

*SOME FEATURES OF SELF-ACTION OF LIGHT IN ABSORBING MEDIA AND CONDITIONS
FOR OBSERVING SELF-FOCUSING DUE TO RESONANT ABSORPTION*

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The threshold characteristics and some features of self-action of light are investigated in the presence of linear absorption and of a nonlinear additive term, quadratic in the field strength, in the dielectric constant. The dependence of the threshold power on the parameters of the medium, the radiation frequency, and the light-beam radius is obtained by means of an approximate analysis of the self-focusing dynamics. The results are applied to the case when the nonlinearity of the dielectric constant is due to resonant absorption of light. This is possible so long as the field is small compared with the field saturating the resonant transition. In strong fields the resonant nonlinearity also becomes saturated and the self-action of the light beams is accompanied by large aberrations. On this basis and on the basis of the threshold condition, an investigation is made of the region of existence of self-focusing due to resonant absorption in the space of the parameters of the medium and of the field.

SELF-action of light, first predicted by Askar'yan^[1], is usually observed in media where the nonlinearity of the dielectric constant ϵ is due either to mechanisms that are not connected directly with absorption, or, as in the case of thermal self-focusing, to a mechanism connected with relatively weak absorption. Analysis for damping of the wave in such cases is therefore not of fundamental importance.

In an earlier paper^[2] we considered the mechanism of variation of the real part of ϵ under the influence of the field of a wave in the case of resonant absorption of the wave. It was shown that the resultant nonlinear increment ϵ_{nl} to the dielectric constant can exceed the value of ϵ_{nl} connected with the known nonlinearity mechanisms. It is natural to assume that effects of self-action of the light can be observed in this case. For example, the possibility of self-focusing following a change in the quantum states of the atoms in molecules of the medium was indicated in^[3]; nonlinear wave propagation in a two-level system on the decreasing part of the resonant curve was discussed in^[4], where, however, no account was taken of all the factors responsible for the formation of ϵ_{nl} in resonant single-photon absorption (for a discussion of these factors, particularly the question of the admissibility of the two-level approximation, see^[2]).

An analysis of the wave interaction due to the resonant absorption raises first of all the problem of whether this self-action can exist in principle, since absorption, on the one hand, gives rise to the nonlinearity ϵ_{nl} , and consequently to the self-action effects themselves, but on the other hand, by decreasing the total power of the beam along the propagation direction, it leads to the suppression of these effects. We note that in the case of thermal self-focusing the absorption necessary to ensure sufficient nonlinearity ϵ_{nl} is usually too small to give rise to a noticeable damping over the self-focusing length^[5].

To resolve this problem it is necessary to investigate

the features of self-action and its threshold characteristics in the presence of absorption. Naturally, the results obtained thereby are applicable also to self-action in media where the absorption and the nonlinearity of the dielectric constants are not connected with each other. An experimental investigation of the threshold power of self-focusing in absorbing media at a finite cell length was carried out in^[6,7] (for a discussion of these results see^[8]). We do not know, however, of studies devoted to the threshold power and to the self-focusing length in unbounded absorbing media (in particular, the threshold of "weak" self-action, see Sec. 2 below).

In accordance with the indicated problems, the present paper consists of three sections. In the first and second sections we consider the threshold characteristics of self-focusing in an absorbing medium regardless of the source of the absorption and of ϵ_{nl} . The results are used in the third section to investigate the regions of existence of self-focusing in the space of parameters of the medium and of the field in the case when the value of ϵ_{nl} is governed by the resonant absorption of the field.

1. THRESHOLD CHARACTERISTICS OF SELF-FOCUSING IN AN ABSORBING MEDIUM

1. As is well known (see, for example,^[9]), the following equations can be written for the eikonal of the complex amplitude s and the field intensity I in the quasi-optical approximation that makes it possible to take diffraction into account in the presence of linear absorption characterized by an exponential-attenuation length L_a of the wave in the medium (in terms of power), and in the case when $\epsilon_{nl} = \epsilon_2 I$:

$$2 \frac{\partial s}{\partial x} + \left(\frac{\partial s}{\partial \rho} \right)^2 = (k^2 \sqrt{I})^{-1} \left(\frac{\partial^2 \sqrt{I}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \sqrt{I}}{\partial \rho} \right) + \frac{\epsilon_2}{\epsilon_0} I, \quad (1)$$

$$\frac{\partial I}{\partial x} + \frac{\partial s}{\partial \rho} \frac{\partial I}{\partial \rho} + I \left(\frac{\partial^2 s}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial s}{\partial \rho} \right) = -\frac{I}{L_a},$$

where $k = 2\pi\sqrt{\epsilon_0}/\lambda$ (λ is the wavelength in vacuum) and x and ρ are the longitudinal and transverse cylindrical coordinates (we consider three-dimensional cylindrically-symmetrical light beams).

We assume that the shape of the beam on entering the medium is Gaussian, and assume further, as is usually done in similar calculations^[6], that during its evolution in space the beam retains approximately the same shape, i.e., $I(\rho, x) = I_c(x) \exp[-\rho^2/a^2(x)]$, where $a(x)$ is the beam radius and $I_c(x)$ is the field intensity at the center of the beam. We assume also that the eikonal is quadratic over the cross section: $s(\rho, x) = s_0(x) + \rho^2\alpha(x)$. We then obtain from (1) an equation describing the behavior of the relative width of the beam $f(x) = a/a_0$ in space ($a_0 = a|_{x=0}$ is the entry radius of the beam):

$$f \frac{d^2 f}{dx^2} = \frac{1}{L_d^2} - \frac{1}{R_n^2} \exp\left(-\frac{x}{L_a}\right), \quad (2)$$

where $L_d = ka_0^2$ is the diffraction length of the beam in the medium and R_n is a nonlinear scale connected with the power:

$$R_n^2 = \frac{a_0^2 \epsilon_0}{\epsilon_2 L_{en}} = \frac{\epsilon_0}{\epsilon_2} \frac{L_d}{k I_{en}} \quad (3)$$

($I_{en} = I_c|_{x=0}$ is the intensity of the field at the center of the beam on entry); R_n is equal to the length of the self-focusing of an initially-parallel beam in a transparent nonlinear medium in the geometrical-optics approximation.

