SPECTRUM OF A POLARON IN A MAGNETIC FIELD. BOUND ELECTRON-PHONON STATES

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Submitted March 18, 1971

Zh. Eksp. Teor. Fiz. 61, 843–858 (August, 1971)

We obtain equations which enable us to evaluate the spectrum of an electron in the energy range close to the threshold for the emission of an optical phonon where the perturbation theory for the mass operator is inapplicable. We solve these equations for the deformation electron-phonon interaction and show that below the threshold there exist new branches of the spectrum which describe bound electron-phonon states.

INTRODUCTION

We study in the present paper the spectrum of an electron in a magnetic field which interacts with optical phonons. It is, in particular, necessary to know this spectrum in order to understand those peculiarities of the magneto-optical absorption which are caused by the participation of optical phonons in the absorption process, as occurs when the peak of the absorption is split[1] and in cyclotron-phonon resonance.[2,3] Moreover, a study of this spectrum is of particular interest in view of the existence of bound electron-phonon states which were mentioned in [4].

For the calculation of the spectrum we make the following basic assumptions: 1) the optical phonons have no dispersion: $\omega(\mathbf{q}) = \omega_0$; 2) the electron-phonon interaction is weak; the dinensionless coupling constant $\alpha \ll 1$; 3) the crystal temperature is low: $T \ll \omega_0$ so that we can neglect phonon absorption processes which are proportional to $e^{-\omega_0/T}$; 4) the electron concentration is small so that we can neglect their effect on the phonons and the electron-electron interaction, and also may assume the electron gas to be non-degenerate; 5) there is no electron-impurity and electron-acoustical phonon interaction.

Notwithstanding the weakness of the electron-phonon interaction the problem of evaluating the spectrum in a magnetic field is not trivial as there is a range of the spectrum where we cannot apply perturbation theory for the mass operator M. Such a situation occurs for energies close to the threshold for the emission of one, two, or more phonons by an electron in one of the Landau zones, i.e., when $\epsilon \approx E_n(0) + \omega_0, E_n(0) + 2\omega_0, \ldots$, where $E_n(0)$ is the bottom of the Landau zone with number $n = 0, 1, 2, \ldots$. An electron which has an energy close to the threshold and which emits the appropriate number of phonons turns out to be close to the bottom of the Landau zone where the density of states is large. The probability for such a transition may thus turn out to be not small even for a small interaction matrix element and this means that the effective electron-phonon coupling in this energy range is not weak. This is just the circumstance which causes the existence of bound states.

Perturbation theory for M formally applied near the threshold diverges: separate diagrams are singular and with increasing order of the diagram the degree of singularity also increases. This is why the results of a calculation of the spectrum in which only the simplest diagram for M is taken into account,[5,6] or in which all diagrams for M without intersections of the phonon lines are summed,[7] are inapplicable in the immediate vicinity of the threshold. Besides, just in that region there occur new branches of the spectrum describing bound electron-phonon states.

The spectrum near the threshold can be found by using a method developed by us using as an example a one-dimensional model of the problem[8] and which is based upon the ideas of studying a spectrum near a decay threshold.[9] Our main consideration is given to the one-phonon threshold in the lowest Landau zone $\epsilon_0 = E_0(0) + \omega_0$ below which there is a true undamped spectrum. To evaluate the spectrum we restrict ourselves to the deformation electron-phonon interaction (nonpolar semiconductor) as there arise additional singularities of a Coulomb nature when we consider the polarization interaction and they are not related to the threshold singularities and cloud the issue.

1. EQUATIONS DETERMINING THE SPECTRUM NEAR THE THRESHOLD

We obtain in this section the basic equations by means of which we can find the spectrum near the one-phonon thresholds $\epsilon_n = E_n(0) + \omega_0$. These equations are obtained by the same method as used in [8] to study the spectrum in the one-dimensional model problem.

It is natural for calculations in a magnetic field to use the Landau representation $lp\beta$, where l is the oscillator quantum number, p the longitudinal (along H) momentum component, and β a quantum number depending on the choice of gauge for the vector potential. The free electron Green function G^0 is diagonal in this representation and independent of β. It is very important that, as was shown in [10], this property is conserved in a uniform and isotropic system also for the "dressed" Green function G. An elementary excitation in the electron-phonon system, found as a pole of G, can thus be characterized by the quantum numbers l and p. We must then, however, bear in mind that the

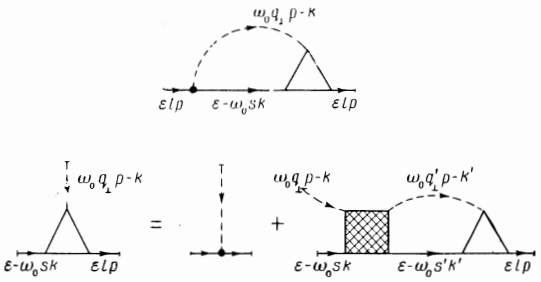


FIG. 1. Mass operator M and equation for the vertex Γ.

quantum number l of such an excitation cannot have anything in common with the Landau zone of number n in which the electron which contributes to the elementary excitation is situated. This is completely analogous to the fact that the longitudinal momentum p can be arbitrarily constructed from the longitudinal electron momentum k and the longitudinal momentum $q_{||}$ of the phonons which contribute to the excitation.

Assumptions 3 and 4 mentioned in the Introduction allow us to evaluate the spectrum of the elementary excitations using the T = 0 diagram technique assuming that there are no electrons in the ground state. The electron Green function will then be retarded. It is convenient to use the diagram technique in the gauge-invariant form [10] where the quantum number β is absent.

