CORRELATION AND RADIATIVE CORRECTIONS TO THE ENERGY OF LOW-TEMPERATURE RELATIVISTIC PLASMA

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A method is proposed for calculating the ground-state energy of relativistic plasma at T = 0. The method is based on the determination of the Fermi momentum using the electron Green function. The expression for the energy of a relativistic Fermi system is obtained in an explicitly renormalized form. The ultraviolet and infrared divergences are removed. The radiative and correlation corrections to the energy are written down explicitly in the lowest order in terms of finite integrals. The nonrelativistic limit is discussed in detail.

1. INTRODUCTION

THE determination of corrections to the energy of a relativistic Fermi system, due to interactions with the quantized electromagnetic field, is an interesting problem, both in the case of homogeneous systems (high-density plasma) and for systems in an external field (heavy atoms). In principle, the problem does not, of course, present any fundamental difficulties. However, practical calculations encounter not only purely computational difficulties but also those connected with renormalization and the removal of ultraviolet and infrared divergences which are characteristic for quantum electrodynamics. Mass renormalization which is necessary for the calculation of correlation terms of order $e^4 \ln e^2$ was discussed to some extent in[1]. However, the standard method used in[1] to calculate the energy corrections, which does not involve the well-known formula in which the integration is carried out over the bare charge, is extremely inconvenient for the analysis of radiative corrections because it requires the introduction in explicit form of the bare mass and bare charge (both infinite), which must be converted in the process of calculation into renormalized quantities.

In this paper, we propose a different method of calculating the radiative corrections to the energy at temperature T = 0. The method is based on the determination of the Fermi momentum and can be used to write down the expression for the energy of a relativistic Fermi system in an explicitly renormalized form. It avoids the use of infinite nonrenormalized quantities. This automatically ensures the removal of ultraviolet divergences, and the entire question of radiative corrections is reduced to a purely numerical problem.

For the sake of simplicity, we confine our attention to the idealized case of a homogeneous system of charged particles of a particular kind (electron plasma). We shall show that for this particular system the calculation of radiative and correlation corrections to the energy at T = 0 in the lowest order reduces to the determination of properly defined and finite integrals, the explicit form of which is reproduced. The integrals are readily evaluated in the logarithmic approximation in e^2 in the case of nonrelativistic plasma, and this leads to the well-known formula of Gell-Mann and Brueckner. In the general case, the evaluation of the integrals is an independent problem and will not be considered here.

It is important to note that problems connected with the renormalization of Green's functions for relativistic Fermi systems were considered in[2], where another method for the determination of the ground-state energy was proposed. However, the analysis of the ultraviolet and infrared divergences and estimates of the contributions of individual diagrams were not given in[2].

2. GROUND STATE ENERGY OF A RELATIVISTIC FERMI SYSTEM IN TERMS OF RENORMALIZED QUANTITIES

A relativistic Fermi system containing N_0 electrons at T = 0 can be described by the usual equations of quantum electrodynamics with a new rule for bypassing the singularities in the electron Green function G(p). This reduces to the replacement of the usual negative imaginary additions to the electron mass, i.e., $-i\epsilon$ ($\epsilon \to 0$), with

$$-i\varepsilon\zeta(p_0) \equiv \begin{cases} -i\varepsilon, & p_0 < 0,\ p_0 > \mu \\ +i\varepsilon, & 0 < p_0 < \mu \end{cases}. \tag{1}$$

In these expressions $\mu > 0$ is the chemical potential (of electrons). The quantity μ is defined in terms of N_0 through the equation

$$N_0 = N(\mu) \equiv iV(2\pi)^{-4} \int\limits_{\tau \to +0} d^4p\, e^{ip_0\tau} \operatorname{Sp}\{\gamma_0(G(p) - G_0(p))\}. \tag{2}$$

The subscript 0 carried by the Green function means that $\mu = 0$, i.e., we have the usual quantum electrodynamic Green function, and V is the volume of the system. The system contains three types of particle, namely, electrons, positrons, and holes. The holes have negative energies.

The ground-state energy E is defined in terms of the function $N(\mu)$ through the well-known thermodynamic formula

$$E = N_0\mu - \int\limits_0^\mu N(\mu')\, d\mu', \tag{3}$$

where μ is the solution of Eq. (2). It is exceedingly important that the expression given by Eq. (2) should be

capable of being transformed to the form known in the theory of nonrelativistic Fermi systems,[3] namely,

$$N(\mu) = 2V(2\pi)^{-3}\int d^3p\,\theta(f^2 - \mathbf{p}^2), \tag{4}$$

where f is the Fermi momentum which is defined in terms of μ by the equations

$$\det G^{-1}(p) = 0 \text{ when } p_0 = \mu, \quad \mathbf{p}^2 = f^2. \tag{5}$$

The proof of Eq. (4) is in all ways quite similar to that given in the nonrelativistic theory, and is outlined in the Appendix. Equations (4) and (5) are convenient for the determination of $N(\mu)$ in the relativistic quantum theory because the main equation given by Eq. (5), which relates f^2 and μ, admits of a simple transition to renormalized quantities. At the same time, Eqs. (3) and (4), which give the number of particles and the energy of the system, are also renormalized.