We note immediately that when $\epsilon_2 > 0$ (i.e., when $R_n^2 > 0$), self-focusing is possible both in an absorbing and in an amplifying medium (i.e., at any sign of L_a). In the case when $\epsilon_2 < 0$ ($R_n^2 < 0$), self-defocusing should always be observed for a Gaussian beam.

2. Let us consider self-focusing of a beam in an absorbing medium ($\epsilon_2 > 0, L_a > 0$).

All the solutions of Eq. (2) can be divided into two types: solutions that vanish at some value $x = x_f > 0$ (Fig. 1, curves 1 and 2), and solutions that vanish nowhere when $x > 0$ (Fig. 1, curves 3 and 4). The first type of solutions corresponds precisely to self-focusing of a beam. We are interested here in the relations that determine those parameters of the beam and of the medium (in particular, the intensity of the beam upon entry) at which the solutions of the second type go over into solutions of the first type, i.e., the relations that define the self-focusing threshold.

To this end, let us prove first that all x_f satisfy the relation

$$x_f \leq x_0, \quad (4)$$

where x_0 is the value of x at which the right-hand side of (2) vanishes, i.e.,

$$x_0 = L_a \ln(L_d/R_n^2). \quad (5)$$

The inequality (4) follows from the fact that when $x > x_0$ the solutions of Eq. (2) can never equal zero and, con-

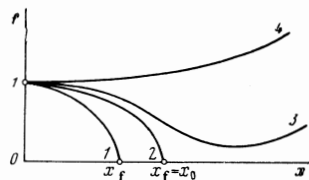


FIG. 1. Dependence of the radius of a self-focusing beam on the distance.

versely, when $x \leq x_0$ there always exist solutions that vanish. Indeed, by finding the solutions of Eq. (2) in a small vicinity of any point $x_1 > x_0$ and by solving to this end first the equation $f^3 f'' = B$ (where $B = \text{const} = L_d^{-2} - R_n^{-2} \exp(-x_1/L_a) > 0$), we find that near x_1 any solution of (2) can be represented in the form $f = \bar{f} + o(\bar{f})$, where $\bar{f}^2 = [B + c_1^2(x + c_2)^2]/c_1$, i.e., $f(x_1) \neq 0$. On the other hand, we can show analogously that any point $x_2 < x_0$ corresponds to solutions that vanish at this point, and all these solutions can be represented in a small vicinity of the point x_2 in the form $f \propto (x_2 - x)^{1/2} + o(\sqrt{x_2 - x})$; finally, there exist solutions that vanish at $x_3 = x_0$, and behave in a small vicinity of the point x_0 like $f \propto (x_0 - x)^{3/4} + o[(x_0 - x)^{3/4}]$. Thus, we have proved the inequality (4).

It is clear therefore that the threshold situation corresponds to the equation

$$(x_f)_{thr} = x_0. \quad (6)$$

Starting from this equation, we can derive the threshold relations of interest to us, for example, in the following manner. Were we to succeed in finding for Eq. (2) a general solution that vanishes at the point $x = x_0$ (i.e., a solution that contains already only one arbitrary constant), then by imposing on this solution two boundary conditions at the point $x = 0$ (on the values $f(0)$ and $f'(0)$), we would obtain relations that determine, besides the aforementioned constant, the value of x_0 , and from this, with allowance for the definition (5), also a relation between the threshold intensity I_{thr} , the attenuation L_a , and the wave parameter of the beam L_d .

Unfortunately, it is impossible to obtain an exact solution of Eq. (2). However, under conditions of relatively weak attenuation ($L_a \gg L_d$) or, conversely, of strong attenuation ($L_a \ll L_d$), we can determine with sufficient accuracy the form of the solution of Eq. (2), $f_{thr}(x)$, corresponding to the threshold situation, and consequently also the value of $(x_0)_{thr}$.

3. Let us investigate first the case of relatively weak attenuation ($L_a \gg L_d$).

To this end we determine first the behavior of the "threshold" solution $f_{thr}(x)$ of Eq. (2) near the focusing point $(x_f)_{thr} = x_0$. Linearizing the right-hand side of (2) near $x = x_0$, we find that in this region the behavior of $f(x)$ can be described by the equation

$$f^2 f'' = -(x_0 - x) / L_a L_d^2. \quad (7)$$

A particular solution of (7), vanishing at $x = x_0$, is

$$f_1 = \frac{2}{\sqrt[3]{3} L_a L_d^2} (x_0 - x)^{3/4}.$$

Making the change of variables $f = f_1 g$ and

$$\xi = -\ln[(x_0 - x) / (L_a L_d^2)^{1/3}],$$

we obtain an equation for $g(\xi)$:

$$16g'' - 8g' - 3(g - 1/g^3) = 0, \quad (8)$$

Whence, putting $p = g'$, we obtain the "phase" equation

$$\frac{dp}{dg} = \frac{1}{16p} \left[3 \left(g - \frac{1}{g^3} \right) + 8p \right], \quad (9)$$

which has two symmetrical singular points: $p = 0$ and $g = \pm 1$. An investigation shows that these are unstable singular points of the saddle type. It is easy to see that