The initial equations for the mass operator M and the vertex Γ are shown in Fig. 1. The full-drawn line denotes the ("dressed") electron Green function. The free Green function is

$$G_l^0(\varepsilon p) = [\varepsilon - E_l(p) + i\eta]^{-1}, \quad \eta \to +0, \tag{1.1}$$

where $E_l(p)$ is the dispersion of the free electron in the Landau zone of number l. For a parabolic zone with effective mass m

$$E_l(p) = \left(l + \frac{1}{2}\right)\omega_c + \frac{p^2}{2m}, \quad \omega_c = \frac{eH}{mc}. \tag{1.2}$$

The dotted line indicates the phonon Green function which we can assume to be a free one, by virtue of assumption 4):

$$D(\omega\mathbf{q}) = (2\pi)^2 B(q)\,[(\omega - \omega_0 + i\eta)^{-1} - (\omega + \omega_0 - i\eta)^{-1}), \tag{1.3}$$

where B(q) is the square of the electron-phonon interaction matrix element. Without restricting ourselves for the moment to a definite form of the interaction we write

$$B(q) = (2\pi)^{-3} B_0 \Phi(q), \quad B_0 = \alpha \cdot 2\pi\omega_0/mp_0, \quad p_0 = \sqrt{2m\omega_0}, \tag{1.4}$$

where α is a dimensionless coupling constant and $\Phi(q)$ a dimensionless form-factor (of the order of unity when $q \approx p_0$) which depends on the form of the interaction. The dot indicates the gauge-invariant part of the vertex

$$\Lambda_{ll'}(\mathbf{q}_\perp) = e^{-i\varphi(l-l')} Q_{ll'}(t), \quad t = (q_\perp/p_c)^2,$$
$$Q_{ll'}(t) = (l!/l'!)^{1/2} t^{(l'-l)/2} e^{-t/2} L_l^{l'-l}(t) = (-1)^{l-l'} Q_{l'l}(t); \tag{1.5}$$

here $\mathbf{q}_\perp$ is the component of the phonon momentum at right angles to H, φ its azimuthal angle around H and $p_c = \sqrt{(2m\omega_c)}$ the magnetic momentum; L is a Laguerre polynomial defined as in [11]. The shaded square indicates the irreducible electron-phonon scattering amplitude Δ which has no intersections with a single electron or a single phonon line.

FIG. 2. Simplest irreducible electron-phonon scattering amplitude.

It is also convenient to use the retarded character of the electron Green functions to carry out the integration over the energy ω of the internal phonon lines using the relation

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} iD(\omega q) F(\omega) = B_0 \Phi(q) F(\omega_0), \tag{1.6}$$

where $F(\omega)$ is a function which is analytical when Im $\omega < 0$. If we now further in the external phonon lines put $\omega = \omega_0$ we obtain the phonon energy parameters shown in Fig. 1. In these graphs we have indicated separately at the phonon lines the transverse and longitudinal phonon momenta $\mathbf{q}_\perp$ and $q_{||}$ (the latter is expressed in terms of the electron momenta). Taking all we have said into account, the equations corresponding to Fig. 1 will have the form

$$M_l(\varepsilon p) = B_0 \sum_{s=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} G_s(\varepsilon - \omega_0, k) \int \frac{d\mathbf{q}_\perp}{(2\pi)^2} \Phi(q_\perp, p - k) \Lambda_{ls}(\mathbf{q}_\perp) \times \Gamma_s^l(\varepsilon p; \mathbf{q}_\perp, k), \tag{1.7}$$
$$\Gamma_s^l(\varepsilon p; \mathbf{q}_\perp, k) = \Lambda_{sl}(-\mathbf{q}_\perp)$$
$$+ B_0 \sum_{s'=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} G_{s'}(\varepsilon - \omega_0, k') \int \frac{d\mathbf{q}_\perp'}{(2\pi)^2} \Phi(q_\perp', p - k') \cdot$$
$$\times \exp\left\{i \cdot \frac{1}{2} \frac{c}{eH} [\mathbf{q}_\perp \mathbf{q}_\perp']\right\} \Delta_{ss'}(\varepsilon p; \mathbf{q}_\perp \mathbf{q}_\perp', kk') \Gamma_{s'}^l(\varepsilon p; \mathbf{q}_\perp', k'). \tag{1.8}$$

We need only add that the appearance of a phase factor in (1.8) is connected with the use of a gauge-invariant technique and is, essentially, the result of integrating over β in the usual technique. The bracket in that factor indicates the longitudinal (along H) component of the vector potential. We have shown $\mathbf{q}_\perp$ and $q_{||}$ separately in Φ.

Further simplification of these equations proceeds in the same way as in [8]. If we are interested in the energy range near the threshold ϵ_n the energy parameter of the Green functions in (1.7) and (1.8) is $\epsilon - \omega_0 \approx E_n(0)$ and, generally speaking, lies in a region where perturbation theory is applicable. We can thus evaluate the function G by an expansion of M in terms of α and if we are not interested in a renormalization of the threshold we can in lowest order replace G by G^0 as was shown in [8]. An exception is the case when the bottom of the Landau zone $E_n(0)$, for which we consider the threshold, coincides with the threshold of another zone n', i.e., $E_n(0) = E_{n'}(0) + \omega_0$, $E_{n'}(0) + 2\omega_0, \ldots$ We note that this situation will never occur if we study the threshold in the lowest Landau zone n = 0.

The quantity Δ in (1.8) is regular for $\epsilon \approx \epsilon_n$ by virtue of the absence from its graphs of the above-mentioned intersections and it can thus be evaluated by an expansion in terms of α and in lowest order can be replaced by the simplest diagram Δ^0 (Fig. 2).

If we assume G and Δ to be known, (1.8) is a set (for different s) of integral equations (in k and $\mathbf{q}_\perp$) for the vertex Γ, the solution of which makes it possible to evaluate the mass operator from (1.7).

Using the fact that we are close to the threshold we can get rid of the summation over s and the integration

over k. It was shown in [8] that in the range of ϵ close to the threshold only the singularities of G contribute in the integration over k. We consider first (1.7). If we replace G by G^0, then it is clear that these singularities are poles

$$\text{for } s=n \text{ when } k\approx 0 \text{ with } \operatorname{Res} G_s \approx m^{1/2}|\varepsilon-\varepsilon_n|^{-1/2},$$
$$\text{for } s\neq n \text{ when } k\approx p_c \text{ with } \operatorname{Res} G_s \approx m^{1/2}\omega_c^{-1/2}.$$

From this it is clear that near the threshold there is only a contribution from the singularity of G_n for k $\approx$ 0. We can thus retain in Eq. (1.7) only one term of the sum with s = n and take in this term from under the integral over k the factors Φ and Γ in the point k = 0.