The mass and charge renormalization in the theory of relativistic Fermi systems can, in general, be performed by analogy with the usual quantum electrodynamics[2,4]. For the mass renormalization we shall assume that

$$m_0 = m + \Sigma_{(0)}(\hat{p} = m), \tag{6}$$

where m is the physical electron mass and the proper mass $\Sigma_{(0)}$ is calculated as already indicated for $\mu = 0$. Let us introduce the renormalized electron Green function G_r, the photon Green function $D_r^{\alpha\beta}$, and the vertex part Γ_r^{α} in accordance with the standard formulas

$$G_r = Z_2^{-1}G, \; D_r^{\alpha\beta} = Z_3^{-1}D^{\alpha\beta}, \; \Gamma_r^{\alpha} = Z_1\Gamma^{\alpha}, \tag{7}$$

where G, $D^{\alpha\beta}$, and Γ^{α} are the renormalized quantities and Z_1, Z_2, and Z_3 are the usual quantum electrodynamic constants corresponding to $\mu = 0$. The proper mass of the electron in our problem contains two irreducible graphs, namely, the usual graph and the graph shown in Fig. 1.

The contribution of all graphs of the form shown in Fig. 1 to Σ is the numerical matrix $\Delta\Sigma$, and it is readily shown that

$$\Delta\Sigma = -e_0^2N_0V^{-1}\gamma_0D^{(0)00}(0) \equiv \mu_1\gamma_0. \tag{8}$$

In this expression N_0 is the number of particles, V is the volume, and $D^{(0)00}(0)$ is the component of the free-photon Green function for zero momentum. In Eq. (8) we can readily transform to the renormalized quantity

$$\mu_1 = -e^2N_0V^{-1}D_r^{(0)00}(0). \tag{9}$$

The quantity μ_1 is the numerical addition to the electron energy. If we substitute $P_0' = P_0 + \mu_1$ it becomes clear that taking $\Delta\Sigma$ into account is equivalent to the replacement of the chemical potential μ by $\mu' = \mu + \mu_1$. Therefore,

$$N(\mu) = N'(\mu + \mu_1), \tag{10}$$

where N′ does not contain diagrams of the form shown in Fig. 1. To determine μ we now have, instead of Eq. (2), an equation containing N_0 on the right-hand side:

$$N_0 = N'(\mu - e^2N_0V^{-1}D_r^{(0)00}(0)). \tag{11}$$

After eliminating diagrams of the form shown in Fig. 1 for the renormalized quantities we obtain a closed system of equations which does not differ in any way

FIG. 1

from the usual renormalized Dyson-Schwinger equations of quantum electrodynamics. All that remains is to remember that all the residue terms must be evaluated for $\mu = 0$, i.e., they are exactly equal to the residue terms in quantum electrodynamics. We note further that because of the existence of a special reference frame, where the resultant 3-momentum of the Fermi system is zero, the relativistic invariance of the theory is violated and all the quantities depend on the additional 4-vector μ which, in this particular special reference frame, has the components $(\mu, 0)$. In particular, we have not one but two polarization operators for the ground state (Π and Π_1) which are related to the polarization tensor $\Pi^{\alpha\beta}(k)$ by

$$\Pi^{\alpha\beta}(k) = (g^{\alpha\beta} - k^{\alpha}k^{\beta}/k^2)\,\Pi(k) + \mu_{\perp}^{\alpha}\mu_{\perp}^{\beta}\Pi_1(k), \tag{12}$$

where $\mu_{\perp}^{\alpha} = \mu^{\alpha} - k^{\alpha}(\mu k)/k^2$. The renormalized Green function for a photon can therefore be chosen in the form

$$D_r^{\alpha\beta}(k) = \frac{g^{\alpha\beta}}{k^2 - \Pi_r} + \frac{\mu^{\alpha}\mu^{\beta}\Pi_{1r}}{(k^2 - \Pi_r)\,[k^2 - \Pi_r - (\mu^2 - (k\mu)^2/k^2)\,\Pi_{1r}]}, \tag{13}$$

and the renormalization of Π_r is performed in the usual way, whereas for Π_{1r} the residues are not involved. The zeros of the second denominator in Eq. (13) correspond to plasma oscillations.