positive solutions of (7), which exist at $x < x_0$ and vanish at $x \rightarrow x_0$ (i.e., precisely at threshold solutions), correspond to a separatrix that enters the singular point $p = 0$, $g = 1$ as $\xi \rightarrow \infty$. From (9) it follows that its slope at the singular point is $dp/dg = 1/4(1 - \sqrt{13})$; representing this separatrix in the region of the saddle in the approximate form

$$p = dg/d\xi \approx 1/8(\sqrt{13}-1)(1-g) \quad (10)$$

and integrating (10), we obtain all the threshold solutions of (7) of interest to us:

$$f_{\text{thr}} \approx 2(3L_a L_d^2)^{-1/2} (x_0 - x)^{3/4} + C(x_0 - x)^{1/4(\sqrt{13}+2)}. \quad (11)$$

By virtue of the assumed condition ($L_a \gg L_d$), the inequality $L_a \gg x_0$ is also satisfied (see relation (12) below). It follows therefore that everywhere on the segment $(0, x_0)$ of interest to us, Eq. (7) differs little from the initial equation (2), and consequently, the function (11) is likewise close to the exact threshold solution of (2) on the segment $(0, x_0)$. Therefore, imposing on the solution (11) the boundary conditions $f|_{x=0} = 1$, $f'|_{x=0} = 0$ (the latter condition means that we are considering here only the case of a plane wave at the entry) and eliminating the quantity C from the resultant equations, we obtain for x_0

$$x_0/L_a = (2A)^{2/3}, \quad (12)$$

where

$$A = L_d/L_a \quad (13)$$

We recall that x_0 is the largest attainable self-focusing length in a medium with absorption for specified L_d and L_a . Thus, the only difference from self-focusing in a transparent medium is that, as already noted, the threshold power corresponds to focusing not at infinity but at a finite distance x_0 .

From (12) with allowance for the definition (5) we get $(R_n)_{\text{thr}}$ (for $A \ll 1$):

$$(R_n)_{\text{thr}}^2/L_a^2 = A^2 \exp[-(2A)^{2/3}] \approx A^2[1 - (2A)^{2/3}]. \quad (14)$$

It is easily seen that when $A \rightarrow 0$ the threshold value of R_n tends to the corresponding value for a transparent medium ($(R_n)_{\text{thr}} = L_d$).

Taking the definition (3) into account, we obtain from (14) the value of the threshold intensity at the center of the beam at entry:

$$I_{\text{thr}} = \frac{\epsilon_0 L_d}{\epsilon_2 k (R_d^2)_{\text{thr}}} = \frac{\epsilon_0}{\epsilon_2 k L_a} A^{-1} \exp[(2A)^{2/3}] \approx \frac{\epsilon_0}{\epsilon_2 k L_a} A^{-1} [1 + (2A)^{2/3}]. \quad (15)$$

4. We consider now the case of "strong" attenuation: $A = L_d/L_a \gg 1$ (for example, at $a_0 = 1$ mm, $L_a = 10$ cm, and $k = 10^{-5}$ cm $^{-1}$ we have $A = 10^2$). It can be shown (see (21) below) that the limiting focal distance x_0 satisfies here the relation $L_a \ll x_0 \ll L_d$, by virtue of which the function $f_{\text{thr}}(x)$ behaves in the following manner. First, over a section on the order of attenuation length, the value of the function f itself remains practically unchanged ($f \sim 1$); all that changes significantly is $f''(x)$, which decreases here to zero approximately like $f'' \approx -R_n^{-2} \exp(-x/L_a)$; then the function $f(x)$ itself decreases almost linearly with constant slope

$$f' \approx \int_0^{\infty} f''(x) dx = -\frac{L_a}{R_n^2}$$

and then begins to behave in accordance with (11) only at small values $f \lesssim A^{-1/2}$, decreasing to zero over a length on the order of L_a . Thus, the course of the rays in this case is determined by a relatively thin entry layer ($x \sim L_a$), which acts like a phase corrector or a thin lens. The focal distance is approximately equal here to the length of the linear decrease of $f(x)$, i.e., to the focal length of a thin lens

$$x_f = -(f')^{-1} = R_n^2/L_a. \quad (16)$$

Since $(x_f)_{\text{thr}} = x_0$, we obtain with allowance for (6) the following equation for $(R_n)_{\text{thr}}$:

$$(R_n)_{\text{thr}}^2/L_a^2 = \ln(L_d/(R_n)_{\text{thr}})^2, \quad (17)$$

whose solution can be represented, for example, in the form of the continued fraction

$$(R_n)_{\text{thr}}^2/L_a^2 = \ln(A^2/\ln(A^2/\ln \dots)), \quad (18)$$

from which, since $A^2 \gg 1$, we have in first approximation

$$(R_n)_{\text{thr}}^2/L_a^2 \approx \ln A^2. \quad (19)$$

From (3) and (19) there follows the value of the threshold intensity of the beam in the case under consideration:

$$I_{\text{thr}} \approx \frac{\epsilon_0}{\epsilon_2 k L_a} \frac{A}{\ln A^2}, \quad (20)$$

and from (5) we obtain the maximum focal length at a given A :

$$\frac{x_0}{L_a} = \ln \frac{A^2}{\ln A^2}. \quad (21)$$

Thus, as $A \rightarrow \infty$, the ratio of the focal length $(x_f)_{\text{thr}}$ to the attenuation length L_a also tends to infinity, and not to a finite limit of the order of unity, as is stated in^[8].

