ANALOGS OF ANISOTROPIC HOMOGENEOUS MODELS OF GENERAL RELATIVITY THEORY IN NEWTONIAN COSMOLOGY

I. S. SHIKIN

Moscow State University

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The problem of Newtonian analogs, defined as limiting states for $c \to \infty$, is considered in a general form for anisotropic homogeneous relativistic models. Expressions for the interval in a synchronous reference frame and in an inertial frame with Eulerian coordinates are compared in the Newtonian approximation. The construction of homogeneous models within the framework of Newtonian cosmology and the associated problem of the Newtonian gravitational paradox are discussed. Newtonian analogs are considered for relativistic models containing matter and possessing transitivity hypersurfaces orthogonal to the 4-velocity, and for empty-space models (particularly for the Bianchi type IX model).

IN general relativity theory (GRT) anisotropic homogeneous cosmological models are considered, which are defined as gravitational fields whose metric permits a group of motions acting transitively on spacelike three-dimensional transitivity hypersurfaces V_3[1,2]. In a synchronous reference frame with geodesic lines of time τ which are orthogonal to V_3 the metric has the form (where c is the velocity of light)

$$ds^2 = (c d\tau)^2 - dl^2, \tag{1}$$

in which dl^2, the element of spatial distance in V_3, is determined by a group of motions (Sec. 1). These models are constructed for empty space as well as for gravitational fields whose sources are matter. It is important that in the general case (1) does not represent a reference frame that is attached to matter.

In the present paper we consider the construction of analogs of these homogeneous models in Newtonian cosmology (Newtonian hydrodynamics with a Newtonian gravitational potential). Such models are known to exist, as has been shown by Milne and McCrae for Friedmann's isotropic models. In[3] the analogs have been constructed for homogeneous models of Bianchi type I, and also type V for the case of transitivity hypersurfaces orthogonal to the 4-velocity. We shall here consider a general formulation of the problem for all possible types of homogeneous models. The structure of the interval in a quasi-Newtonian approximation is discussed (Sec. 2). Newtonian analogs are considered for homogeneous GRT models containing matter in the case where the frame of (1) is also attached, and for empty-space models. The Newtonian gravitational paradox is also discussed (Sec. 3).

1. NECESSARY RELATIONS IN GENERAL RELATIVITY THEORY

In GRT the element of spatial distance dl^2 in V_3 [(Eq. 1)] in the case where the group of motions G_3 for the metric (1) operates in V_3 in a simply transitive manner has the form[1]

$$dl^2 = \gamma_{\alpha\beta} dy^\alpha dy^\beta = \gamma_{ab}(\tau) l_\alpha{}^a l_\beta{}^b dy^\alpha dy^\beta \tag{1.1}$$

where the Greek indices of the spatial coordinates y^α take values from 1 to 3, and the basis vectors l_α^a are labeled with the Latin letters a, b, ..., which also take values from 1 to 3. The components of γ_{ab} depend on τ; l_α^a depends on y^α. We also introduce the basis l_a^α through the relations

$$l_a{}^\alpha l_\beta{}^a = \delta_\beta{}^\alpha, \quad l_a{}^\alpha l_\alpha{}^b = \delta_a{}^b; \quad \gamma^{\alpha\beta} = \gamma^{ab}(\tau) l_a{}^\alpha l_b{}^\beta; \quad \gamma^{ab}\gamma_{bc} = \delta_c{}^a. \tag{1.2}$$

The basis vectors in (1.1), combined with infinitesimal operators of their common group, satisfy relations with the structural constants c_{ab}^c[1,4]

$$\partial l_\alpha{}^c / \partial y^\beta - \partial l_\beta{}^c / \partial y^\alpha = c_{ab}{}^c l_\alpha{}^a l_\beta{}^b. \tag{1.3}$$

Nine nonisomorphic structural types are known for the simple transitive groups G_3 classified by Bianchi.[1,5,6] A model with a metric permitting a given type of G_3 group will be called the model of the corresponding Bianchi type.