The above equations for the ground-state energy of a Fermi system can readily be written in terms of renormalized quantities. It is sufficient to rewrite Eq. (5) in the form

$$\det G_r^{-1}(p) = 0, \quad p_0 = \mu, \quad \mathbf{p}^2 = f^2. \tag{14}$$

The solution of this renormalized equation enables us to relate μ and f and then, with the aid of Eq. (4), to find $N'(\mu)$. After that we find $N(\mu)$ in accordance with Eq. (10), and then use Eqs. (3) and (11) to obtain $E = E(N_0)$. The entire problem thus reduces to finding the renormalized proper mass Σ_r of the electron for $P_0 = \mu$. We note, by the way, that this also yields the spectrum of single-fermion excitations of the system near the Fermi surface.

3. GENERAL ANALYSIS OF SECOND- AND FOURTH-ORDER CONTRIBUTIONS

The proper mass of the electron evaluated to the order of e^2 and e^4 ($\Sigma^{(2)}$ and $\Sigma^{(4)}$) is formally specified by the usual set of Feynman graphs (Fig. 2) in which, however, the electron Green function must be taken with allowance for Eq. (1). In the zero-order $G^{(0)} = G_{(0)}^{(0)} + G_{(1)}^{(0)}$, where $G_{(0)}^{(0)}$ is the free Green function for $\mu = 0$, and

$$G_{(1)}^{(0)} = -2\pi i(m + \hat{p})\,\delta(m^2 - p^2)\,\theta(p_0)\,\theta(\mu - p_0). \tag{15}$$

We can now isolate in each of the graphs that part which corresponds to quantum electrodynamics at $\mu = 0$, i.e., the quantity $\Sigma_{r(0)}$, and hence represent the entire proper mass in the form $\Sigma_r = \Sigma_{r(0)} + \Sigma_{r(1)}$ (henceforth we shall be dealing exclusively with renormalized quantities and, for the sake of brevity, we shall omit the subscript r). The quantity $\Sigma_{(1)}$ is directly the contribution of diagrams 2a–2d in which at least one of the electrons is replaced by $G_1^{(0)}$ and, moreover, residue terms have been added and are necessary for the renormalization of the internal vertices or the proper energy parts. Care must be taken in this procedure in the case of diagram 2b which must, in fact, be considered separately.

In the lowest-order approximation, when Σ is not taken into account at all, the solution of Eq. (14) is clearly of the form

$$f^{2(0)} = \mu^2 - m^2 \equiv \varkappa^2. \tag{16}$$

In this expression m is the renormalized electron mass. When Eq. (16) is satisfied, we have $P^2 = m^2$ and, therefore, $\Sigma_{(0)} = 0$. Hence, it follows that the quantity $\Sigma_{(0)}^{(2)}$ provides a contribution to Eq. (14) but only beginning with the order of e^4 because the electron momentum does not, in fact, lie on the mass shell and the deviation from the mass shell $x \equiv m^2 - \mu^2 + f^2$ is of the order of e^2. The contribution due to $\Sigma_{(0)}^{(4)}$ is important only to the order of e^6. Therefore, in the second order, the quantity $\Sigma^{(2)}$ on the mass shell is identical with $\Sigma_{(1)}^{(2)}$ and corresponds to diagram 2a with the electron propagator $G_{(1)}^{(0)}$. Taking $\Sigma_{(1)}^{(2)}$ into account is equivalent to allowing for the exchange interaction in the system. In the explicit form,

$$\Sigma_{(1)}^{(2)}(p) = -\frac{2e^2}{(2\pi)^3}\int \frac{d^4k}{k^2}(2m - \hat{p} + \hat{k})\,\delta(m^2 - (p-k)^2) \times \theta(p_0 - k_0)\,\theta(\mu - p_0 + k_0). \tag{17}$$

The quantity $\Sigma_{(1)}^{(2)}$, just like $\Sigma_{(0)}^{(2)}$, has the matrix structure

$$\Sigma^{(2)} = \hat{b} + a, \tag{18}$$

where b is the 4-vector with components (c, dp), and a, c, d are certain scalar functions p_0 and p^2. Therefore, to within second-order terms, the electron Green function can be written in the form

$$G = \frac{m - a + \hat{p} + \hat{b}}{(m-a)^2 - (p+b)^2}, \tag{19}$$

and the Fermi momentum can be found with the same accuracy from the equation

$$(m-a)^2 - (\mu + c)^2 + f^2(1+d)^2 = 0. \tag{20}$$

Let us begin with the simplest contribution $\Sigma_{(0)}^{(2)}$. This can be taken directly from quantum electrodynamics (see, for example,[4]) and, when the fact that the deviation x from mass shell is small is taken into account, we have

$$\Sigma_{(0)}^{(2)} = \frac{-e^2}{16\pi^2}\left\{\frac{x}{m}\left(\ln\frac{x^2}{m^4} - 1\right) + 2(\hat{p} - m)\left(\ln\frac{\lambda^2}{m^2} - 1\right)\right\}. \tag{21}$$

