SCATTERING OF A STRONG WAVE BY AN ELECTRON IN A MAGNETIC FIELD

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The problem of scattering of a strong electromagnetic wave ($E > mc\omega/e$) by a plasma electron is considered, the circularly polarized wave propagating along a constant magnetic field. Taking into account radiation reaction, the scattering cross section is determined as a function of the amplitude of the wave and the strength of the magnetic field. Conditions for wave propagation in a plasma are derived.

THE problem of motion of an electron in a strong electromagnetic wave has been considered in detail in several papers.[1-4] The problem of motion of an electron in the presence of a constant magnetic field along the direction of wave propagation has also been considered.[5] In doing this, the authors of the cited papers were obliged to neglect the presence of radiation reaction.

One usually considers the electron to be at rest until the wave arrives. The wave imparts to such an electron a definite velocity along the direction of propagation. If one takes into account the radiation reaction this velocity varies with time, so that, strictly speaking, there is no stationary regime of motion with a well defined scattering cross section.

In a plasma however, the motion of the electrons creates a space charge and a longitudinal electric field which automatically attains a value which ensures that the motion of the electron along the propagation direction ceases. If the problem is posed in this way one finds the asymptotic solution for $t \rightarrow \infty$ of the kinetic equation in which the motion of the electron along the direction of wave propagation is taken into account, together with the field which appears in the plasma due to this.

The problem of scattering on such a semifixed electron[6] is interesting both from the physical and from the methodological points of view, the latter being related to the simplification of the computations. In the classical case such a simplification is attained only when one considers a circularly polarized wave. In this case it is possible to take into account exactly the radiation reaction.

An electron in a field of a circularly polarized wave will move along a circle situated in a plane perpendicular to the propagation direction of the wave. A longitudinal magnetic field will obviously not affect the character of the motion and can be introduced without complications, as done in the present note. For a frequency of the wave close to the cyclotron frequency of the electron phenomena occur which are characteristic for nonlinear systems: "dragging" of the resonance and hysteresis. For given input data: magnetic field and wave amplitude, more than one stationary solution become possible. The realization of one of the two possible stable solutions depends on the history of the switching-on of the wave and the magnetic field.

We note that for relativistic motion of the electron in a strong wave field the problem is nonlinear, and so one cannot transpose our results, which refer to circularly polarized waves, to the case of linearly polarized waves. It is remarkable that some qualitative results are nevertheless general: first, there is the absence of harmonics in the forward scattered wave. This implies that a strong sine wave does not become a shock wave in a plasma, in the same manner as it does not get deformed in any approximation in vacuo.

In addition to the scattering cross section it is interesting to determine the spectral composition of the scattered radiation. The spectrum of the scattered radiation is obtained directly from the theory of synchrotron radiation, since in the latter case the electron is moving in a circle too.

The wave scattered in the forward direction determines the damping and phaseshift of the transmitted wave. The damping depends on the imaginary part of the forward scattering amplitude and is trivially related to the total scattering cross section. The real part of the forward scattering cross section can also be easily computed; it characterizes the index of refraction of a tenuous plasma for a strong wave. We also note that in the case of a weak wave field one naturally obtains the known formulas of the linear theory of scattering by a magnetized electron. It is curious that in the absence of the constant magnetic field an increase of the wave amplitude leads to an increase of the cross section, then it reaches a maximum and later decreases, whereas the index of refraction falls monotonically, approaching unity.

One can glean a certain analogy between the computation carried out in the present paper and the well-known classroom derivation of the formula for the frequency of oscillations of a pendulum by considering a rotation which is equivalent to two perpendicular oscillations which have a phase shift of $\pi/2$.