Einstein's gravitational equations (without the cosmological term)

$$R_i{}^k = (8\pi k / c^4)[T_i{}^k - (T/2)\delta_i{}^k] \tag{1.4}$$

(in which the notation of[7] is used) for (1) and (1.1) are written in basis components.[1,4,8] Then for an ideal fluid, in (1.4) we have (with the notation e—energy density, p—pressure, and u^i—4-velocity)

$$T_i{}^k = (e+p)u_i u^k - p\delta_i{}^k, \quad e = e(\tau), \quad u^0 = u_0(\tau), \quad u^\alpha = u^a(\tau) l_a{}^\alpha. \tag{1.5}$$

In the case of dustlike matter [when in (1.5) we have $p = 0$, $e = \rho c^2$, $\rho = mn$, and n is the particle number density] the conservation laws $(T_i^k)_{;k} = 0$ lead to the continuity equation

$$(nu^i)_{;i} = 0 \tag{1.6}$$

and to the equations of hydrodynamics (matter moves along geodesics)

$$u^i (u_k)_{;i} = 0. \tag{1.7}$$

The motion of matter is characterized by isotropic expansion with the expansion scalar Θ, the shear tensor $Q_{ik}(Q_i^i = 0)$, and the rotation tensor Ω_{ik}. We now split $u_{i;k}$ as follows:

$$c u_{i;k} = D_{ik} + \Omega_{ik}; \quad D_{ik} = c\frac{u_{i;k} + u_{k;i}}{2}$$

$$= \Theta(g_{ik} - u_i u_k) + Q_{ik}; \quad \Theta = \frac{cu^i_{;i}}{3}. \tag{1.8}$$

Using (1.1), (1.3), and (1.5), we obtain (denoting differention with respect to τ by dots)

$$3\Theta = \dot{u}^0 + \frac{1}{2}\varkappa_a{}^a u^0 + \overset{a}{c}_{ab} c u^b, \; \dot{u}_0 = -\frac{\varkappa_{ab}u^a u^b}{2u^0} = D_{00}; \; \varkappa_{ab} = \dot{\gamma}_{ab}, \; \varkappa_a{}^b = \gamma^{bd}\dot{\gamma}_{da};$$

$$2D_{0\alpha} l_0{}^\alpha = \varkappa_{ab} u^b + \dot{u}_a, \quad 2D_{\alpha\beta} l_a{}^\alpha l_b{}^\beta = -\varkappa_{ab} u_0 - c u^d (\gamma_{ac} c_{bd}{}^c + \gamma_{bc} c_{ad}{}^c);$$

$$2\Omega_{0\alpha} l_a{}^\alpha = -\dot{u}_a = c_{ab}{}^c c u_c u^b / u^0, \quad 2\Omega_{\alpha\beta} l_a{}^\alpha l_b{}^\beta = c_{ab}{}^d c u_d. \tag{1.9}$$

For empty-space models with $T_{ik} = 0$, (1.8) is conserved, with u^i representing a 4-vector that is tangent to a geodesic.

2. THE INTERVAL IN THE QUASI-NEWTONIAN APPROXIMATION

It is the purpose of this investigation to study the behavior of homogeneous GRT models in the limit $c \to \infty$. These limiting states will be called Newtonian analogs of the corresponding GRT models. The coordinates and the metric will be transformed as follows:

$$y^\alpha = k^\alpha + z^\alpha / c, \quad k^\alpha = \text{const};$$
$$g^{ab} = c^2 \gamma^{ab}, \quad g_{ab} = \gamma_{ab} / c^2, \quad dl^2 = g_{ab} l_\alpha{}^a l_\beta{}^b dz^\alpha dz^\beta. \tag{2.1}$$

The components of the 4-velocity $\tilde{u}^i$ in the coordinates τ, z^α are

$$\tilde{u}^0 = u^0, \quad \tilde{u}^a = \tilde{u}^\alpha l_\alpha{}^a = c u^a, \quad \tilde{u}_a = -g_{ab}\tilde{u}^b = u_a / c. \tag{2.2}$$

Einstein's equations (1.4) with $i = \alpha$, $k = 0$ and $i = \alpha$, $k = \beta$ in the basis components with the metric g_{ab} (2.1) become for dustlike matter in virtue of (1.3) (ref. 7, Sec. 99; ref 4)

$$c^2 \tilde{R}_\alpha{}^0 l_a{}^\alpha = \varkappa_a{}^d c_{bd}{}^b + \varkappa_c{}^d c_{ad}{}^c = 8\pi k \rho \tilde{u}_a \tilde{u}^0; \quad \varkappa_a{}^b = g^{bd}\dot{g}_{da}; \tag{2.3a}$$

$$-c^2 \tilde{R}_\alpha{}^\beta l_a{}^\alpha l_\beta{}^b = \tilde{P}_a{}^b + {}^1/{}_4 \varkappa_a{}^b \varkappa_d{}^d + {}^1/{}_2 \dot{\varkappa}_a{}^b = 8\pi k \rho (g_{ac}\tilde{u}^b \tilde{u}^c + {}^1/{}_2 \delta_a{}^b). \tag{2.3b}$$