5. A more detailed calculation, which will not be presented here, yields I_{thr} also at certain points where $A \sim 1$; for example, for $A = 3$ we obtain $I_{\text{thr}} \approx 2\epsilon_0/\epsilon_2 k L_a$. Combining this result with relations (15) and (20) for I_{thr} on the edges of the region, an estimating formula can be obtained for the interpolation of the function $I_{\text{thr}}(A)$ over the entire region of its definition (Fig. 2, curve 1):

$$I_{\text{thr}} \frac{\epsilon_2 k L_a}{\epsilon_0} \approx \frac{1}{A} \left\{ 1 + \frac{2A^2}{3 \ln[1 + 2/3(A^2/2)^{2/3}]} \right\}, \quad (22)$$

From this, with allowance for the definitions (5) and (3), follows also the value of the limiting focal length

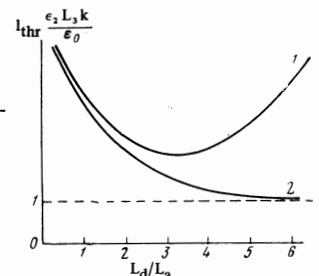


FIG. 2. Plot of the threshold intensity of the beam against the ratio of the diffraction length L_d to the attenuation length L_a .

$$\frac{x_0}{L_a} = \ln \left\{ 1 + \frac{2A^2}{3 \ln [1 + \sqrt[3]{A^2/2}]^{2/3}} \right\}. \quad (23)$$

It is easily seen that when $A \ll 1$ and $A \gg 1$, Eq. (23) goes over into formulas (12) and (21), respectively.

From (15), (20), and (22) it is seen that the threshold intensity of self-focusing in an absorbing medium becomes minimal at a certain optimal value of A (corresponding to the optimal beam width at entry, $a_0 = \sqrt{L_d/k}$, for a given attenuation length L_a). It follows from (22) that this occurs at

$$A_{opt} \approx 3; \quad L_d \approx 3L_a, \quad a_{opt} \approx \sqrt[3]{3L_a/k}; \quad (24)$$

corresponding to

$$(I_{thr})_{min} \approx 2\epsilon_0/\epsilon_2 k L_a, \quad x_f = x_0 \approx 2L_a. \quad (25)$$

6. It is of interest to determine the total beam power

$$P = \frac{c \sqrt{\epsilon_0}}{4\pi} \int I dS.$$

For a Gaussian beam we have

$$P = \frac{c \sqrt{\epsilon_0}}{4} a_0^2 = \frac{c \sqrt{\epsilon_0}}{4} \frac{I_{en} L_d}{k}, \quad (26)$$

whence, using (22), we obtain the total threshold power of the beam (Fig. 3, curve 1)

$$\frac{4}{c \sqrt{\epsilon_0}} P_{thr} = I_{thr} \frac{L_d}{k} = \frac{\epsilon_0}{\epsilon_2 k^2} \left\{ 1 + \frac{2A^2}{3 \ln [1 + \sqrt[3]{A^2/2}]^{2/3}} \right\}. \quad (27)$$

At small damping ($A \rightarrow 0$), Eq. (27) gives a well-known result—the threshold self-focusing power in a transparent medium $P_{cr} \propto (\epsilon_2 k^2)^{-1}$, which does not depend on the beam radius (curve 3, Fig. 3).

We note one more characteristic feature of self-focusing in the presence of absorption. By calculating the wave power P_f reaching the focusing point x_f in the case when the power at the entry has the threshold value P_{thr} (i.e., when $x_f = x_0$), we obtain

$$P_f = P_{thr} \exp [-x_f/L_a],$$

or by virtue of (5)

$$P_f = P_{thr} (R_n/L_d)^2.$$

Taking the definitions (3) and (26) into account, we get from this

$$P_f = cn_0^3/4\epsilon_2 k^2, \quad n_0 = \sqrt{\epsilon_0}. \quad (28)$$

Thus, under threshold conditions, the power reaching the focusing point always has the same value P_f , which does not depend on the entry power, the beam radius, or the attenuation, and this power is exactly equal to the critical self-focusing power in a transparent nonlinear medium.

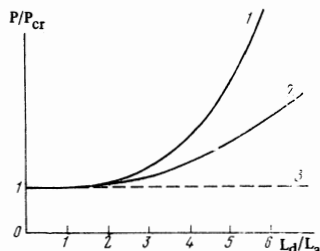


FIG. 3. Plot of the total threshold power of the beam against the ratio of the diffraction length L_d to the attenuation length L_a .

2. "WEAK" SELF-FOCUSING AND SELF-DEFOCUSING OF A GAUSSIAN BEAM IN AN ABSORBING MEDIUM

1. We consider now the case when the entry intensity of the beam is smaller than the threshold value but larger than the critical intensity required for self-focusing in a transparent nonlinear medium:

$$I_{thr} > I > I_{cr} = \frac{\epsilon_0}{\epsilon_2 k L_d} = \frac{\epsilon_0}{\epsilon_2 k L_a} \frac{1}{A}. \quad (29)$$

Then the solution of (2) no longer vanishes anywhere, but at first there occurs a certain narrowing of the beam, after which the beam begins to diverge (curve 3, Fig. 1)¹⁾.