FORMULATION OF THE PROBLEM AND RESULTS

Thus, let a plane monochromatic electromagnetic wave with circular polarization propagate along a constant magnetic field. A constant electric field is created in the plasma, preventing the electron from moving in the direction of propagation of the wave. The electron moves in a stationary manner along a circle in a

plane perpendicular to the direction of the magnetic field with the frequency of the driving force equal to the frequency ω of the wave. Then the motion of the electron will be determined by only two quantities. One of them is the energy of the electron

$$\varepsilon = \gamma mc^2, \text{ where } \gamma = (1-\beta^2)^{-\frac{1}{2}}, \quad \beta = v/c.$$

The second is the phaseshift φ of the electron relative to the electric field of the wave. Since the electron moves in a stationary manner the energy absorbed by the electron from the wave over 1 second, $evE \sin \varphi$ will all be reemitted.

The total intensity of radiation from an electron moving on a circular orbit with frequency ω and energy ϵ is taken from the theory of synchrotron radiation. The expression for the radiation intensity with all quantities expressed in terms of ω and γ has the form

$$Q = \frac{2}{3} \frac{e^2\omega^2}{c} \beta^2\gamma^4 \text{ [erg/s]} \tag{1}$$

In the relativistic case the effective frequency of the harmonics will be $\omega_m = \omega\gamma^3$ and the radiation will be emitted essentially in the orbit plane. The eigenfrequency of the motion of an electron of energy ϵ in a magnetic field is

$$\omega_0 = eH / \gamma mc = \omega_H / \gamma.$$

If the frequency and intensity of the wave are such that $\omega \approx \omega_H/\gamma$ a resonance phenomenon will occur. This is, of course, valid only for a polarization of the wave which turns the electron in the same direction as the magnetic field. Such a wave is usually called extraordinary. For the opposite circular polarization, i.e., for the ordinary wave, such a phenomenon does not occur. Near the resonance one must take into account the friction forces, which in this case will be the radiation reaction force.
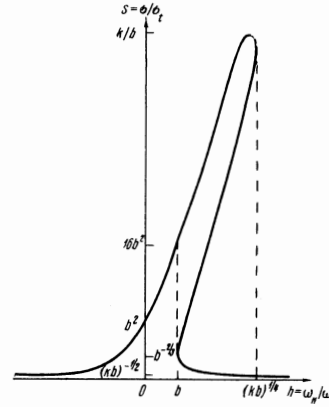
The problem is solved in the classical approximation. One may neglect quantum effects if the following conditions are fulfilled: a) in the rest frame of the electron the longitudinal field is smaller than the critical value $\alpha m^2 c^4/e^3$, where $\alpha = e^2/\hbar c$; b) the energy carried away by the scattered photon is smaller than the electron energy, i.e., $\hbar\omega\gamma^3 < \epsilon$.

We first quote the basic results of [6] which refer to the case when the magnetic field is absent.

We introduce the following notation:

$$b = eE / mc\omega, \quad k = 3\lambdabar / 2r_0,$$
$$s = \sigma / \sigma_t,$$

where E and ω are the field strength and frequency of the wave, r_0 is the classical electron radius, $\lambdabar = c/\omega$, σ is the scattering cross section of radiation, and σ_t the cross section for Thomson scattering. In a strong electromagnetic wave ($b > 1$) the electron becomes relativistic. As the parameter b increases the cross section behaves in the following manner: as long as $b < k^{1/3}$ the cross section increases: $\sigma = \sigma_t(1 + b^2)$; for $b \sim k^{1/3}$ it attains the magnitude $\sigma \approx \sigma_t k^{2/3}$; and in the region $b > k^{1/3}$ the scattering cross section decreases, $\sigma \approx \sigma_t k/b = 4 \pi e/E$. In the region where the cross section decreases the electron moves with velocity c directed antiparallel to the electric field of the wave, i.e., the electron absorbs from the wave the maximally possible energy.