In this expression λ is the photon mass. The quantity $m - \hat{p}$ must be replaced by $\Sigma_{(1)}^{(2)}$ on the mass shell and, therefore, we have finally

$$\Sigma_{(0)}^{(2)} = -\frac{e^2}{16\pi^2}\left\{\frac{x}{m}\left(\ln\frac{x^2}{m^4} - 1\right) - 2\Sigma_{(1)}^{(2)}\left(\ln\frac{\lambda^2}{m^2} - 1\right)\right\}. \tag{22}$$

The dependence on the photon mass in this expression

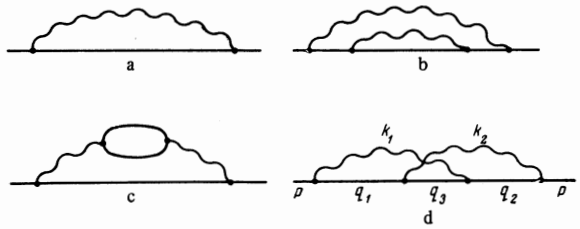

FIG. 2

is due to the residue term $(1 - Z_2)\Sigma_{(1)}^{(2)}$. We shall see later that this dependence can be removed from the complete expression for Σ, since it cancels out with the analogous dependence in the fourth-order diagrams (Figs. 2b and 2d).

Let us now consider the fourth-order diagrams. We note that when polarization is taken into account this leads to the appearance of Eq. (13) as the photon Green function. It is clear from its structure that when the photon and plasma momenta are small, the main contribution is due to the region $|k| \sim e$ and, therefore, in the electron propagator $G = (m - \hat{p} - \Sigma)^{-1}$ the quantity $\Sigma(\sim e^2)$ is always smaller than $(m - \hat{p})$ $(\sim e)$. Therefore, we are entitled to expand the electron Green function in the perturbation-theory series subject to the condition that the photon Green function is taken into account exactly. The only exception is diagram 2b, where the electron Green function must be taken into account in a more rigorous fashion. Having used Eq. (19), we find that

$$G = (m - \hat{p} - \Sigma)^{-1}|_{p_0 \to p_0 + i\eta\varepsilon(p_0)} + \theta(p_0)\,\theta(\mu - p_0)\,[(m - \hat{p} - \Sigma^{(-)})^{-1} - (m - \hat{p} - \Sigma^{(+)})^{-1}], \tag{23}$$

where $\Sigma^{(\pm)}$ is the proper mass on the upper (+) and lower (–) edges of the cut. Hence, it follows that

$$G = G_{(0)} + G_{(1)}^{(0)} + G_{(1)}^{(2)}, \tag{24}$$

where $G_{(0)}$ corresponds to $\mu = 0$, $G_{(1)}^{(0)}$ is given by Eq. (15), and

$$G_{(1)}^{(2)} = G_{(0)}^{(0)}\Sigma_{(1)}^{(2)}G_{(0)}^{(0)} - 2\pi i\theta(p_0)\,\theta(\mu - p_0)\Big\{(m + \hat{p})\,[\delta(m^2 - p^2 - 2\mathrm{Re}(ma_{(1)} + pb_{(1)})) - \delta(m^2 - p^2)] - 2(m + \hat{p})\,\mathrm{Re}(ma_{(0)} + pb_{(0)})\,\delta'(m^2 - p^2 - 2\,\mathrm{Re}\,(ma_{(1)} + pb_{(1)})) + \mathrm{Re}(\hat{b} - a)\,\delta(m^2 - p^2) + \frac{1}{\pi}\mathrm{Im}(\hat{b}^{(+)} - a^{(+)})\,\mathrm{P}(m^2 - p^2)^{-1} + \frac{2}{\pi}\mathrm{Im}(ma^{(+)} + pb^{(+)})\,\mathrm{P}(m^2 - p^2)^{-2}\Big\}. \tag{25}$$

In these expressions the scalar a, $a_{(0)}$, $a_{(1)}$ and the vectors b, $b_{(0)}$, and $b_{(1)}$ are related, respectively, to $\Sigma^{(2)} = \Sigma_{(0)}^{(2)} + \Sigma_{(1)}^{(2)}$, $\Sigma_{(0)}^{(2)}$, and $\Sigma_{(1)}^{(2)}$ through Eq. (18). The symbol P denotes the principal value of the integral. Equation (25) cannot be simplified further because of the logarithmic singularities of Σ on the mass shell. The contribution of the diagram of Fig. 2b thus turns out to be

$$A_{\mathrm{II}} = -\frac{ie^2}{(2\pi)^4}\int d^4k\,D^{\alpha\beta}(k)\gamma_\alpha G_{(1)}^{(2)}(p-k)\gamma_\beta. \tag{26}$$