The Ricci 3-tensor in (2.3b) is[1,8]

$$\tilde{P}_\alpha{}^\beta l_a{}^\alpha l_\beta{}^b \equiv \tilde{P}_a{}^b = -c_{de}{}^c c_{gh}{}^f [2g^{eb}\delta_a{}^h(\delta_c{}^g \delta_f{}^d + g^{gd} g_{cf})$$
$$- g^{dg} g^{eh} g_{fa} \delta_c{}^b + 2g^{eh}\delta_c{}^d (\delta_c{}^g \delta_f{}^b + g^{gb} g_{af})] / 4. \tag{2.3c}$$

The component $R^0_0 - (R/2)$ of Einstein's equations with g_{ab} of (2.1) has for dust the form[1)]

$${}^1/{}_8[-\varkappa_a{}^b \varkappa_b{}^a - (\varkappa_a{}^a)^2] + {}^1/{}_2 \tilde{P}_a{}^a = 8\pi k \rho \tilde{u}^0 \tilde{u}_0. \tag{2.3d}$$

In accordance with (2.3a) we have $u^a = 0$ (and $\Omega_{ik} = 0$) identically only for the type I model ($c^c_{ab} = 0$); for the other types $u^a = 0$ only in special cases.

In the coordinates z^α, τ the equations of (1.9) are conserved with g_{ab}, $\tilde{u}^a$, and c^c_{ab}/c instead of c^c_{ab}, as a consequence of (2.1) and (1.3).

In accordance with (2.1) the components of the vectors l^a_α in the limit $c \to \infty$ for groups of all Bianchi types are transformed into the constant quantities L^a_α which equal l^a_α for $y^\alpha = k^\alpha$ and depend, in general, on the constants k^α. Then the L^a_α correspond to an Abelian group of motions in three-dimensional Euclidean space. In accordance with (1.2) the vectors l^α_a are also reduced to the constants L^α_a for $c \to \infty$. In accordance with (1.3) and (2.1) we have, accurately to $1/c$ terms,

$$l_\alpha{}^a = L_\alpha{}^a + \frac{1}{c}\Lambda_\alpha{}^a, \quad l_a{}^\alpha = L_a{}^\alpha - \frac{1}{c}\Lambda_\beta{}^b L_b{}^\alpha L_a{}^\beta; \quad L_\alpha{}^a L_b{}^\alpha = \delta_b{}^a;$$

$$\Lambda_\alpha{}^a = {}^1/{}_2 c_{fg}{}^a L_\alpha{}^f L_\beta{}^g z^\beta + S_{\alpha\beta}{}^a z^\beta, \quad S_{\alpha\beta}{}^a = S_{\beta\alpha}{}^a. \tag{2.4}$$

The value of the constant symmetric tensor $S^a_{\alpha\beta}$ in (2.4) is associated with the allowed transformations of z^α that conserve (1.3). The equations (2.4) and the relations that will be presented below in the quasi-Newtonian approximation are valid in GRT for values of y^α that are close to a fixed value k^α, and can be interpreted as local relations in the vicinity of the τ geodesic that passes through the point $y^\alpha = k^\alpha$ of a transitivity hypersurface.

In a synchronous frame the metric, according to (1), (2.1), and (2.4), is given with the same accuracy by

$$ds^2 = (cd\tau)^2 - \left[\lambda_{\alpha\beta} + \frac{1}{c} g_{ab}(\Lambda_\alpha{}^a L_\beta{}^b + \Lambda_\beta{}^b L_\alpha{}^a)\right] dz^\alpha dz^\beta, \tag{2.5}$$

$$\lambda_{\alpha\beta} = g_{ab} L_\alpha{}^a L_\beta{}^b, \quad \lambda^{\alpha\beta} = g^{ab} L_a{}^\alpha L_b{}^\beta. \tag{2.6}$$

In three-dimensional Euclidean space we shall use Euler's Cartesian coordinates $x^\alpha = x_\alpha$ with the element of distance in the form

$$dL^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \equiv \eta_{\alpha\beta} dx^\alpha dx^\beta = \eta_{ab} dx^a dx^b. \tag{2.7}$$