Similar considerations are also applicable to Eq. (1.8) where we can retain in the sum over s' only the term s' = n and take Φ, Γ, and Δ from under the integral over k' in the point k' = 0. After that we put, to close the equation, k = 0 and s = n.

As a result we get the following equations for the energy range $|\epsilon - \epsilon_n| \ll \omega_c$:

$$M_l(\varepsilon p)=\tilde{M}_n(\varepsilon)\int\frac{d\mathbf{q}_\perp}{2\pi/a^2}\Phi(q_\perp,p)\Lambda_{ln}(\mathbf{q}_\perp)\Gamma_n^{\,l}(\varepsilon p;\mathbf{q}_\perp,0), \quad (1.9)$$

$$\Gamma_n^{\,l}(\varepsilon p;\mathbf{q}_\perp,0)=\Lambda_{ln}(-\mathbf{q}_\perp)+\tilde{M}_n(\varepsilon)\int\frac{d\mathbf{q}_\perp'}{2\pi/a^2}\Phi(q_\perp',p)$$
$$\times\exp\left\{i\cdot\frac{1}{2}\frac{c}{eH}[\mathbf{q}_\perp\mathbf{q}_\perp']\right\}\Delta_{nn}(\varepsilon p;\mathbf{q}_\perp\mathbf{q}_\perp';00)\Gamma_n^{\,l}(\varepsilon p;\mathbf{q}_\perp'0), \quad (1.10)$$

where

$$\tilde{M}_n(\varepsilon)=\frac{B_0}{2\pi a^2}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}G_n^{\,0}(\varepsilon-\omega_0,k), \quad (1.11)$$

$(2\pi a^2)^{-1}$ is the degree of degeneracy of the Landau level, and $a=\sqrt{2/p_c}$ is the magnetic length. The quantity $\tilde{M}_n$ is the product of the interaction matrix element and the level density near the bottom of the zone and determines thus the effective interaction. Straightforward calculation gives

$$\tilde{M}_n(\varepsilon)=-i\tilde{\alpha}\omega_0(w+i\eta)^{-1/2}, \quad \operatorname{Im}\tilde{M}_n(\varepsilon)<0, \quad (1.12)$$

where

$$w=\frac{\varepsilon-\varepsilon_n}{\omega_0} \qquad \tilde{\alpha}=\frac{1}{2}\frac{\omega_c}{\omega_0}\alpha. \quad (1.13)$$

Comparing Eqs. (1.9) and (1.10) with the corresponding equations in [8] we can see that in a magnetic field the role of the effective coupling constant is played by $\tilde{\alpha}$. In other words, perturbation theory is valid only if $|w| \gg \tilde{\alpha}$ and taking into account diagrams without intersections of phonon lines is sufficient only when $|w| \gg \tilde{\alpha}^2$. Of course, this follows also from a direct comparison of the diagrams calculated in [5]. If the latter condition is satisfied the vertex can be replaced by the simplest one, i.e., the second term on the right-hand side of (1.10) is unimportant. However, in the immediate vicinity of the threshold, when $|w| \lesssim \tilde{\alpha}^2$ we must take into account diagrams with phonon line intersections, i.e., take the complete vertex. To do this we must solve Eq. (1.10) which in contrast to the one-dimensional problem [8] is not an algebraic but an integral equation (in $\mathbf{q}_\perp$).

We study now the elimination of the angles from the integral equation. To do this we change to polar coordinates t, φ. Noting that the free term in the equation for Γ depends on φ as $e^{i(l-n)\varphi}$ we look for Γ with the same angular dependence:

$$\Gamma_n^{\,l}(\varepsilon p;\ \mathbf{q}_\perp,0)=e^{i(l-n)\varphi}\Gamma_n^{\,l}(\varepsilon p;\ t). \quad (1.14)$$

Substituting this expression into (1.9) and (1.10) we can integrate over φ and obtain

$$M_l(\varepsilon p)=\tilde{M}_n(\varepsilon)\int_0^\infty dt\,\Phi(t,p)Q_{ln}(t)\Gamma_n^{\,l}(\varepsilon p;t), \quad (1.15)$$

$$\Gamma_n^{\,l}(\varepsilon p;t)=Q_{ln}(t)+\tilde{M}_n(\varepsilon)\int_0^\infty dt'\Phi(t',p)\,\Delta_n^{\,l}(\varepsilon p;tt')\Gamma_n^{\,l}(\varepsilon p;t'), \quad (1.16)$$

where

$$\Delta_n^{\,l}(\varepsilon p;tt')=\oint\frac{d\varphi'}{2\pi}e^{i(l-n)(\varphi'-\varphi)}\exp\left\{i\frac{1}{2}\frac{c}{eH}[\mathbf{q}_\perp\mathbf{q}_\perp']\right\}\Delta_{nn}(\varepsilon p;\mathbf{q}_\perp\mathbf{q}_\perp';00). \quad (1.17)$$

Replacing Δ by Δ^0 we have

$$\Delta_{nn}^{0}(\varepsilon p;\mathbf{q}_\perp\mathbf{q}_\perp';00)=$$
$$=\exp\left\{i\frac{1}{2}\frac{c}{eH}[\mathbf{q}_\perp\mathbf{q}_\perp']\right\}\sum_{s=0}^{\infty}G_s^{\,0}(\varepsilon-2\omega_0,-p)\Lambda_{ns}(\mathbf{q}_\perp')\Lambda_{sn}(-\mathbf{q}_\perp). \quad (1.18)$$

We substitute (1.18) into (1.17) and integrate over the angles. Moreover, as the integral term in (1.16) is important only near the threshold and Δ is regular there, we can replace ϵ by ϵ_n in Δ. As a result we find

$$\Delta_n^{\,l}(\varepsilon p;\ tt')=\Delta(p)K_n^{\,l}(\sigma(p);\ tt'), \quad (1.19)$$

where

$$\Delta(p)=-\left(\omega_0+\frac{p^2}{2m}\right)^{-1}=-\frac{1}{\omega_0}(1+u^2)^{-1}, \quad u=\frac{p}{p_0} \quad (1.20)$$

and K is a series of Bessel functions

$$K_n^{\,l}(\sigma;tt')=\sum_{s=0}^{\infty}\frac{\sigma}{s-n+\sigma}Q_{ns}(t)Q_{ns}(t')J_{s+l-2n}(2\sqrt{\sigma tt'}) \quad (1.21)$$

with the parameter

$$\sigma(p)=-\frac{1}{\Delta(p)\omega_c}=\sigma(0)(1+u^2), \qquad \sigma(0)=\frac{\omega_0}{\omega_c}. \quad (1.22)$$