The contribution due to the ground-state polarization diagram (Fig. 2c) is obviously given by

$$A_{\mathrm{II}} = -\frac{ie^2}{(2\pi)^4}\int d^4k\,(D^{\alpha\beta}(k) - D_{(0)}^{\alpha\beta}(k))\gamma_\alpha G^{(0)}(p-k)\gamma_\beta. \tag{27}$$

The subscript (0) indicates, as usual, that $\mu = 0$. The

superscript (0) represents the zero order of the perturbation theory. The polarization operator $\Pi^{\alpha\beta}$ in the expression for $D^{\alpha\beta}$ is given by the usual (second-order) formula

$$\Pi^{\alpha\beta} = -\frac{ie^2}{(2\pi)^4}\int d^4q\ \mathrm{Sp}\{\gamma^\alpha G^{(0)}(q)\gamma^\beta G^{(0)}(p-q)\}. \qquad (28)$$

The denominators in $D^{\alpha\beta}$ [see Eq. (13)] should retain only that part of the functions Π and Π_1 which is nonzero for $k \to 0$. The remaining part should be taken into account in accordance with the perturbation theory and only in Eq. (27).

To write down the remaining contribution due to the vertex parts (Fig. 2d), let us introduce the following notation. We shall use $\Gamma^{(2)}_{(0)}$ to represent the vertex part in the second order, in which the electron propagators are free and of the Feynman type ($\mu = 0$), but the photon propagator is taken in accordance with Eq. (13), taking $\Pi^{\alpha\beta}$ in the lowest order. The complete vertex part in the second order is $\Gamma^{(2)} = \Gamma^{(2)}_{(0)} + \Gamma^{(2)}_{(1)}$, where $\Gamma^{(2)}_{(1)}$ contains the contribution of diagrams in which at least one of the electron propagators is replaced by $G^{(0)}_{(1)}$. The quantity $\Gamma^{(2)}_{(1)}$ consists of three terms, namely,

$$\Gamma^{(2)}_{(1)} = \Gamma^{(2)}_{(1i)} + \Gamma^{(2)}_{(1e)} + \Gamma^{(2)}_{(12)}.$$

The contribution $\Gamma^{(2)}_{(12)}$ corresponds to the diagram in which both electron propagators are replaced by $G^{(0)}_{(1)}$. The contributions $\Gamma^{(2)}_{(1i)}$ and $\Gamma^{(2)}_{(1e)}$ correspond to diagrams in which $G^{(0)}_{(1)}$ replaces only one of the electron propagators touching, respectively, the internal (i) or external (e) vertex in Σ. The total contribution to Σ of vertex-part diagrams can then be written in the form

$$A_\Gamma = -\frac{ie^2}{(2\pi)^4}\int d^4k\, D^{(0)}\{(\Gamma^{(2)}_{(0)} + Z_1 - 1)G^{(0)}_{(1)}\gamma + \gamma G^{(0)}_{(1)}(\Gamma^{(2)}_{(0)} + Z_1 - 1)$$
$$+ \gamma G^{(0)}_{(0)}\Gamma^{(2)}_{(1i)} + \gamma G^{(0)}_{(1)}\Gamma^{(2)}_{(1i)} + \Gamma^{(2)}_{(1i)}G^{(0)}_{(1)}\gamma + \gamma G^{(0)}_{(1)}\Gamma^{(2)}_{(1e)} + \gamma G^{(0)}_{(1)}\Gamma^{(2)}_{(12)}\}. \qquad (29)$$

For simplicity, we have omitted the arguments and some of the vector indices of photons. The photon propagator in its explicit form can be taken to be free, since the vertex parts have no poles for photon momenta $k \to 0$. However, the vertex parts themselves should be evaluated with the photon propagator which takes into account the ground-state polarization in accordance with Eq. (13). The complete contribution to $\Sigma^{(4)}_{(1)}$ is the sum of three terms given by Eqs. (26), (27), and (29), i.e.,

$$\Sigma^{(4)}_{(1)} = A_\Sigma + A_\Pi + A_\Gamma. \qquad (30)$$

It is readily verified that the integrals in Eqs. (26), (27), and (29) are finite and satisfactorily converge for high and low momenta. The dependence on the photon mass is given by Eqs. (26) and (29). The infrared contribution to $G^{(2)}_{(1)}$ is of the form $(1 - Z_2)G^{(0)}_{(1)}$ and, therefore, the infrared contribution to A_Σ is $(1 - Z_2)\Sigma^{(2)}_{(1)}$. The infrared contribution to A_Γ is due to terms with Z_1 and is equal to $2(Z_1 - 1)\Sigma^{(2)}_{(1)}$. Altogether these contributions cancel out with the infrared contribution due to $\Sigma^{(2)}_{(0)}$ considered at the beginning of this section. Therefore, the dependence on the photon mass is, in fact, removed completely.