In the Newtonian approximation the interval, expressed by means of the Eulerian space coordinates x^α and Newtonian time t, is represented with the same accuracy by[7] [2)]

$$ds^2 = \left(1 + \frac{2\varphi}{c^2}\right)(cdt)^2 + \frac{1}{c^2} F_\alpha dx^\alpha cdt + \left(-\eta_{\alpha\beta} + \frac{1}{c} P_{\alpha\beta}\right) dx^\alpha dx^\beta. \tag{2.8}$$

Here φ is the Newtonian gravitational potential, and $\eta_{\alpha\beta}$ is defined by (2.7). The coefficient of $dx^\alpha cdt$ contains no $1/c$ term; in the presence of such a term we would have inertial forces due to rotation of the coordinate system.[7] The coefficient of $dx^\alpha dx^\beta$ contains $1/c$ terms because of (2.4).

A transformation from the coordinates z^α, τ of the synchronous frame in (2.5) to the coordinates x^α, t of the metric (2.8) is achieved with the same accuracy by the equations

$$\tau = t + c^{-2}\Psi(x^\alpha, t), \tag{2.9}$$

$$x^\alpha = M^\alpha{}_\beta(t) z^\beta, \quad z^\beta = (M^{-1})^\beta{}_\gamma x^\gamma, \tag{2.10}$$

$$(M^{-1})^\alpha{}_\beta = \eta_{\beta\gamma}\lambda^{\alpha\delta}(t) M^\gamma{}_\delta, \quad M^\alpha{}_\gamma M^\beta{}_\delta \lambda^{\gamma\delta}(t) = \eta^{\alpha\beta}, \tag{2.10a}$$

$$M^\gamma{}_\alpha M^\delta{}_\beta \eta_{\gamma\delta} = \lambda_{\alpha\beta}(t) \tag{2.10b}$$

where in the matrix M^α_β the first index is lowered by means of the tensor $\eta_{\alpha\gamma}$ and the second index is raised by means of $\lambda^{\beta\gamma}(t)$ in (2.6).

The fact that the coefficient of $dx^\alpha cdt$ in (2.8) contains no $1/c$ term leads to

$$\partial\Psi / \partial x^\alpha + K_{\alpha\beta} x^\beta = 0, \tag{2.9a}$$

where (differentiation with respect to t is denoted hereinafter by a dot)

$$K^\alpha{}_\beta(t) = \dot{M}^\alpha{}_\gamma (M^{-1})^\gamma{}_\beta = \dot{M}^\alpha{}_\gamma M^\mu{}_\delta \eta_{\beta\mu}\lambda^{\gamma\delta}(t), \quad K_{\alpha\beta} = \eta_{\alpha\gamma} K^\gamma{}_\beta. \tag{2.10c}$$

From (2.9a) and the integrability conditions we obtain

$$K_{\alpha\beta} = K_{\beta\alpha}, \tag{2.11}$$

$$\Psi = -K_{\alpha\beta}(t) x^\alpha x^\beta / 2 + \Psi_1(t). \tag{2.12}$$

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1)The equations (2.3a), (2.3b), and (2.3d) are valid for an arbitrary metric in a synchronous frame under the transformation (2.1).

2)The possibility of a transition from (2.5) to (2.8) is associated with the fact that the expressions for R_{ik} in the coordinates z^α, τ contain a $1/c^2$ factor.

For φ in (2.8), in accordance with (2.9), (2.10), and (2.12) we have

$$\varphi=\frac{\partial\Psi}{\partial t}-\frac{1}{2}\eta^{\gamma\delta}K_{\gamma\alpha}K_{\gamma\beta}x^\alpha x^\beta=-\frac{1}{2}(\dot K_{\alpha\beta}+\eta^{\gamma\delta}K_{\gamma\alpha}K_{\gamma\beta})x^\alpha x^\beta+\varphi_0(t). \quad (2.13)$$

We obtain expressions for the 4-velocity U^i in the coordinates t, x^α through the components of $\tilde{u}^i$ in the coordinates τ, z^α, using (2.10) and t = τ. Since, on the basis of (2.2) and (2.4),

$$c\tilde{u}^\alpha=c\tilde{u}^a L_a^\alpha-({}^1/_2 c_{fg}{}^b L_\beta{}^f L_\gamma{}^g+S_{\beta\gamma}{}^b)L_b{}^\alpha\tilde{u}^a L_a{}^\beta z^\gamma,\quad \tilde{u}^a=\tilde{u}^a(t), \quad (2.14)$$

we have, with the previous accuracy,

$$U^0=\tilde{u}^0,\quad cU^\alpha=K^\alpha{}_\beta U^0x^\beta+R^\alpha{}_\beta x^\beta+M^\alpha{}_\beta c\tilde{u}^a L_a{}^\beta,$$
$$R^\alpha{}_\beta(t)=-M^\alpha{}_\delta\left(\frac{1}{2}c_{fg}{}^b L_\xi{}^f L_\gamma{}^g+S_{\xi\gamma}{}^b\right)L_b{}^\delta\tilde{u}^a L_a{}^\xi(M^{-1})^\gamma{}_\beta. \quad (2.15)$$