After these calculations we can write the integral equation for the vertex in the following final form:

$$\Gamma_n^{\,l}(\varepsilon p;t)-\lambda(\varepsilon p)\int_0^\infty dt'\Phi(t',p)K_n^{\,l}(\sigma(p);t,t')\Gamma_n^{\,l}(\varepsilon p;t')=Q_{ln}(t), \quad (1.23)$$

where

$$\lambda(\varepsilon p)=\tilde{M}_n(\varepsilon)\Delta(p). \quad (1.24)$$

This is a Fredholm-type equation of the second kind with polar kernel ΦK which depends on p and a parameter λ which depends on ϵ and p. The solution of this equation gives $\Gamma_n^l(\lambda; p, t)$ which allows us to evaluate

$$\bar{\Gamma}_n^{\,l}(\lambda;p)=\int_0^\infty dt\Phi(t,p)Q_{ln}(t)\Gamma_n^{\,l}(\lambda;p;t) \quad (1.25)$$

and to find the mass operator near the threshold

$$M_l(\varepsilon p)=\tilde{M}_n(\varepsilon)\bar{\Gamma}_n^{\,l}(\lambda;\ p). \quad (1.26)$$

After this we find the spectrum of the elementary excitations with quantum numbers l and p, which lie close to the threshold in the Landau zone of number n, from the equation

$$\varepsilon-E_l(p)-\tilde{M}_n(\varepsilon)\bar{\Gamma}_n^{\,l}(\lambda;\ p)=0, \quad (1.27)$$

which is solved together with (1.24).

2. CALCULATION OF THE VERTEX NEAR THRESHOLD IN THE LOWEST LANDAU ZONE

In what follows we shall consider the spectrum close to the threshold in the lowest Landau zone, i.e., the case n = 0, restricting ourselves to the deformation electron-photon interaction for which $\Phi(q) = 1$. In the case n = 0 the kernel of the integral equation for the vertex can be somewhat simplified and takes the following form:

$$K_0^l(\sigma; t, t') = e^{-(t+t')/2} S^l(\sigma; 2\sqrt{tt'}),$$

$$S^l(\sigma, z) = \sum_{s=0}^{\infty} \frac{\sigma}{s+\sigma} \frac{1}{2^s s!} z^s J_{l+s}(z). \tag{2.1}$$

For small σ we have from this immediately

$$S^l(0, z) = J_l(z), \tag{2.2}$$

while for large σ it is convenient to use an expression in decreasing powers of σ in the form (see Appendix 1)

$$S^l(\sigma, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{2m+l} \frac{\Pi(\sigma)}{\Pi(\sigma+m)\Pi(l+m)} \tag{2.3}$$

(here Π is the factorial function).

As $\Phi = 1$, the equation for the vertex contains apart from ω only one dimensionless parameter σ the magnitude of which is determined by the strength of the magnetic field H and the momentum p. If the field is weak, $\omega_c \ll \omega_0$, we have $\sigma \gg 1$ for all $p \gg p_c$. If, however, the field is strong, $\omega_c \gg \omega_0$, we have $\sigma \ll 1$ for $p \ll p_c$, and $\sigma \gg 1$ for $p \gg p_c$. It is therefore convenient to evaluate Γ first as function of λ and σ and then to solve Eq. (1.27) for the spectrum. As the equation for Γ is a Fredholm equation, $\bar{\Gamma}$ is a meromorphic function and can be written in the form [12]

$$\bar{\Gamma}^l(\lambda) = \bar{\Gamma}^l(0) + \sum_{j=0}^{\infty} \left\{ \frac{A_j}{\lambda - \lambda_j} + \frac{A_j}{\lambda_j} \right\}, \tag{2.4}$$

where λ_j are the eigenvalues of the integral equation. It is clear that A_j and λ_j depend on l and σ. Bearing in mind that the functions Q are normalized, we find from (1.23)

$$\bar{\Gamma}^l(0) = 1. \tag{2.5}$$

Solution of the Integral Equation for the Vertex at $\sigma = 0$.

In this case, using (1.5) and making the substitution

$$\Gamma(t) = (-1)^l (l!)^{1/2} t^{-l/2} \hat{\Gamma}(t), \tag{2.6}$$

we get Eq. (1.23) in the form

$$\hat{\Gamma}(t) - \lambda \int_0^{\infty} dt' e^{-(t+t')/2} \left(\frac{\sqrt{t}}{\sqrt{t'}}\right)^l J_l(2\sqrt{tt'}) \hat{\Gamma}(t') = (l!)^{-1} t^l e^{-t/2}, \tag{2.7}$$

while the quantity of interest to us is

$$\bar{\Gamma} = \int_0^{\infty} dt\, e^{-t/2} \hat{\Gamma}(t). \tag{2.8}$$

After a Laplace transformation $\hat{\Gamma}(t) \to g(z)$ the integral Eq. (2.7) becomes a functional equation:

$$g(z) - \lambda \frac{1}{(z+1/2)^{l+1}} g\left(\frac{1}{z+1/2} + \frac{1}{2}\right) = \frac{1}{(z+1/2)^{l+1}}. \tag{2.9}$$

Here g(z) is analytical when Re $z > 0$, as the kernel of the equation, and hence also its solution, are integrable.