We shall define such a case as "weak" self-focusing (there is no strong contraction of the beam); it is essentially analogous to external self-focusing^[9], since it can actually be regarded as focusing by a finite nonlinear layer (of thickness $\sim L_a$). If $L_d \gg L_a$, the situation can be regarded as focusing by a thin lens (see Sec. 1, Subsection 4) with a focal length x_f determined by formula (16). At sufficiently large distances ($x \gg L_a$), the behavior of the beam can be described by Eq. (2), in the right-hand side of which the exponential term is set equal to zero. The solution in this case (for a plane wave at the entry to the lens) is given by the formula

$$f^2 \approx (1 - x/x_f)^2 + x^2/L_d^2. \quad (30)$$

Assuming in analogy with^[9] that the threshold intensity of weak self-focusing is the intensity at which the narrowing point (i.e., the point at which f is minimal and $f' = 0$) is located at the largest distance from the origin, we obtain the threshold condition $x_f = L_d$ (at which $f_{min}^2 = 1/2$ and $x(f_{min}) = L_d/2$), from which it follows, when account is taken of (16), that

$$(R_n^2)_{thr} = L_d L_a. \quad (31)$$

or, using (3),

$$I_{thr} = I_c = \epsilon_0/\epsilon_2 k L_a. \quad (32)$$

Comparing with (20), we see that when $L_d \gg L_a$ the threshold of weak self-focusing (curve 2 in Fig. 2) is much lower than the threshold of self-focusing with a sharp focus; it can be shown that when $L_d < L_a$, these two thresholds almost coincide with each other and with the threshold of focusing in a transparent medium (29). The total threshold beam power P is determined by formula (26) with allowance for (32) (curve 2 on Fig. 3).

2. Let us consider the case when the nonlinearity is negative ($\epsilon_2 < 0$, $R_n^2 < 0$). Self-defocusing of the beam should then occur. Let us estimate its angular divergence. As seen from (2), the beam first broadens monotonically (if $f'(0) = 0$), and then, at $x \gg L_a$, it behaves as in a linear medium, propagating in the far zone ($x \gg L_d$) with a constant divergence angle.

We confine ourselves here for simplicity only to the case $L_d \gg L_a$ and of not very high intensity ($|R_n|^2 > L_a^2$), and using arguments analogous to those in subsection 3 of Sec. 1, we find that the nonlinear medium acts in this case like a thin defocusing lens with a negative focal length x_f determined by formula (16), where

¹⁾We note that the solution corresponding to I_{cr} (29) diverges cubically from the very outset (curve 4, Fig. 1), as follows directly from (2) at $R_n^2 = L_d^2$ and $f'|_x=0 = 0$.

$R_n^2 < 0$. The dynamics of the beam at $x \gg L_a$ is therefore described by relation (30), where $x_f < 0$, from which we find that when $x^2 \gg x_f^2$ the total beam divergence angle φ_∞ , defined by the formula $\tan(\varphi_\infty/2) = af'(\infty)$, equals, when account is taken of (16) and (3),

$$\varphi_\infty = \varphi_d \sqrt{1 + (L_d L_a / R_n^2)^2} = \varphi_d \sqrt{1 + (I/I_a)^2}, \quad (33)$$

where $\varphi_d = 2/ka_0 = 2a_0/L_d$ is the diffraction divergence angle of a beam with entry radius a_0 in a linear medium. It is interesting that in the case of weak self-focusing ($\epsilon_2 > 0$) the beam, after passing the narrowing point (where $f' = 0$) and expanding beyond it, acquires at $x^2 \gg x_f^2$ a divergence determined by the same formula (33). Thus, in both cases the effect can be detected or the constants of the medium can be measured by determining the beam divergence angle in the far zone, taking into account here, however, the fact that the behavior of the beam near the entry point ($0 < x < |x_f|$) is entirely different in these cases. (We are not referring here to the case of self-focusing with a sharp focus, since the solution of Eq. (2) with $R_n^2 > 0$, which vanishes at a certain $x = x_f$, cannot be continued further if we remain in the region of real values of the function f . On the other hand, the question of what occurs in fact in this case when $x > x_f$, and, in particular, the question of formation of the nonlinear channel, cannot of course, be solved, within the framework of an analysis that makes use of Eq. (2)).

It is seen from (33) that the nonlinear divergence of the beam noticeably exceeds the diffraction divergence under a condition analogous to the condition for observing weak self-focusing: $I > I_c$ (32); in this case it is necessary to take in (32) the absolute value of ϵ_2 .

Thus, as can easily be seen from (26) and (32), the necessary condition for observing self-action of any type in an absorbing medium is that the wave intensity exceed a certain threshold value (32) characteristic of the given medium. We shall show that this condition has a simple physical meaning. Indeed, writing down the complete expression for the complex dielectric constant $\epsilon = \epsilon_0 + i\epsilon' + \epsilon_{nl}$, where $\epsilon' = \epsilon_0/kL_a$ and $\epsilon_{nl} = \epsilon_2 I$, we find that it turns out that $\epsilon_{nl} = \epsilon_0/kL_a = \epsilon'$ when $I = I_c$. It follows therefore that self-action in an absorbing medium can be observed only if the nonlinear increment of ϵ_0 exceeds in absolute magnitude the imaginary part of ϵ .

3. CONDITIONS FOR THE EXISTENCE OF SELF-FOCUSING DUE TO RESONANT ABSORPTION

1. We now discuss the conditions for the existence of self-focusing caused by resonant absorption. In this case the nonlinear increment to the dielectric constant is^[2]

$$\epsilon_{nl} = 4\pi N\eta_0 \frac{|p|^2 T}{\hbar} m \frac{q[\theta(1+\Delta^2) - (1-\Delta^2)] + 2\theta\Delta + \Delta q^2 m}{(1+\Delta^2)[1+2\theta m + (\Delta + qm)^2]}. \quad (34)$$

We have introduced here the dimensionless intensity $m = |p|^2 T^2 I / \hbar^2$, where p is the matrix element of the dipole moment of the resonant transition and T is the reciprocal half-width of the line of this transition. In addition, (34) includes the following parameters: $\Delta = (\omega - \omega_0)T$ —the dimensionless difference between the field frequency ω and the transition frequency ω_0 ,

N —the concentration of the resonant particles of the medium, η_0 —the equilibrium population difference of the resonant levels (in the absence of a field) per particle, and $\theta = 2\tau/T$ —the product of the lifetime τ of the particle in the excited state by the transition line width. Finally, $q = (\kappa_2 - \kappa_1)/|p|^2 \hbar^{-1} T$ is a parameter characterizing the "repolarizability" of the given resonant transition, i.e., the change of the nonlinear polarizability of the molecule as it goes from the lower level, which has a linear nonresonant polarizability κ_1 , to the upper level with polarizability κ_2 (expressions for κ_2 and κ_1 in terms of the dipole moments and the frequencies of the remaining transitions of the particle can be found in^[2]).

