of frequencies and can be described in an averaged manner.

The quantity characterizing this averaged state can be the soliton amplitude distribution function $f(\eta)$, which is given by

$$f(\eta) = \frac{\partial n}{\partial \eta} = \frac{1}{2\pi}\int \frac{\eta}{\sqrt{|q|^2 - \eta^2}} dx.$$

Since the dimension of the soliton is uniquely connected with its amplitude, it follows that $f(\eta)$ is likewise a distribution function with respect to the reciprocal dimension. If the initial condition is such that $u(x, 0)$ does not vanish as $|x| \to \infty$, and is only bounded, then we can introduce the average soliton amplitude distribution function

$$\tilde{f}(\eta) = \lim_{L\to\infty}\frac{1}{2\pi L}\int_{-L}^{L}\frac{\eta\, dx}{\sqrt{|q|^2 - \eta^2}}.$$

Both the function $f(\eta)$ and the "average" distribution function $\tilde{f}(\eta)$ are independent of the time.

Using these facts, we can assess the character of the nonlinear stage of development of the instability of a plane monochromatic wave with amplitude u_0, within the framework of Eq. (3). The instability gives rise to a "reversible" one-dimensional turbulence characterized by an average distribution function with respect to the reciprocal lengths,

$$\tilde{f}(\eta) = 1/2\pi\sqrt{\varkappa|u_0|^2 - \eta^2}.$$

The development of the instability will not be accompanied by any systematic energy transfer from some scales to others.

8. CONSERVATION LAWS

Equation (3) has an infinite set of conservation laws—this follows at least from the fact that the quantity $a(\zeta)$ is constant in time. Among the conservation laws connected with $a(\zeta)$ is an enumerable set of so-called polynomial conservation laws. They have the form of an integral, with respect to x, of a polynomial expression in terms of the function u(x, t) and its derivatives with respect to x. Let us describe a regular method of calculating such conservation laws.

The function

$$a(\zeta)=\varphi_1\psi_2-\varphi_2\psi_1,\quad \operatorname{Im}\zeta\geqslant 0$$

specifies the asymptotic form, as $x\to+\infty$, of the solution

$$\varphi=\begin{Bmatrix}\varphi_1\\ \varphi_2\end{Bmatrix}$$

of Eq. (8):

$$\varphi_1(x,\zeta)e^{i\zeta x}\to a(\zeta)\ \text{ as }\ x\to+\infty;\quad \operatorname{Im}\zeta>0,\ \ \zeta\neq\zeta_j.$$

Alternately, putting $\varphi_1 \exp i\zeta x = e\phi$, we have

$$\phi(x,\zeta)\to\ln a(\zeta)\ \text{ as }\ x\to\infty,\ \ |\zeta|>R.$$

From (8) we obtain the following equation for ϕ:

$$2i\zeta\phi'=|q|^2+\phi'^2+q\frac{d}{dx}\left(\frac{1}{q}\phi'\right),$$

which makes it possible to calculate, in recurrent fashion, the coefficients of the asymptotic expansion of the function $\phi'(x,\zeta)$ in powers of $1/\zeta$:

$$\phi'(x,\zeta)\approx\sum_{n=1}^{\infty}f_n/(2i\zeta)^n, \tag{34}$$

$$f_{n+1}=q\frac{d}{dx}\left(\frac{1}{q}f_n\right)+\sum_{j+k=n}f_jf_k,\quad f_1=|q|^2. \tag{35}$$

When $|\zeta|\to\infty$, and $\operatorname{Im}\zeta\geq 0$, the function $\ln a(\zeta)\to 0$ and also admits of an asymptotic expansion in powers of $1/\zeta$:

$$\ln a(\zeta)\approx\sum_{n=1}^{\infty}C_n/\zeta^n. \tag{36}$$

From (33), (34), and (36), it follows that

$$(2i)^nC_n=\int_{-\infty}^{+\infty}f_n(x)\,dx,\quad n=1,2,\ldots \tag{37}$$

By virtue of the recurrence formula (35), the integrands f_n of the conservation laws (37) are polynomials of the function q(x, t) and its derivatives with respect to x. We present the first five conservation laws:

$$2iC_1=\frac{\varkappa}{2}\int_{-\infty}^{+\infty}|u(x,t)|^2\,dx,$$

$$(2i)^2C_2=-\frac{\varkappa}{4}\int_{-\infty}^{+\infty}(u^*u_x-uu_x^*)(x,t)\,dx,$$

$$(2i)^3C_3=-\frac{\varkappa}{2}\int\left(|u_x|^2-\frac{\varkappa}{2}|u|^4\right)(x,t)\,dx,$$

$$(2i)^4C_4=\frac{\varkappa}{2}\int\left(uu_{xxx}^*+3\frac{\varkappa}{2}uu_x^*|u|^2\right)(x,t)\,dx,$$

$$(2i)^5C_5=\frac{\varkappa}{2}\int\left[|u_{xx}|^2+\frac{\varkappa^2}{2}|u|^6-\frac{\varkappa}{2}\left(\frac{\partial}{\partial x}|u|^2\right)^2-3\varkappa|u_x|^2|u|^2\right](x,t)\,dx.$$

The first three integrals have a simple physical meaning if we interpret Eq. (3) as a nonlinear Schrödinger equation. The constants of motion C_1, C_2, and C_3 then coincide, apart from coefficients, with the number of particles, the momentum, and the energy.