The quantity in which we are interested is $\bar{\Gamma} = g(\frac{1}{2})$. We first find $g(\frac{1}{2})$ for small λ, integrating Eq. (2.9) over λ from which follows the chain

$$g(z_m) = \frac{1}{(z_m + 1/2)^{l+1}} + \lambda \frac{1}{(z_m + 1/2)^{l+1}} g(z_{m+1}),$$

$$z_{m+1} = \frac{1}{z_m + 1/2} + \frac{1}{2}, \qquad z_0 = \frac{1}{2}. \tag{2.10}$$

Expressing $g(z_0)$ in terms of $g(z_1)$, $g(z_1)$ in terms of $g(z_2)$, and so on, we get from this chain

$$g(1/2) = \sum_{k=0}^{\infty} g_k \lambda^k, \tag{2.11}$$

where

$$g_k^{-1} = [(z_0 + 1/2)(z_1 + 1/2) \ldots (z_k + 1/2)]^{l+1}. \tag{2.12}$$

The numbers z_n can be expressed in terms of the Fibonacci numbers c_n. Writing

$$z_m = \frac{c_m}{c_{m-1}} - \frac{1}{2}, \tag{2.13}$$

we find from (2.10) that

$$c_{m+1} = c_m + c_{m-1}, \quad c_0 = c_{-1}, \tag{2.14}$$

which is satisfied by the equation

$$c_m = \rho^{-m} + (-1)^{m+1} \rho^{m+1}, \quad \rho = 1/2(\sqrt{5} - 1) = 0.618\ldots \tag{2.15}$$

Substituting (2.15) into (2.13) and then into (2.12) we find

$$g_k = \left(\frac{3\rho - 1}{c_k}\right)^{l+1} \tag{2.16}$$

The radius of convergence of (2.11) is $\lambda_0 = \rho^{-(l+1)}$ but we can easily find its analytical continuation by expanding it into elementary fractions. To do this we must write

$$g_k = (3\rho - 1)^{l+1} \rho^{k(l+1)} [1 + (-1)^{k+1} \rho^{2k+1}]^{-(l+1)} = = (3\rho - 1)^{l+1} \rho^{k(l+1)} \sum_{j=0}^{\infty} \binom{-l-1}{j} (-1)^{(k+1)j} \rho^{(2k+1)j}, \tag{2.17}$$

substitute it into (2.11), change the order of summation, and sum over k. As a result we get the representation (2.4) where

$$\lambda_j = (-1)^j \rho^{-(2j+l+1)}, \quad A_j = -(\sqrt{5})^{l+1} \frac{(l+j)!}{l! j!} \lambda_j^{-1}. \tag{2.18}$$

We note that then

$$\sum_{j=0}^{\infty} \frac{A_j}{\lambda_j} = -1. \tag{2.19}$$

Solution of the Integral Equation for the Vertex as $\sigma \to \infty$.

Using the expansion (2.3) of the kernel we can change the integral equation into a set of algebraic equations. If we write

$$\Gamma_k = (-1)^l (l!)^{1/2} \frac{1}{(l+k)!} \int_0^{\infty} dt\, e^{-t/2} t^{l/2+k} \Gamma(t), \tag{2.20}$$

the system will have the form

$$\sum_{k'=0}^{\infty} a_{kk'} \Gamma_{k'} = 1, \qquad k = 0, 1, 2, \ldots, \tag{2.21}$$

where

$$a_{kk'} = \delta_{kk'} - (-1)^k \lambda \frac{(l+k+k')!}{(l+k)!} \frac{\Pi(\sigma)}{\Pi(\sigma+k')}. \tag{2.22}$$

In the notation of (2.20) the quantity of interest to us is $\bar{\Gamma} = \Gamma_0$. It is clear from (2.22) that if $|\lambda| \ll \sigma^{m+1}$, we have $|a_{kk'}| \ll 1$ when $k' > m$ and we need therefore retain in the set only $m+1$ equations. This means that in the range $|\lambda| \ll \sigma^{m+1}$ there are $m+1$ eigenvalues λ_j and, hence, $|\lambda_j| \approx \sigma^j$.

We study now the calculation of $\bar{\Gamma}(\lambda)$ in the range $\sigma^{m-1} \ll |\lambda| \ll \sigma^{m+1}$, $m = 1, 2, \ldots$ Using determinants to solve the set, we have

$$\bar{\Gamma}(\lambda) = D_0(\lambda) / D(\lambda), \tag{2.23}$$

where

$$D(\lambda) = \mathrm{Det}\, \|a_{kk'}\|_0^m, \tag{2.24}$$

and $D_0(\lambda)$ is obtained from $D(\lambda)$ by replacing the first column a_{k0} by the column from the unit matrix. We note now that

$$a_{k0} = -\lambda + \delta_{k0}. \tag{2.25}$$

Therefore

$$D(\lambda) = -\lambda D_0(\lambda) + D'(\lambda), \tag{2.26}$$

where $D'(\lambda)$ is obtained from $D(\lambda)$ by replacing a_{k0} by δ_{k0}. We can now write

$$\bar{\Gamma}(\lambda) = [-\lambda + f(\lambda)]^{-1}, \quad f(\lambda) = D'(\lambda) / D_0(\lambda). \tag{2.27}$$

Evaluating all quantities in the leading order in σ we write

$$a_{kk'} = -(-1)^{k'} \frac{\lambda}{\sigma^{k'}} \left\{ \frac{(l+k+k')!}{(l+k)!} - (-1)^{k'} \frac{\sigma^{k'}}{\lambda} \delta_{kk'} \right\} \tag{2.28}$$

Substituting now $a_{kk'}$ into D' and D_0 we can take from the columns the factors in front of the braces and cancel them in the ratio f. Furthermore, the second term in the braces is not small only in the coefficient a_{mm}. Expanding the last column in a sum of two we are led to the determinant

$$(\Delta_0)_m = \mathrm{Det}'' \frac{(l+k+k')!}{(l+k)!} \Big\|_0^m \tag{2.29}$$

and to a determinant $(\Delta')_m$ which is obtained from $(\Delta_0)_m$ by replacing the first column by δ_{k0}. The evaluation of these determinants (see Appendix 2) gives

$$(\Delta_0)_m = 1!2! \ldots m!,$$

$$(\Delta')_m = 1!2! \ldots (m-1)! \frac{(l+m+1)!}{(l+1)!}. \tag{2.30}$$

finally we find

$$f(\lambda) = \frac{(l+m+1)!}{(l+1)!m!} \frac{\lambda - a_m}{\lambda - b_m}, \quad b_m = (-1)^m \frac{\sigma^m}{m!}, \quad a_m = \frac{m}{l+m+1} b_m,$$

$$\sigma^{m-1} \ll |\lambda| \ll \sigma^{m+1}, \quad m = 1, 2, \ldots \tag{2.31}$$