The quantity ϵ_{nl} can be regarded as quadratic in the field provided the field intensity is small compared with the saturating intensity (for a given frequency deviation Δ):

$$m < m_{sat}(\Delta) = (1 + \Delta^2) / 2\theta. \quad (35)$$

Here $\epsilon_{nl} \approx \bar{\epsilon}_2 m$, where

$$\bar{\epsilon}_2 = 4\pi N\eta_0 \frac{|p|^2 T}{\hbar} \frac{\theta}{1 + \Delta^2} \left[q + \frac{2\Delta - q(1 - \Delta^2)/\theta}{1 + \Delta^2} \right] \quad (36)$$

($\bar{\epsilon}_2$ in this case, naturally, likewise has no dimensionality, and the simultaneous substitutions $\epsilon_2 \rightarrow \bar{\epsilon}_2$ and $I \rightarrow m$ do not change the results of the preceding sections, since they always contain only the product $\epsilon_2 I = \bar{\epsilon}_2 m$).

Satisfaction of the condition (35) also makes it possible to assume the resonant absorption to be linear and to characterize it by an attenuation length

$$L_a = (1 + \Delta^2) / 4\pi N\eta_0 \frac{k}{\epsilon_0} \frac{|p|^2 T}{\hbar}. \quad (37)$$

As seen from (34), when condition (35) for the intensity at the center of the entering beam is violated, we deal with a complicated distribution of ϵ in the cross section, namely, ϵ is practically constant in the center and experiences changes only near the edges of the beam, thus giving rise to strong aberration. Under such conditions, the pattern of the behavior of the rays in the medium becomes quite different from what is usually meant when speaking of self-focusing. It can therefore be assumed that the condition (35), written for the intensity at the center of the entering beam m_{en} , determines the upper limit of existence of the self-focusing due to resonant absorption. The lower limit is the threshold of "weak" self-focusing. Allowance for saturation in the calculation of this threshold cannot greatly change (32), since the condition (35) is satisfied both in the focusing layer and with further propagation of the beam. Thus, the entry power at the center of the beam should satisfy the following conditions, which are necessary to effect self-focusing:

$$m_{sat}(\Delta) > m_{en}(\Delta) > M = \epsilon_0 / \bar{\epsilon}_2 k L_a. \quad (38)$$

This leads to a condition for the parameters of the medium and the field frequency:

$$m_{sat}(\Delta) > M = \left\{ \theta \left[q \left(1 - \frac{1 - \Delta^2}{\theta(1 + \Delta^2)} \right) + 2 \frac{\Delta}{1 + \Delta^2} \right] \right\}^{-1}. \quad (39)$$

(We have substituted here $\bar{\epsilon}_2$ and L_a from (36) and (37).) The latter condition can also be rewritten in the form

$$G = q[1 - \vartheta^{-1} + \Delta^2(1 + \vartheta^{-1})] + 2\Delta > 2. \quad (40)$$

This inequality determines the region of frequency deviations Δ and medium parameters q and ϑ at which self-focusing can be observed. We point out first that the region of the parameters q and ϑ where the non-linearity is generally positive ($\bar{\epsilon}_2 > 0$) at least in some range of frequency deviations is governed by satisfaction of the condition $G > 0$ (for a medium with absorption $\eta_0 > 0$) for at least one value of the frequency deviation, hence

$$q > -\vartheta / \sqrt{\vartheta^2 - 1}. \quad (41)$$

On the diagram of Fig. 4, this corresponds to the regions II–IV ($q_{\min} \rightarrow -1$ as $\vartheta \rightarrow \infty$); near the lower edge of the region II, $\bar{\epsilon}_2 > 0$ in the frequency-deviation range $\Delta = \sqrt{(\vartheta - 1)/(\vartheta + 1)}$. In region I we have $\bar{\epsilon}_2 < 0$ at any frequency deviation; it is easy to verify that $\bar{\epsilon}_2 > 0$ at any value of Δ in a region that is a mirror image of region I about the ϑ axis. Analogously, we can determine from (40) that region in the (q, ϑ) plane where the self-focusing is possible in at least a small range of frequency deviations

$$q > \vartheta \frac{\vartheta + 1 - \sqrt{2\vartheta(\vartheta + 1)}}{\vartheta^2 - 1} \quad (42)$$

(regions III–IV on Fig. 4); as $\vartheta \rightarrow 1$ we have $q_{\min} \rightarrow -1/4$ and as $\vartheta \rightarrow \infty$ we have $q_{\min} \rightarrow 1 - \sqrt{2}$; near the lower edge of region III we have $G > 2$ at

$$\Delta = \frac{\vartheta - 1}{[2\vartheta(\vartheta + 1)]^{1/2} - (\vartheta + 1)}.$$

Finally, if we stipulate satisfaction of (40) for all values of Δ , we obtain a region where self-focusing is possible at any point of the range of possible frequency-deviation values²⁾:

$$q > \vartheta \frac{\vartheta + 1 + \sqrt{2\vartheta(\vartheta + 1)}}{\vartheta^2 - 1} \quad (43)$$

(Region IV on Fig. 4; $q_{\min} \rightarrow 1 + \sqrt{2}$ as $\vartheta \rightarrow \infty$).