In the case $\tilde{u}^a = 0$, from (2.15) in the limit c $\to\infty$ the dependence of the three-dimensional velocity v^α on Eulerian coordinates is

$$v^\alpha=K^\alpha{}_\beta(t)x^\beta \quad (2.16)$$

with the Hubble—"constant" tensor $K^\alpha_\beta(t)$ in (2.10c).

We introduce the Lagrangian coordinates ξ^α which "individualize" particles. In GRT the frame of reference with τ and ξ^α, in accordance with

$$\xi^\alpha=z^\alpha-\Phi^\alpha(\tau),\quad \dot\Phi^\alpha=c\tilde{u}^\alpha/\tilde{u}^0 \quad (2.17)$$

is attached to matter; the interval given by (1) and (2.1) becomes

$$ds^2=(cd\tau/\tilde{u}^0)^2+2\tilde{u}_a d\xi^\alpha(cd\tau/\tilde{u}^0)-g_{ab}l_\alpha{}^a l_\beta{}^b d\xi^\alpha d\xi^\beta. \quad (2.17a)$$

In the present approximation the coordinates ξ^α given by (2.17) [with $\Phi^\alpha(t)$] are Lagrangian coordinates. When $\tilde{u}^a \neq 0$, (2.17) and (2.14) define ξ^α as a linear function of z^β. When $\tilde{u}^a = 0$ the coordinates z^α coincide with the Lagrangian coordinates, so that v^α $\partial x^\alpha(\xi^\beta, t)/\partial t$, and from (2.10) we have

$$x^\alpha=M^\alpha{}_\beta(t)\xi^\beta. \quad (2.16a)$$

This last equation elucidates the physical meaning of the matrices M^α_β, which are related to the metric tensor (2.6) by (2.10b).

When $u^a = 0$ the synchronous frame of (2.5) coincides with the attached frame of (2.17a), and the transitivity hypersurfaces are orthogonal to the 4-velocity [in (2.1) the constant fixes a "fluid" particle of the medium). Then in (2.15) we have $U^0 = 1$. In this case, for c $\to\infty$ (1.7) with k = α assumes the form of Euler's nonrelativistic hydrodynamic equations with a Newtonian potential φ, Eq. (1.4) with i = k = 0 assumes the form of Poisson's equation for φ, and (1.6) assumes the form of the Newtonian continuity equation [Eqs. (2.2), (2.3), and (2.1) in[3]]. These nonrelativistic equations comprise a system of equations of Newtonian cosmology for dust in the x^α, t inertial system.[9,10,3] Thus the homogeneous GRT models with $u^a = 0$ in the Newtonian approximation are described by the equations of Newtonian cosmology and Hubble's law [(2.16), (2.16a)]. Relativistic homogeneous models with $u^a = 0$ have been classified and discussed in[11-13].

When $u^a \neq 0$, then $U^0 \neq 1$ in (2.15). But (1.4) with i = k = 0, (1.7) with k = α, and (1.6) are not then reduced for c $\to\infty$ to nonrelativistic equations of Newtonian cosmology, so that the relativistic models with u^a $\neq 0$ do not have direct Newtonian analogs; to obtain the corresponding Newtonian analogues we must for c $\to\infty$ set $U^0 = 1$, $\tilde{u}^0 = 1$, $\tilde{u}^a = 0$. However, for $u^a \neq 0$ Hubble's law also holds true locally in accordance with (2.15).