In the remaining unstudied interval $|\lambda| \lesssim 1$ we must clearly retain in the set one equation which gives

$$f(\lambda) = 1, \quad |\lambda| \lesssim 1. \tag{2.32}$$

It is now easy to write $\bar{\Gamma}(\lambda)$ in the form (2.4), and we have

$$\lambda_j = (-1)^j \frac{\sigma^j}{j!}, \qquad A_j = -\frac{(l+j)!}{l!j!} \lambda_j^{-1}. \tag{2.33}$$

In leading order in σ we have then

$$\sum_{j=0}^{\infty} \frac{A_j}{\lambda_j} \approx \frac{A_0}{\lambda_0} = -1. \tag{2.34}$$

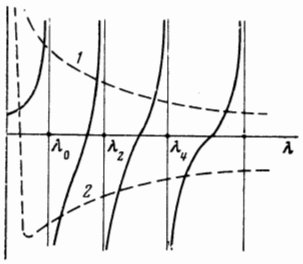


FIG. 3. Graphical solution of Eq. (3.1): curve 1: graph of the right-hand side of the equation for $a_l > 0$; 2: for $a_l < 0$.

Comparing the results for large and small σ we see that in both cases the signs of the eigenvalues alternate, that their absolute values increase as a power function, and that the signs of the A_j are the opposite of those of the λ_j. It is very natural to assume that the same situation holds for $\sigma \approx 1$.

3. SPECTRUM BELOW THE THRESHOLD IN THE LOWEST LANDAU ZONE

We change in Eqs. (1.27) and (1.24) to the dimensionless variables w and u and introduce the dimensionless magnetic momentum $u_c = p_c / p_0$. We also take into account that as we are interested in the region $w < 0$, we have $\lambda > 0$. The equation for $n = 0$ can then be written in the form

$$\bar{\Gamma}(\sigma, \lambda) = \frac{a_l(u)}{\lambda} + \frac{\bar{\alpha}^2}{(1+u^2)^3} \frac{1}{\lambda^3}, \tag{3.1}$$

where

$$a_l(u) = -\frac{1 - u^2 - l u_c^2}{1+u^2}, \qquad \sigma = \frac{1+u^2}{u_c^2} \tag{3.2}$$

and

$$w = -\bar{\alpha}^2 / \lambda^2 (1+u^2)^2. \tag{3.3}$$

Equation (3.1) determines the roots λ which, after substitution into (3.3), give the different branches of the spectrum.

It is convenient to consider Eq. (3.1) graphically. The graph of $\bar{\Gamma}$ consists of one isolated branch and an infinite number of "tangent-like" branches, shown by the full drawn curve in Fig. 3. The dashed curves 1 and 2 in the figure show the graph of the right-hand side of the equation for $a_l > 0$ and $a_l < 0$. From this it is clear that there is a root $\lambda^{(0)}$ from the intersection with the isolated branch and an infinite set of roots $\lambda^{(r)}$, $r = 1, 2, \ldots$ from the intersections with the tangent-like branches. It is also clear from the figure that the roots with $r = 1, 2, \ldots$ are not small, i.e., $\lambda^{(r)} \gtrsim 1$ and therefore the second term on the right-hand side of (3.1) is unimportant for their determination. The order of magnitude of the root $r = 0$ depends on the sign of a_l. If $E_l(0) > \epsilon_0$ i.e., $1 - l u_c^2 < 0$, $a_l(u) > 0$ for all u and then $\lambda^{(0)} \gtrsim 1$ and the second term on the right-hand side is unimportant. However, if $E_l(0) < \epsilon_0$, $a_l(u)$ changes sign when $u = u_l \equiv \sqrt{(1 - l u_c^2)}$ which corresponds to the threshold momentum for the emission of a phonon by an electron in the Landau zone of number l. When $u > u_l$ we have $a_l(u) > 0$ and, as before, $\lambda^{(0)} \gtrsim 1$. If, however, $u \le u_l$, $a_l(u) \le 0$ and then the intersection with the isolated branch determines the second term on the right-hand side and because it is small $\lambda^{(0)} \ll 1$. In that case we may assume that $\bar{\Gamma} = 1$ for all σ and we find

$$\lambda^{(0)} = \frac{\bar{\alpha}}{|a_l(u)|^{1/2} (1+u^2)^{3/2}} \text{ when } u < u_l, \quad |u^2 - u_l^2| \gg \bar{\alpha}^{2/3}. \tag{3.4}$$

$$\lambda^{(0)} = \frac{\tilde{\alpha}^{2/3}}{1+u_l^2} \qquad \text{when } |u^2-u_l^2| \ll \tilde{\alpha}^{2/3}. \tag{3.5}$$

Substitution of (3.4) into (3.3) gives the spectrum

$$w_0(u) = -1 + lu_c^2 + u^2, \tag{3.6}$$

which after changing to quantities with dimensions can be seen to be the same as the spectrum of a free electron. As far as the branches of the spectrum which are determined by the roots $\lambda \gtrsim 1$ are concerned, they lie close to the energy threshold. To obtain a more complete picture of these branches we consider the cases of weak and strong magnetic fields.

Weak Magnetic Field ($\omega_c \ll \omega_0$)

In this case $\sigma \gg 1$ for all u and we must use (2.36). As all A_j with $j \neq 0$ are small, $\bar{\Gamma}$ will almost everywhere, except in small intervals in the vicinity of the poles λ_{2r}, r = 1, 2, ..., be equal to $-(\lambda-1)^{-1}$.