The evaluation of Eqs. (26), (27), and (29) is, in general, quite a complicated independent problem. Here we shall confine our attention to a brief discussion of the limiting cases, namely, the nonrelativistic ($\kappa/m \ll 1$) and ultraviolet ($\mu/m \gg 1$). This will be done in the next section.

4. NONRELATIVISTIC AND ULTRAVIOLET LIMITS

The nonrelativistic limit is characterized by the inequality $\kappa/m \ll 1$. The relative value of the two small parameters $\alpha = e^2/4\pi$ and κ/m may be quite different. We shall now largely confine our attention to the case where $\alpha \ll \kappa/m$ (high-density cases). In the nonrelativistic limit the proper mass in the second order [Eq. (17)] can be evaluated without difficulty, and is given by

$$\Sigma^{(2)}_{(1)}(p) = \frac{2\alpha\varkappa}{\pi}\left(1 - \frac{1}{2}\gamma_0\right)\left(1 + \frac{\varkappa^2 - n^2}{2\varkappa|p|}\ln\frac{\varkappa + |p|}{\varkappa - |p|}\right). \qquad (31)$$

When γ_0 is replaced by unity this becomes identical with the well-known formula in the nonrelativistic theory (see, for example,[5]). Equation (20) gives the following relation between f and κ in the exact form in the nonrelativistic limit:

$$f^2 = \varkappa^2 + 2\alpha\varkappa m/\pi. \qquad (32)$$

Hence, after simple transformations involving Eqs. (2), (3), and (4), we obtain the following well-known expression for the exchange energy in the nonrelativistic theory:

$$E_* = -\frac{\alpha}{4\pi^3}V(3\pi^2 N_0 V^{-1})^{4/3}. \qquad (33)$$

Let us now estimate the contribution, in the nonrelativistic limit, of the various fourth-order terms, neglecting logarithmic terms. The quantity x in Eq. (22) is equal to the difference $(f^2 - \kappa^2)$ and, according to Eq. (32), it is of the order of $e^2\kappa m$. Therefore, the contribution of Eq. (22) is of the order of $e^4\kappa$. When A_Σ is estimated with our adopted accuracy, we can expand all the δ-functions in Eq. (25) around the neighborhood of $p^2 = m^2$. The principal contribution to the integral in Eq. (26) is then due to the region of small $k \sim \kappa$. The contribution due to the region of large k is nonzero only for the first term in Eq. (25), and is of the order of $e^4\kappa^3$ because well away from the mass shell $\Sigma^{(2)}_{(1)}$ is of the order of $e^2\kappa^3$. For $k \sim \kappa$ both $\Sigma^{(2)}_{(0)}$ and $\Sigma^{(2)}_{(1)}$ in Eq. (25) are of the order of $e^2\kappa$. The integrand in Eq. (26) also contains the photon propagator which is of the order of κ^{-2} and either two electron propagators or $\delta'(m^2 - p^2)$ which, in any case, gives a further κ^{-2}. If we take into account the phase volume ($\sim\kappa^4$) we find that the contribution of A_Σ is of the order of $e^2\kappa$.

In precisely the same way we can estimate the contribution of the vertex parts A_Γ. In the diagram of Fig. 2d, at least one of the internal electron 3-momenta should be small ($\sim\kappa$). Suppose that it corresponds to, say, the line touching the external electron (momentum q_1 in Fig. 2d). The photon momentum k_1 is then also small. If k_2, which is the momentum of the other photon, is large, the denominator referring to q_3 is also large and the overall estimate for the integral is determined only by the denominators connected with k_1 and q_1. The phase volume gives κ^4, and the two propagators contribute κ^{-3}. The entire integral is of the order of $e^4\kappa$. If the second photon momentum k_2 is also small, the estimate for the integral consists of the κ^8 due to the phase volume and κ^{-7} due to the two-photon and three-electron

propagators. As a result, we again obtain $e^4\kappa$. All that remains is to consider the case when q_3 is small and the remaining momenta are not small. The contribution will, of course, be small and of the order of $e^4\kappa^3$. Thus, all the diagrams of the vertex part A_Γ, like the diagrams of the proper mass A_Σ, and the contribution due to $\Sigma^{(2)}_{(0)}$, are of the same order in the nonrelativistic limit, i.e., $e^4\kappa$.