For N-soliton solutions it is easy to calculate all the constants of motion C_n. In our case it follows from (11) that

$$a(\xi)a^*(\xi)=1.$$

It is clear therefore that the function $a(\zeta)$, which is analytic in the upper half-plane, is continued analytically into the lower half-plane, where it has simple poles at points conjugate to the zeroes of ζ_j, $j = 1, \ldots, N$. Reconstructing the rational-fraction function $a(\zeta)$ from its zeroes and poles, we obtain, recognizing that $a(\zeta)\to 1$ as $|\zeta|\to\infty$,

$$a(\zeta)=\prod_{p=1}^{N}\frac{\zeta-\zeta_p}{\zeta-\zeta_p^*}.$$

It is easy to verify now that

$$C_n=\frac{1}{n}\sum_{k=1}^{N}(\zeta_k^n)^*-\zeta_k^n.$$

The presence of an infinite number of polynomial conservation laws is a characteristic property of the equations to which the inverse-problem method is applicable. For the Korteweg–de Vries equation, the theory of the conservation laws was developed in detail in [13].

APPENDIX

Let $\zeta_1, \ldots, \zeta_N$ be arbitrary complex numbers such that $\zeta_j\neq\zeta_k$ at $j\neq k$, and $\zeta_j\neq\zeta_k^*$ for all j and k. We consider a system of linear algebraic equations of the type (25):

$$\sum_{k=1}^{N}\frac{1}{\zeta_k-\zeta_j^*}f_k=g_j,\quad j=1,\ldots,N. \tag{A.1}$$

Let

$$a(\zeta)=P(\zeta)/Q(\zeta),\quad P(\zeta)=\prod_{p=1}^{N}(\zeta-\zeta_p),\quad Q(\zeta)=\prod_{p=1}^{N}(\zeta-\zeta_p^*),$$

$$a_k=\frac{1}{a'(\zeta_k)}=\prod_{p=1}^{N}(\zeta_k-\zeta_p^*)\Big/\prod_{p=1}^{N}{}'(\zeta_k-\zeta_p).$$

Here and henceforth Π' denotes that the factor equal to zero has been left out from the product. It is easy to see that

$$a(\zeta)=1+\sum_{k=1}^{N}a_k^*/(\zeta-\zeta_k^*).$$

Putting here $\zeta=\zeta_j$, we get

$$\sum_{k=1}^{N}\frac{a_k^*}{\zeta_k^*-\zeta_j}=1=\sum_{k=1}^{N}\frac{a_k}{\zeta_k-\zeta_j^*}. \tag{A.2}$$

We have thus obtained the solution of the system (1) for $g_1 = \ldots = g_N = 1$.

We introduce also the polynomials

$$P_k(\zeta) = \frac{1}{\zeta - \zeta_k} P(\zeta), \quad Q_n(\zeta) = \frac{1}{\zeta - \zeta_k^*} Q(\zeta).$$

It is easy to verify that any polynomial $H(\zeta)$ of degree $N - 1$ satisfies the identity

$$H(\zeta) = \sum_{k=1}^{N} \frac{H(\zeta_k)}{Q(\zeta_k)} a_k P_k(\zeta).$$

Putting here $\zeta = \zeta_j^*$, we get

$$\frac{H(\zeta_j^*)}{P(\zeta_j^*)} = \sum_{k=1}^{N} \frac{H(\zeta_k)}{Q(\zeta_k)} a_k \frac{1}{\zeta_j^* - \zeta_k}$$

For the special choice of the polynomial $H(\zeta) = Q_l(\zeta)$, this formula takes the form

$$\sum_{k=1}^{N} \frac{a_k}{\zeta_j^* - \zeta_k} \frac{1}{\zeta_k - \zeta_l^*} = \begin{cases} 0 & \text{for} \quad l \neq j \\ 1/a_l^* & \text{for} \quad l = j \end{cases}. \qquad (A.3)$$

It follows from (A.3) that the matrix inverse to the matrix of the system (1) is

$$\left\| \frac{a_j^* a_k}{\zeta_j^* - \zeta_k} \right\|.$$

Returning to the system (25), we see that we have to write out the solution of the system (1) for $g_j = 1/(\zeta_j - \zeta_0^*)$, where ζ_0 is a certain complex number different from $\zeta_1, \ldots, \zeta_N$. Let

$$\tilde{a}_k = \prod_{p=0}^{N} (\zeta_k - \zeta_p^*) \Big/ \prod_{p=0}^{N}{}' (\zeta_k - \zeta_p), \qquad k = 0, \ldots, N.$$

Then

$$\tilde{a}_k = \frac{\zeta_k - \zeta_0^*}{\zeta_k - \zeta_0} a_k, \quad k > 0.$$

In analogy with (A.3), we have

$$\sum_{k=0}^{N} \frac{\tilde{a}_k}{\zeta_j^* - \zeta_k} \frac{1}{\zeta_k - \zeta_0^*} = \begin{cases} 0, & j > 0, \\ 1/\tilde{a}_0^*, & j = 0. \end{cases}$$

Hence, replacing $\tilde{a}_k$ by a_k with $k > 0$, we get

$$\sum_{k=1}^{N} \frac{a_k}{(\zeta_j^* - \zeta_k)} \frac{1}{\zeta_k - \zeta_0} = -\frac{\tilde{a}_0}{2i\eta_0} \frac{1}{\zeta_j^* - \zeta_0}, \qquad (A.4)$$

and

$$\sum_{k=1}^{N} \frac{a_k}{(\zeta_0^* - \zeta_k)} \frac{1}{\zeta_k - \zeta_0} = \frac{1}{\tilde{a}_0^*} - \frac{\tilde{a}_0}{4\eta_0^2}. \qquad (A.5)$$

In connection with (24), we indicate also a generalization of formula (A.2):

$$\sum_{k=1}^{N} \frac{a_k}{\zeta_k - \zeta_0} = 1 - \frac{\tilde{a}_0}{2i\eta_0}, \qquad (A.6)$$

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