3. ANALOGS OF MODELS CONTAINING MATTER WITH 4-VELOCITY ORTHOGONAL TO THE TRANSITIVITY HYPERSURFACES, AND OF EMPTY-SPACE MODELS, IN NEWTONIAN COSMOLOGY. THE GRAVITATIONAL PARADOX

We shall now consider the construction of homogeneous models within the framework of Newtonian cosmology. Here the law

$$v_\alpha=h_{\alpha\beta}(t)x^\beta \quad (3.1)$$

follows from the requirement of invariance under three translations in ξ^α space [forming a simple transitive group of motions for (2.7)] [9,10,14,15] in accordance with the relativistic group criterion of homogeneity. The tensor of deformation rates $\partial v_\alpha/\partial x^\beta = h_{\alpha\beta}(t)$ is split into a spherical tensor with the expansion scalar Θ, a shear tensor $q_{\alpha\beta}$ with zero trace ($q^\alpha_\alpha = 0$), and a rotational tensor $\omega_{\alpha\beta}$ to form a Newtonian analog of (1.8), as follows[9,10,3]:

$$h_{\alpha\beta}=d_{\alpha\beta}+\omega_{\alpha\beta};\qquad d_{\alpha\beta}=\frac{h_{\alpha\beta}+h_{\beta\alpha}}{2}=\Theta\eta_{\alpha\beta}+q_{\alpha\beta};\qquad \Theta=\frac{h_\alpha{}^\alpha}{3}=\frac{\dot R}{R}. \quad (3.1a)$$

By inserting (3.1) into the equations of Newtonian cosmology [Eqs. (2.1), (2.2), and (2.3) in[3]] in the general case of a nonsymmetric tensor $h_{\alpha\beta}(t)$, we obtain for the density ρ and the quantities in (3.1a)

$$\rho(t)=3M/4\pi R^3,\ M=\text{const}, \quad (3.2)$$
$$\dot\omega_{\alpha\beta}+2\Theta\omega_{\alpha\beta}+\omega_{\alpha\gamma}q_\beta{}^\gamma+\omega_{\gamma\beta}q_\alpha{}^\gamma=0, \quad (3.3)$$
$$3(\dot\Theta+\Theta^2)+q_{\alpha\beta}q^{\alpha\beta}+\omega_{\alpha\beta}\omega^{\beta\alpha}+4\pi k\rho=0. \quad (3.4)$$

Equation (3.3) results from the integrability condition of the expression for $\partial\varphi/\partial\chi^\alpha$.[9,10] Equation (3.4), obtained from Poisson's equation, serves as the Newtonian analog of Raychaudhuri's identity in GRT (the expression for $R_{ik}u^i u^k$).[16,8,17,3] These two equations (3.3) and (3.4) exhaust the information furnished by the equations of Newtonian cosmology for the tensor $h_{\alpha\beta}(t)$ in (3.1); five components of the shear tensor thus remain arbitrary. For the gravitational potential φ we obtain

$$\varphi=\varphi_1+\varphi_2,\quad \varphi_1=\frac{2\pi}{3}k\rho\eta_{\alpha\beta}x^\alpha x^\beta=\frac{kM}{2B^3}(x^2+y^2+z^2), \quad (3.5)$$
$$\varphi_2=-{}^1/_2(\dot q_{\alpha\beta}+2\Theta q_{\alpha\beta}+q_{\alpha\gamma}q_\beta{}^\gamma-{}^1/_3\eta_{\alpha\beta}q_{\gamma\delta}q^{\gamma\delta}+\eta^{\gamma\delta}\omega_{\alpha\gamma}\omega_{\delta\beta}-{}^1/_3\eta_{\alpha\beta}\omega_{\gamma\delta}\omega^{\delta\gamma})x^\alpha x^\beta+\varphi_0(t). \quad (3.5a)$$

The expression for φ is a superposition of two terms. The term φ_1 in (3.5) satisfies Poisson's equation and has the form of a Newtonian potential for a spherically symmetric distribution of matter with constant density, derived for spheres of finite radius.[7] The term φ_1, which is essentially associated with the presence of matter, determines the isotropic part of φ.[3]

In (3.5a) the coefficient of $x^\alpha x^\beta$ is a tensor with zero trace, so that φ_2 satisfies Laplace's equation:

$$\partial^2\varphi_2/\partial x^\alpha\partial x_\alpha=0. \quad (3.5b)$$

This equation corresponds to Newtonian analogs of homogeneous empty-space GRT models.[3,1] In accordance with the foregoing discussion the expression for φ_2 contains five arbitrary functions of time. This result is associated with the content of the Newtonian gravitational paradox[18] and is derived from the specific form of the Newtonian problem when φ_2 is unbounded at infinitely distance points. The general solution of the linear Poisson equation consists of a particular solution of this equation in the form of Newton's potential for a distribution of masses and the general solution of the homogeneous equation (3.5b) for the harmonic function φ_2. In problems where φ_2 is bounded at infinity the general solution of (3.5b) is reduced, in accordance with Liouville's theorem, to a constant value (not depending on the coordinates) of φ_2. In the problems that we are considering here φ_2 remains unbounded, but is subject to the condition of homogeneity. Thus φ_2 becomes a quadratic function of x^α with coefficients that are arbitrary functions of time. To determine the latter, expressions must be given for $q_{\alpha\beta}(t)$ as substitutes for the boundary conditions of Newtonian problems in a finite region. For the Newtonian analogs of relativistic models the expressions for $q_{\alpha\beta}(t)$ are determined by the equations of GRT. We note that within the framework of a pure Newtonian treatment we cannot exclude forms of $q_{\alpha\beta}(t)$ that are inconsistent with the equations of GRT; the corresponding Newtonian solutions cannot serve as analogs of relativistic models.[17]

For the here considered Newtonian analogs of homogeneous GRT models, with $u^a = 0$ rotation is absent in accordance with (2.16) and (2.11), and in (3.1) we have

$$h_{\alpha\beta} \equiv K_{\alpha\beta}(t), \qquad d_{\alpha\beta} = h_{\alpha\beta}, \qquad \omega_{\alpha\beta} = 0. \tag{3.1b}$$

Then (with $\omega_{\alpha\beta} = 0$) Eqs. (3.2), (3.4) and (3.5), (3.5a) are valid, consistently with (2.13). As a consequence of (2.10a), (2.10c), and (3.1a) we have relations for the quantities in (2.3a), (2.3b), and (2.3d):

$$L_a{}^\alpha L_\beta{}^b \varkappa_b{}^a = 2 d_{\delta\gamma} M^\delta{}_\beta M^\gamma{}_\xi \lambda^{\xi\alpha}, \quad \varkappa_a{}^a = 6\Theta, \quad \varkappa_a{}^b \varkappa_b{}^a = 4 d_\alpha{}^\beta d_\beta{}^\alpha. \tag{3.6}$$

Equation (2.3d), in virtue of (3.6), becomes

$$6\Theta^2 - q_\alpha{}^\beta q_\beta{}^\alpha + \tilde{P} = 16\pi k\rho, \qquad \tilde{P} \equiv \tilde{P}_a{}^a, \tag{3.6a}$$

and can be interpreted as an extension of the familiar expression for isotropic models that relates the density to its critical value.[7,3] The expressions for the shear tensor $q_{\alpha\beta}(t)$ in the Newtonian equations are determined by the GRT equations (2.3a), (2.3b), and (2.3d); we are thus led to (3.6a) in virtue of (3.6). In accordance with (3.6) the spatial components of (1.9) for $u^a = 0$, when transformed in accordance with (2.10), are reduced to (3.1b).

The foregoing statements regarding Newtonian analogs of GRT models containing matter with $u^\alpha = 0$, $u^0 = 1$ apply also to homogeneous empty-space GRT models corresponding to $\rho = 0$ in (2.3a), (2.3b), (2.3d), and (3.6a).[3] In the Newtonian theory the continuity and momentum equations determine the parameters of test particles moving without rotation (along z^α = const geodesics) in a field defined by (3.5b).

We are also interested in anisotropic homogeneous GRT models with 3-space of constant curvature:

$$\tilde{P}_a{}^b = \tilde{P}\delta_a{}^b / 3. \tag{3.7}$$

Equation (3.7) holds true in the anisotropic case for $P = 0$ and $P < 0$ ($P > 0$ is not realized).[11] The case of $P = 0$ corresponds to a type I model (and to a special case of type VII[11]). The case of $P < 0$ corresponds to a type V model[19-23] $\lfloor c^c_{ab} = \delta^c_b a_a - \delta^c_a a_b$, from (2.3c) $\tilde{P} = -6g^{ab}a_a a_b \rfloor$; for $u^a = 0$, in accordance with (2.3a), we have a special case of this model.[8,11,3] For (3.7) we have, from (2.3b), (2.3d), and (3.4) and utilizing (1.8) and (1.9) ($u^a = 0$),

$$\varkappa_a{}^b/2 - \Theta\delta_a{}^b \equiv Q_a{}^b = A k_a{}^b / R^3, \qquad A = \text{const}, \tag{3.8}$$

where k_a^b is a constant matrix which by a transformation of the basis is diagonalized (along with g_{ab}) with the elements $\alpha_1, \alpha_2, \alpha_3$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$. In virtue of (3.1a) and (3.6) we have

$$q_0{}^0 = q_a{}^0 = 0, \quad q_a{}^b = Q_a{}^b = A(\alpha_1 \delta_a{}^1 \delta_1{}^b + \alpha_2 \delta_a{}^2 \delta_2{}^b + \alpha_3 \delta_a{}^3 \delta_3{}^b) / R^3. \tag{3.8a}$$

Equation (3.8a) $\lfloor$with $\alpha_1 = 0$ for $P < 0$, $u^a = 0$, arising out of (2.3a) for $a_2 = a_3 = 0\rfloor$ determines the shear tensor for the Newtonian analogs of models with (3.7) that were discussed in detail in[3].