Let us now consider the polarization diagram (Fig. 2c) which, in the nonrelativistic limit, provides the principal contribution to the proper mass. The region of large k in the integral given by Eq. (27) provides a small contribution because, in this region, the polarization operator is of the order of $e^2\kappa^3$ and the overall estimate for A_Π is $e^4\kappa^3$. In this region, where $k \ll m$, the polarization operator can be calculated in an explicit form[1]. Using the results reported in[1], we find that

$$\Pi = (2\alpha\varkappa^3/\pi\mu)[1 + (\eta^2 - 1)f(\eta^2)], \tag{34}$$

$$\Pi + (\mu^2 - (k\mu)^2/k^2)\Pi_1 = -4\alpha\varkappa\mu\pi^{-1}f(\eta^2)(k^2/\mathbf{k}^2), \tag{35}$$

where

$$f(\eta^2) = 1 - \frac{1}{2}\eta\ln\frac{\eta+1}{\eta-1}, \tag{36}$$

and $\eta = \mu k_0/\kappa|\mathbf{k}|$. We note that the second denominator in Eq. (13) in the region where $k \ll m$ assumes the form which is well known in the nonrelativistic theory of plasma oscillations. In the nonrelativistic limit, the quantity Π is clearly of the order of $e^2\kappa^3$. On the other hand, Π_1 is much greater in the region where $\eta \sim 1$ (i.e., $k_0 \sim \kappa|\mathbf{k}|/m$), where it is of the order of $e^2\kappa$. It is precisely this region which provides the principal contribution to the integral given by Eq. (27).

When $|\mathbf{k}| \sim \kappa$ and $k_0 \sim \kappa^2$ the electron propagator in Eq. (27) is of the order of $\min\{1/\kappa^2, 1/e^2\kappa m\}$. The phase volume provides κ^5. Finally, the difference $D - D^{(0)}$ is of the order of $e^2\kappa^{-3}m$ ($e^2\kappa m$ due to Π_1 in the numerator and κ^{-4} due to $|\mathbf{k}|^4$ in the denominator). The overall estimate for Eq. (27) yields $\min\{e^4m, e^2\kappa\}$, i.e., a result which is greater than that provided by the other diagrams in the ratio of m/κ (for high density) or $1/e^2$ (for low density $e^2 \gg \kappa/m$). Therefore, in the nonrelativistic limit the principal contribution to the proper mass is due to the polarization diagram (Fig. 2c) and the region of integration is $\eta \sim 1$. In the usual nonrelativistic theory this corresponds to taking into account the correlation interaction in the low-density approximation. Corrections due to more precise allowance for correlation effects, and those due to radiative effects are clearly of the same order of small quantities, i.e., κ/m or e^2, as compared with the principal term and should, in principle, be taken into account at the same time.

Each special calculation of the contribution due to the polarization diagram in the above nonrelativistic region immediately reduces to the corresponding problem in the nonrelativistic theory. The integral given by Eq. (27) is written in terms of the variables $k_1 = \kappa^{-1}\mathbf{k}$ and η, and when all the terms containing κ/m are neglected we obtain

$$A_\Pi = -\frac{ie^4m}{(2\pi)^4\pi^2}(1+\gamma_0)\int d^3k_1\,d\eta\,f(\eta^2)\Big/|\mathbf{k}_1|(\mathbf{k}_1^2 + \beta f(\eta^2))\left(\frac{1}{2}\beta + \mathbf{k}_1^2 + 2\eta|\mathbf{k}_1| - 2|\mathbf{k}_1|\cos\theta\right), \tag{37}$$

where $\beta = e^2m/\pi^2\kappa$ and the singularity in the second parentheses in the denominator lie in the second and fourth quadrants of the plane of the complex η. In Eq. (37) we have substituted $p_0 = \mu$ and $\mathbf{p}^2 = \kappa^2$.

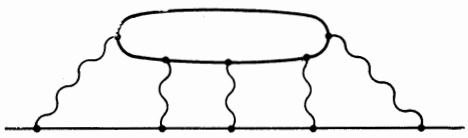

FIG. 3

When f is calculated in accordance with Eq. (20) to the order of e^4 we can replace the matrix γ_0 in Eq. (37) by unity, in which case this equation becomes identical with the expression for the correlation proper mass in nonrelativistic plasma on the Fermi surface when only polarization diagrams are taken into account. Therefore, the energy calculated from Eqs. (2)—(4) will also be equal to the corresponding values in the nonrelativistic theory. In particular, when $\beta \to 0$ (high-density approximation) the correlation energy is given by the usual Gell-Mann—Brueckner formula

$$E_c = \alpha^2(1 - \ln 2)\pi^{-2}N_0m\ln(\alpha mV^{1/3}N_0^{-1/3}). \tag{38}$$

When $\beta \to \infty$ (low-density approximation) we must, in addition to fourth-order diagrams, take into account diagrams of higher order, which describe multiple interactions between the particles, i.e., diagrams of the form shown in Fig. 3.

We emphasize that the parameters m and e in the nonrelativistic formulas turn out to be automatically renormalized in our approach. No intermediate divergent quantities need to be used, which is in contrast to the method described in[1].