Thus, in the case of liquids and solids (where $\vartheta \gg 1$), it suffices to have q exceed $1 + \sqrt{2}$ in order to be able to assume the condition for the observation of self-focusing to be satisfied for any frequency close to resonance. In the case $\vartheta \gg 1$ (when $G \approx 2\Delta + q(1 + \Delta^2)$), the regions of existence of self-focusing in the plane of q and Δ have the simplest form. The region where $\bar{\epsilon}_2 > 0$ ($G > 0$) is determined by the relation $q > -2\Delta/(1 + \Delta^2)$ (regions II and III in the diagram of Fig. 5; $\bar{\epsilon}_2 < 0$ in the region I), and the region of possible observation of self-focusing ($G > 2$) is determined by the relation $q > 2(1 - \Delta)/(1 + \Delta^2)$ (region III of Fig. 5).

The question of the width and position of the frequency band, at which self-focusing can be observed, is important because it is not always possible to tune the lasing frequency accurately to the center of the absorption line of the substance employed. Furthermore, the multimode character of the radiation or jumps of the generation from one mode to another do not always make it possible to regard the laser as an emitter of strictly fixed fre-

²⁾We recall that we assume a Lorentz line shape, and therefore the results presented here are valid for values of Δ at which the Lorentz curve describes the real line sufficiently well.

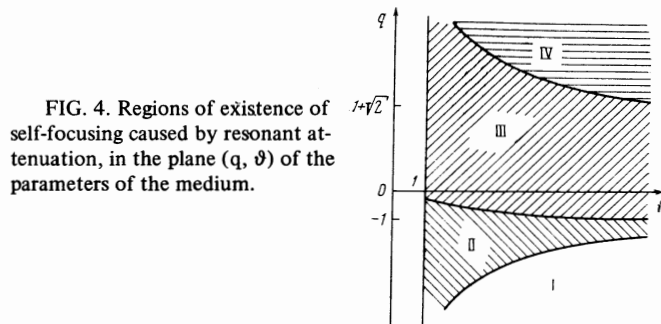
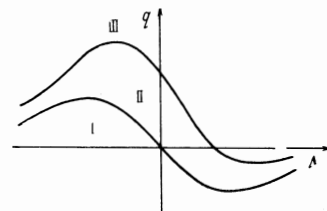


FIG. 4. Regions of existence of self-focusing caused by resonant attenuation, in the plane (q, ϑ) of the parameters of the medium.

FIG. 5. Regions of existence of self-focusing in the plane of the parameters of the medium q and of the field Δ at $\vartheta \gg 1$ (Δ —deviation of field frequency from resonance).



quency, a particularly important factor in our case, when the nonlinearity has a strong dispersion within the limits of the line width. This is precisely why particular interest attaches to substances that fall in region IV of the diagram of Fig. 4.

3. We note in this connection one interesting feature of the band characteristic of the self-focusing threshold in a resonantly absorbing medium, connected with the contribution made by the difference $\kappa_2 - \kappa_1 \sim q$ between the linear polarizabilities of the molecule at the upper and lower levels. As seen from (39), when $q > 0$ the threshold intensity M tends, with increasing deviation of the emission frequency from resonance ($\Delta^2 \gg 1$) to the constant value

$$M_0 = q^{-1}(\vartheta - 1)^{-1} \quad (44)$$

or if $\vartheta \gg 1$

$$M_0 \approx M_\infty = (q\vartheta)^{-1} = \frac{|p|^2 T^2}{2\hbar\tau(\kappa_2 - \kappa_1)} = \left(\frac{|p|T}{\hbar}\right)^2 |\mathcal{E}_\kappa|^2, \quad (45)$$

i.e., the absolute value of the threshold intensity $I_\kappa = |\mathcal{E}_\kappa|^2$ coincides exactly in this case with the intensity of the resonant repolarization field $|\mathcal{E}_\kappa|^2 = \hbar/2\tau(\kappa_2 - \kappa_1)$, which was introduced by us in^[2]. As shown in^[2], the intensity I_κ at realizable values of q and ϑ is quite small (we note that the saturating intensity $m_{\text{sat}}(\Delta)$ increases with increasing Δ like $1 + \Delta^2$, and therefore the inequality $M \ll m_{\text{sat}}(\Delta)$ holds when $\Delta^2 \gg 1$).

Thus, when $q > 0$ self-focusing can always be observed on the skirts of the absorption line at sufficiently large distances from the center (regardless of whether self-focusing occurs at the line center); the self-focusing threshold is in this case practically independent of the frequency deviation and is determined by the characteristic intensity $I_\kappa = |\mathcal{E}_\kappa|^2$ ^[2]. This is illustrated in Fig. 6, where we choose the region of existence of self-focusing in the plane of m_{en} (entry intensity) and the frequency deviation Δ under the condition $\vartheta \gg 1$. It is seen from (39) that the lower limit of the region is in this case $M = (1 + \Delta^2)/\vartheta[q(1 + \Delta^2) + 2\Delta]$. In accordance with this and with (35), the cross-hatched region corre-

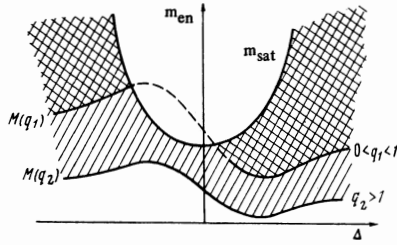


FIG. 6. Regions of existence of self-focusing in the plane of the field parameters (intensity m_{en} and frequency deviation Δ) at different values of the medium parameter q .

sponds to $0 < q_1 < 1$, and the entire shaded region to $q_2 > 1$.