Great interest attaches to the Bianchi type IX model[24-32] (the model of a "mixmaster universe;" $c^c_{ab} = e_{abd}\eta^{dc}$, e_{abc} is a completely antisymmetric tensor) containing Friedmann's closed model[4] as a special case. For type IX in (2.3b), (3.6a) we have, from (2.3c),

$$4\tilde{P}_a{}^b = 2g^{bc}\eta_{ac}(4 - g^{ed}\eta_{ed}g_{fg}\eta^{fg}) + g^{cd}g^{fg}\eta_{cg}\eta_{fd}g_{ea}\eta^{eb} + g_{ac}\eta^{cb}(g^{fg}\eta_{fg})^2 + 2g_{fg}\eta^{fg}g^{bc}g^{ed}\eta_{ad}\eta_{ce}. \tag{3.8b}$$

Analytic solutions of (2.3a), (2.3b), and (2.3d) for the type IX model were not obtained in the general case. For the type IX metric

$$dl^2 = (R_1^2 l_\alpha{}^1 l_\beta{}^1 + R_2^2 l_\alpha{}^2 l_\beta{}^2 + R_3^2 l_\alpha{}^3 l_\beta{}^3) dz^\alpha dz^\beta \tag{3.9}$$

the behavior of $R_\alpha(\tau)$ in the empty-space case and the oscillatory mode of approach to the singularity were investigated qualitatively for GRT in[24-26]. In the Newtonian model corresponding to (3.9) we have, in accordance with (2.16a) and (2.10b),

$$x^1 = R_1(t)L_1{}^1\xi^1, \quad x^2 = R_2(t)L_2{}^2\xi^2, \quad x^3 = R_3(t)L_3{}^3\xi^3, \quad h_\alpha = d\ln R_\alpha / dt, \tag{3.10}$$

with the functions $R_\alpha(t)$ from (3.9). Equations (3.10) determine the law of motion of particles of the medium and of test particles in the empty-space case. The Newtonian potential φ in accordance with (2.13) and (3.10) is given by

$$\varphi = -[(\ddot{R}_1/R_1)(x^1)^2 + (\ddot{R}_2/R_2)(x^2)^2 + (\ddot{R}_3/R_3)(x^3)^2]/2 + \varphi_0(t) \tag{3.11}$$

with coefficients that are related to the deceleration parameter $q = -\ddot{R}R/\dot{R}^2$.[9] In the empty universe case (3.11) has no isotropic part and the equipotential surfaces are hyperboloids. As the singularity is approached $R_\alpha(t)$ in (3.10) decreases or oscillates against the background of decrease in accordance with[25,24,4].

An analytic solution is known for the type IX empty universe model with $R_2 = R_3$ in (3.9). Then in (3.10) and (3.11) we have[1,27]

$$R_1^2 = \frac{2k^2}{\operatorname{ch}[2k^2(\eta - \eta_1)]}, \qquad R_2^2 = R_3^2 = \frac{k^2 \operatorname{ch}[2k^2(\eta - \eta_1)]}{2\operatorname{ch}^2[k^2(\eta - \eta_2)]}, \qquad dt = R^3 d\eta, \qquad k = \text{const}. \tag{3.12}$$

Newtonian analogs of homogeneous GRT models also exist for matter with the equation of state $e = e(p)$ and the tensor (1.5) in the right-hand members of (2.3a), (2.3b) and (2.3d). In the Newtonian theory, instead of the equation of continuity and Poisson's equation we then use the corresponding equations from[3]. Consequently, (3.2) is replaced as specified in[3] and the last term in (3.4) is replaced by $4\pi k(e + 3p)/c^2$,[3] with a corresponding change for φ_1. The general scheme of our discussion and the equations (2.16), (2.17), (2.9)--(2.13) are conserved.

In the approximation (2.8) we can also consider the propagation of light signals[15] by means of the GRT equations for an isotropic 4-vector in the approximation corresponding to (2.5).

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46