In the ultrarelativistic case, the small parameters are α and m/μ. The proper mass in the second order can again be evaluated in an elementary fashion. Near the mass shell it is given by

$$\Sigma^{(2)}_{(1)} = -\frac{\alpha}{\pi}\left\{\frac{1}{2}p\gamma + \hat{p}\left[\frac{1}{4}\ln\frac{4\varkappa^2}{m^2} + \frac{1}{4}\left(\frac{2-\xi}{1-\xi} - \frac{2}{(1-\xi)^2}\ln(1-\xi) + \frac{\xi(2-\xi)}{(1-\xi)^2}\ln\xi\right)\right]\right\}, \tag{39}$$

where $x = m^2 - p^2$ and $\xi = x/m^2$. It is assumed in Eq. (39) that ξ is substantially different from unity. When p^2/m^2 is large, the term proportional to p does not contribute to the equation for f^2 given by Eq. (20), and independently of ξ we obtain

$$f^2 = \varkappa^2(1 - \alpha/\pi), \tag{40}$$

from which it also follows that $x = -\kappa^2\alpha/\pi$. From Eq. (40) we find that the exchange energy in the ultrarelativistic case is given by

$$E_e = 3\alpha(8\pi)^{-1}N_0(3\pi^2N_0/V)^{1/3}, \tag{41}$$

which agrees with the results reported in[1].

Let us now consider the higher orders. Assuming that the electron momentum is $p = \kappa p_1$, and performing the analogous replacement for all the momenta involved in the integration, we find that, to within logarithmic terms, any proper mass diagram is proportional to κ. Therefore, the contribution of all the diagrams in fourth order is of the order of $e^4\kappa$ and, moreover, it contains $\ln e^2$ and $\ln(m^2/\kappa^2)$. If the quantities e^2 and m^2/κ^2 are of a substantially different order of small quantities, then,

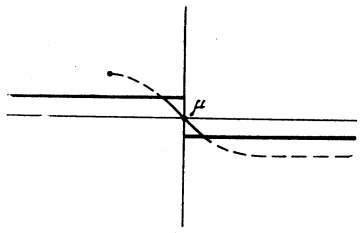

FIG. 4

of course, one must retain the dependence on the logarithm of the larger of the two. An exception is the contribution of $\Sigma_{(0)}^{(2)}$. When $x \ll m^2$, it follows from Eqs. (22) and (40) that $\Sigma_{(0)}^{(2)}$ is of the order of $e^4\kappa^2/m$ and, therefore, it is greater than all the other fourth-order contributions in the ratio of κ/m. However, in Eq. (20) the matrix structure in the relativistic case ensures that the contribution of $\Sigma_{(0)}^{(2)}$ is suppressed in the same ratio and, therefore, is not a dominant factor. The final result is that in the ultrarelativistic limit all the fourth-order diagrams provide a comparable contribution to the proper mass and must be taken into account at the same time. Therefore, the expression for the correlation energy of an ultrarelativistic system which was found in[1] and which took into account only the polarization diagram cannot be regarded as valid.

APPENDIX

DERIVATIONS OF EQS. (4) AND (5) IN THE RELATIVISTIC THEORY

From the analytic properties of $d(p) \equiv \det G^{-1}(p)$ as a function of the complex p_0 it follows that the equation $d(p) = 0$ has, in general, a solution on the nonphysical sheet of the complex plane with the exception of the case when $p_0 = \mu$, and the three-dimensional momentum is fixed by Eq. (5). When $p^2 > f^2$ ($p^2 < f^2$) the zero of $d(p)$ moves under the cut to the left (right). Because of the zero of $d(p)$ the function $\ln d(p)$ has an additional cut which for $p^2 < f^2$ intersects the real axis [see Fig. 4 which shows the cuts for $\ln d(p)$]. Therefore, the function $\ln d(p)$ undergoes a discontinuity on the real p_0 axis at $p_0 = \mu$, which amounts to $-2\pi i g\theta(f^2 - p^2)$. In this expression g is the multiplicity of the zero of $d(p)$, which is determined by the degeneracy with respect to the internal quantum numbers. For electrons $g = 2$.

Let us now use the equation $\ln d = \mathrm{Sp} \ln G^{-1}$. If we take into account the discontinuity in $\ln d$ on the real axis, we have

$$\frac{\partial}{\partial p_0}\ln d = -4\pi i\delta(p_0-\mu)\theta(f^2-\mathbf{p}^2) - \mathrm{Sp}\,\gamma_0 G - \mathrm{Sp}\,G\frac{\partial}{\partial p_0}\Sigma. \quad (42)$$

If we subtract from Eq. (42) its value for $\mu = 0$, and integrate with respect to all the 4-momenta p with a weight function $\exp(ip_0\tau)$ for $\tau \to 0$, we find that the left-hand side becomes zero and on the right-hand side the third term vanishes for the same reasons as in the nonrelativistic theory.[3] The remaining two terms yield the formula given by Eq. (4).

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