We study first the determination of the roots $\lambda^{(r)}$ with r = 1, 2, ... It is clear from the figure that for all u except the very small ones these roots are almost the same as the poles, i.e., $\lambda^{(r)} \approx \lambda_{2r}$, r = 1, 2, ... For small u the right-hand side of Eq. (3.1) is close to $-\lambda^{-1}$ and in the region $\lambda \gg 1$ differs little from the left-hand side and it is therefore necessary to consider it more carefully. Using for $\bar{\Gamma}$ the representation (2.30) we have the equation

$$f(\lambda) = -\lambda(2u^2 + lu_c^2). \tag{3.7}$$

Using (2.34) and (2.35) we can find the roots of this equation

when $l \neq 0$:

$$\lambda \approx b_{2r}, \quad r = 1, 2, \ldots;$$

when $l = 0$:

$$\lambda \approx a_{2r}, \quad u \ll (u_c)^r, \quad r = 1, 2, \ldots, \tag{3.8}$$

We need find the root $\lambda^{(0)}$ only for $u > u_l$. As it does not lie close to the poles λ_{2r}, r = 1, 2, ..., we easily find that $\lambda^{(0)} = (u^2-1)/2u^2$. Knowing the roots $\lambda^{(r)}$ we can write out the spectrum. For r = 0 and all l we have:

$$w_0(u) = -1 + lu_c^2 + u^2, \quad u < u_l,$$
$$w_0(u) = -\alpha^2 \frac{u_c^4 u^4}{(u^4-1)^2}, \quad u > u_l; \tag{3.9}$$

for r = 1, 2, ...

when $l \neq 0$,

$$w_r(u) = -\alpha^2 \frac{[(2r)!]^2}{4}\left(\frac{u_c^2}{1+u^2}\right)^{4r+2},$$

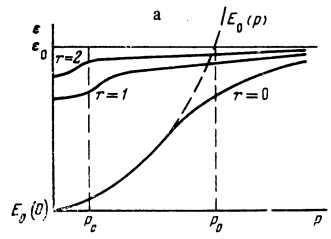
when $l = 0$

$$w_r(u) = -\alpha^2 \frac{[(2r)!]^2}{4}\left(\frac{2r+1}{2r}\right)^2 (u_c^2)^{4r}, \quad u \ll (u_c)^r,$$
$$w_r(u) = -\alpha^2 \frac{[(2r)!]^2}{4}\left(\frac{u_c^2}{1+u^2}\right)^{4r+2}, \quad u \gg (u_c)^r. \tag{3.10}$$

The nature of the spectrum is illustrated by Fig. 4, where we show an additional rearrangement of the spectrum in the small momentum region for $l = 0$.

Strong Magnetic Field ($\omega_c \gg \omega_0$)

In this case $\sigma \ll 1$ when $u \ll u_c$ and $\sigma \gg 1$ when $u \gg u_c$. We consider first the first range of momenta.



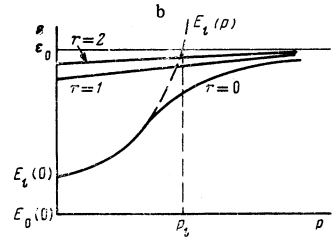
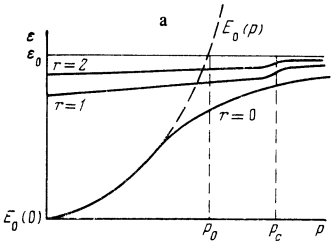


FIG. 4. Electron spectrum in a weak magnetic field: a: $l = 0$; b: $l \neq 0$.



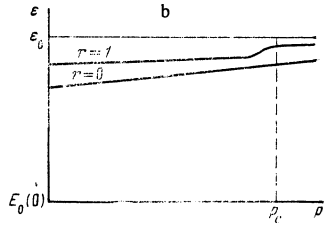


FIG. 5. Electron spectrum in a strong magnetic field: a: $l = 0$; b: $l \neq 0$.

When $l = 0$, it is impossible to find an analytical expression for the roots $\lambda^{(r)}$. We can only see that for all u in that region

$$\lambda_{2r-2} < \lambda^{(r)} < \lambda_{2r}, \quad r = 1, 2, \ldots \qquad \lambda^{(0)} < 1. \tag{3.11}$$

A different situation occurs when $l \neq 0$. In that case the right-hand side of Eq. (3.1) contains the large parameter u_c^2 and the roots are therefore close to the poles, i.e., $\lambda^{(r)} \approx \lambda_{2r}$, r = 1, 2, ...

We now turn to the second range of momenta. Using the results obtained when considering a weak field we have for all l

$$\lambda^{(r)} = b_{2r}, \quad r = 1, 2, \ldots, \qquad \lambda^{(0)} = 1/2. \tag{3.12}$$

The nature of the spectrum is illustrated by Fig. 5 in which it is shown that in the region of large momenta (of the order of the magnetic momentum) there is an additional rearrangement of the spectrum for the branches r = 1, 2, ... which, as follows from (3.12) with increasing number approach the asymptote very closely.

On the basis of our study of the spectrum in strong and weak fields we can give a qualitative picture of the spectrum for any H. In the region below the threshold when $\epsilon < \epsilon_0$ the spectrum is undamped and consists of two kinds of branches.

The first kind of branch is for a given l important when the bottom of the Landau zone of number l lies below the threshold, i.e., $E_l(0) < \epsilon_0$; the equation $E_l(p) = \epsilon_0$ then determines the threshold momentum p_l. The spectrum of this kind of branches is for $p < p_l$ in the lowest approximation used by us in the coupling constant α the same as the free electron spectrum. In the next approximation, of course, there occurs a correction of order α which can be evaluated using perturbation theory but it does not change the qualitative nature of the spectrum in this range of momenta. When p approaches p_l there begins an essential rearrangement of the spectrum. The true spectrum does not reach the threshold energy ϵ_0 at $p = p_l$ but remains below it by an amount of order $\alpha^{2/3}$. When $p > p_l$ the spectrum asymptotically approaches the threshold remaining below it by an amount of order α_l. Even if it exists there is for a given l only one such branch.