At $q = 0$, the nonlinearity of ϵ is connected only with the saturation of the populations and, as seen from (40), self-focusing occurs only on the high-frequency side $\Delta > 0$; at $\Delta < 0$, the radiation should become defocused. Although the threshold intensity M becomes lower than the saturating intensity with increasing frequency deviation ($m_{sat} \propto 1 + \Delta^2$; $M \sim (1 + \Delta^2)/\Delta$), it no longer tends to a constant value, but (when $\Delta^2 \gg 1$) increases in proportion to the frequency deviation

$$M = \frac{m_{sat}(\Delta)}{\Delta} = \frac{1 + \Delta^2}{2\theta\Delta}. \quad (46)$$

We see therefore that at $q = 0$ the minimum deviation at which self-focusing is still possible is $\Delta = 1$ (see also Fig. 5).

The diagrams in Figs. 2, 4, 5, and 6 make it possible to determine the possibility of observing weak or strong self-focusing at given parameters of the medium and to choose the optimal experimental conditions for the observation of the effect. For example, let the parameters q and \mathcal{A} correspond to a point in region III of Fig. 4; then, using diagrams analogous to those in Fig. 6, we can determine the frequency-deviation region Δ (consequently also the radiation-frequency region) at which self-focusing is possible, as well as the order of magnitude of the threshold power. Knowing the latter, we can use Fig. 2 to select the beam parameters at which either weak or strong self-focusing can be observed.

4. There is no need to consider especially the regions of self-defocusing. Indeed, it is easily seen that everything said in the preceding subsections concerning focusing is applicable also to defocusing if it is recognized that all the formulas and conclusions obtained here (including the diagrams of Figs. 3, 4, 5, and 6) are invariant against a simultaneous "focusing-defocusing" substitution (i.e., $\bar{\epsilon}_2 \rightarrow -\bar{\epsilon}_2$, $q \rightarrow -q$, and $\Delta \rightarrow -\Delta$).

5. Let us stop to discuss briefly the case of an amplifying medium (the population is inverted, $\eta_0 < 0$, and consequently the attenuation is negative, $L_a < 0$). In this case there is apparently no threshold, and the beam should be focused at any entrance power³⁾. The question of self-focusing (or defocusing) is resolved here only by the sign of $\bar{\epsilon}_2$. Taking into account the fact that now $\eta_0 < 0$ in (36), the self-focusing condition is $G < 0$. From this we find, for example, that the region of the parameters q and \mathcal{A} where self-focusing occurs at any fre-

quency deviation Δ is the region I on the diagram of Fig. 4; the region of defocusing at all Δ is the mirror image of region I about the \mathcal{A} axis. In the intermediate region, some frequency deviations correspond to self-focusing and others to defocusing; in particular, at $q = 0$ self-focusing is possible at all the lower frequencies ($\Delta < 0$) and defocusing at all the higher frequencies ($\Delta > 0$).

Finally, let us formulate the main conclusions of our work.

1) Observation of self-focusing due to resonant absorption is possible, just as observation of self-action, in media where the absorption and the nonlinearity of ϵ are not connected with each other. The threshold power corresponds in this case to focusing at a finite distance, and not at infinity (as in transparent media). It is also interesting to note that there exist light beams with dimensions optimal for the observation of self-focusing in an absorbing medium; these dimensions are determined by the wavelength and the attenuation of the wave. Such beams have a minimal threshold power for the given medium.

2) Under threshold conditions, the power arriving at the focusing point is always the same, regardless of the entrance power, the beam radius, or the attenuation length. This power is exactly equal to the critical power of the self-focusing in a transparent medium having the same nonlinearity of ϵ .

3) A distinguishing feature of the self-action in an absorbing medium is the existence of so-called "weak" self-focusing, when the beam first contracts to finite dimensions and then expands. The threshold intensity of the "weak" self-action is determined by a simple relation, namely, the nonlinear increment to the real part of ϵ should be equal to the imaginary linear component ϵ' , which characterizes the absorption.

4) In the space of the parameters of the medium and of the field, there exist regions in which self-focusing due to resonant absorption can be observed, and regions in which there is no such self-focusing. In such a subdivision into regions, an important parameter of the medium is the quantity q , which characterizes the repolarizability of the resonant transition. In particular, self-focusing can almost always be realized on the "tails" of the absorption line, where its threshold intensity is equal to the magnitude of the so-called "repolarizing intensity" $|\mathcal{E}_\kappa|^2 \sim 1/q^{[2]}$.

5) Self-focusing can be realized both in absorbing and amplifying media. The latter case is of undisputed interest in the study of the process of generation in lasers and in amplifiers, causing under suitable conditions the "pinching" of the amplified or generated radiation. It should be noted here that everywhere above we have considered the dynamics of self-focusing without allowance for the effect of saturation in the region where the light intensity is already sufficiently high. In the calculation of the threshold characteristics, allowance for saturation is of little importance, but when the threshold is noticeably exceeded, in a region close to the calculated point of the focusing of the beam, the increased intensity gives rise to saturation of both the absorption (amplification) and of the nonlinearity of ϵ . Allowance for this circumstance should lead to elimination of the focusing of the beam, and possibly to the formation of a quasi-

³⁾In this case, the question of the threshold power for focusing over a limited length can be raised, but it will not be considered here.

waveguide channel with very weak attenuation of the wave.

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