The second kind of branches lies entirely, i.e., for all momenta, near the threshold. Such branches lie for $p = 0$ below the threshold by an amount of the order of α^2 and approach the threshold as $p \to \infty$. For a given l there are infinitely many branches of this kind and they form a sequence bunching towards the threshold. Branches of the second kind can, at least for $p < p_l$, be interpreted as bound electron-phonon states. The following facts speak in favor of this interpretation. It was clear when calculating the mass operator that for an energy close to the threshold the main contribution was given by an electron state with $k = 0$ and $n = 0$, i.e., by a state at the bottom of the lowest Landau zone with energy $E_0(0)$. Since the energy of an elementary excitation can be written in the form

$$\varepsilon(p) = E_0(0) + \omega_0 - W(p), \tag{3.13}$$

where W is a small positive quantity, proportional to α^2, it is natural to interpret the energy ϵ as the sum of the energies of a free electron and of a free phonon with a deficit W which is the electron-phonon binding energy. The momentum of the elementary excitation p is almost completely due to the phonon as the electron momentum $k \approx 0$.

A different situation occurs for branches of the first kind, more precisely, for those sections of them which lie far from the threshold in energy. Since perturbation theory is valid along those sections the main contribution is made by electron states with $k = p$ and $n = l$, i.e., the contributions of the phonons to the energy and momentum of the elementary excitation are small. We shall call such excitations polaron excitations.

Since the phonon is a particle the number of which is not conserved the concept of a bound state is not completely rigorous. However, this concept is sufficiently sharp if the total energy and momentum of the electron and the phonon lie sufficiently far from the mass surface of the electron since in that case the conservation laws make it impossible for the phonon to disappear. In other words, a spectral branch can be interpreted as a bound electron-phonon state if it lies far from a branch which can be interpreted as an electron. This clearly occurs when $p < p_l$. When $p > p_l$ such an important difference between first and second kind branches is absent and the interpretation in that region is therefore not so unambiguous.

At most interest is the binding energy for $p = 0$. As to order of magnitude we have

$$W(0) \approx \alpha^2\omega_0(\omega_c/\omega_0)^6, \quad \omega_c \ll \omega_0,$$
$$W(0) \approx \alpha^2\omega_0(\omega_c/\omega_0)^2, \quad \omega_c \gg \omega_0. \tag{3.14}$$

From this it is clear that the binding energy depends strongly on H especially in the weak field region.

We must note that the appearance of bound states is not evident. First of all, it is a priori unclear whether the electron-phonon interaction is "repulsive" or "attractive." Secondly, even in the "attractive" case a weak interaction does not guarantee the existence of a bound state in a three-dimensional problem. From this point of view the role of the magnetic field consists in the production of "quasi-one-dimensionality." This terminology emphasizes the fact that the appearance of bound states is not a trivial result of the transformation of a three-dimensional shallow well to a one-dimensional one, as in that case there must always appear one level.[13]

The case of a resonance magnitude of the magnetic field requires special consideration, when $E_l(0) = \epsilon_0$, i.e., when a sharp difference between first and second kind branches vanishes. Such a situation occurs for $l = 1$, for instance, in the effects of the splitting of the absorption peak.[1] We consider now in our scheme the lowest branch with $l = 1$ paying particular attention to the level $p = 0$. When $\omega_c < \omega_0$ this is a first kind branch and the level $p = 0$ is the bottom of the electron (more precisely, the polaron) zone which is raised with increasing H. When $\omega_c > \omega_0$ there are no branches of the first kind with $l = 1$. Therefore, when H crosses a resonance value from the side of lower fields the branch under consideration becomes a second kind branch, i.e., a polaron state with $p = 0$ becomes a bound electron-phonon state with $p = 0$. In [14], attention was already drawn to the fact that a state with $l = 1$ and $p = 0$ for $\omega_c = \omega_0$ is connected with bound electron-phonon states.

We note also that from the point of view of Eq. (3.1) the resonance case for $l = 1$ and $p = 0$ is characterized by the fact that $a_1 = 0$. Therefore $\lambda^{(0)} \approx \tilde{\alpha}^{2/3}$ and $|w| \approx \tilde{\alpha}^{2/3}$. This means that in a resonance field the lowest level with $l = 1$ turns out to lie in the region where perturbation theory is valid. For this reason the contribution from the phonons to that state is small even though it is larger than the contribution in the normal polaron state. This is clear from the fact that the shift in the level $E_1(0)$ is of order $\tilde{\alpha}^{2/3}\omega_0$ and not $\tilde{\alpha}\omega_0$ as in the usual non-resonance level.

The authors express their gratitude to L. P. Pitaevskiĭ and É. I. Rashba for manifold discussions both directly concerned with the present paper and also about problems connected with it.

APPENDIX 1

The sum (2.1) of interest to us can be written in the form

$$S^l(\sigma, z) = \sigma z^{-\sigma} \int_0^z dx\, x^{\sigma-1} \sum_{s=0}^{\infty} \frac{1}{2^s s!} J_{l+s}(z) x^s. \tag{A.1}$$

After this it can be summed using the expansion[15]

$$\sqrt{(z+h)}^{-l} J_{-l}(\sqrt{z+h}) = \sum_{m=0} \frac{1}{m!} \left(\frac{h}{2}\right)^m z^{-(l-m)/2} J_{-l-m}(\sqrt{z}), \quad |h| < |z|, \tag{A.2}$$

which after changing the variables becomes

$$\sum_{m=0}^{\infty}\frac{1}{2^m m!}v^m J_{l+m}(t)=t^l(t^2-tv)^{-l/2}J_l(\sqrt{t^2-tv}),\quad |v|<|t|. \quad (A.3)$$

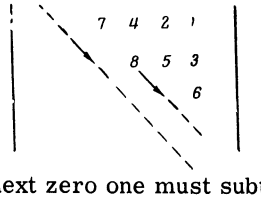
Using (A.3) we have

$$S^l(\sigma,z)=\sigma\int_0^1 d\xi\,\xi^{\sigma-1}(1-\xi)^{-l/2}J_l(z\sqrt{1-\xi}). \quad (A.4)$$

Expanding J_l here into a power series and integrating over ξ we get (2.3).

APPENDIX 2

We can calculate the determinants Δ_0 and Δ' by linear transformations of the columns in which all elements above the main diagonal progressively are made equal to zero. The order of the generation of the zeroes is shown in the following scheme:



For forming the next zero one must subtract from the column in which the zero must be generated the preceding column with the necessary factor.

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Translated by D. ter Haar
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