We now consider in more detail the case where there exists a longitudinal magnetic field. The dependence of the total cross section for the scattering of the wave on the magnitude of h = ω_H/ω, where $\omega_H = eH/mc$, will have the shape of a resonance curve, the parameters of which vary dependent on the magnitude of the wave field (cf. the figure). The electron in the wave moves in such a manner that the scattering cross section is closely related to the energy of its motion. As shall be seen in the following this relationship has the form

$$\gamma^2(\gamma^2 - 1) = \sigma b^2 / \sigma_t = s b^2,$$

i.e., in the nonrelativistic case $\gamma \approx 1 + sb^2/2$ and in the relativistic case $\gamma \approx (sb^2)^{1/4}$.

At the maximum of the resonance curve for nonrelativistic ($b < 1/k$) electron motion the scattering cross section corresponds to a maximal cross section for the classical oscillator $\sigma_{max} = \sigma_t/k^2 = 6 \pi \lambdabar$. In the case $b > 1/k$ the motion of the electron at the resonance is relativistic and the cross section at the maximum decreases with increasing magnitude of the field strength of the wave: $\sigma_{max} = \sigma_t k/b$. The maximum of the resonance curve corresponds to a motion of the electron when its velocity is directed against the E field of the wave, i.e., the wave gives up to the electron the maximally possible energy ecE[erg/s] in the relativistic case and $ecEkb$ [erg/s] for nonrelativistic motion. The nonrelativistic resonance occurs when $\omega_H/\omega \approx 1$ and the relativistic one for $\omega_H/\omega \approx (kb)^{1/4}$. The region where $\omega_H/\omega < 0$ corresponds to motion of the electron in the ordinary wave, whereas $\omega_H/\omega > 0$ to the motion in the extraordinary wave. On both wings of the resonance curve the scattering cross section decreases like $\sigma_t(\omega_H/\omega)^{-2}$. These "wings" correspond to nonrelativistic motion of the electron even when the wave is strong ($b > 1$).

For $b > (3k^3)^{1/2}$ there is a well-defined region of magnetic fields for which three different energies of motion are possible, i.e., there are three cross sections. The motion corresponding to the middle one is unstable. The real motion of the electron is described either by the largest value of the cross section or by the smallest one. If the magnetic field is increased slowly starting with $H \approx 0$ then at a point near the resonance the amplitude "jumps" and the scattering cross section falls sharply. The sharpness of this effect depends on the speed with which the stationary motion of the electron and the longitudinal electric field in the

plasma are established. If, conversely, one reduces the magnetic field starting from large values, a sharp increase in the cross section is possible. In the interesting case $1 < b < k^{1/3}$ the region of magnetic field strengths for which there are two scattering cross sections is sufficiently wide: $b < \omega_H/\omega < (kb)^{1/4}$. The scattering cross section can fall off near the resonance for $\omega_H/\omega = (kb)^{1/4}$ from the magnitude $\sigma_t k/b$ to $\sigma_t(kb)^{-1/2}$, and where $\omega_H/\omega = b$, the cross section can increase from $b^{-2/3}\sigma_t$ to $16b^2\sigma_t$.

For $b > k^{1/3}$ the resonance curve degenerates. For the region $|\omega_H/\omega| < b$ the magnetic field is inessential and there we have everywhere $\sigma \approx \sigma_t k/b$. In the region of large magnetic fields $|\omega_H/\omega| > b$ the scattering cross section decreases with the increase of the parameter ω_H/ω like $\sigma_t(\omega_H/\omega)^{-2}$.

THE EQUATION OF MOTION

The equation for the centrifugal force acting on the electron, i.e., the equation of motion, projected on the perpendicular to the velocity in the plane of motion will have the form

$$\omega p = \pm eH\beta + eE\cos\varphi, \qquad (2)$$

where the plus sign refers to the extraordinary wave for which $\omega_H/\omega > 0$, and the minus sign corresponds to the ordinary wave; φ is the angle between E and the radius-vector r of the electron.

For stationary motion of the electron the energy acquired by the electron from the wave is also reemitted by it. Using Eq. (1), we thus obtain

$$Q = evE\sin\varphi = \frac{2}{3}\frac{e^2\omega^2}{c}\beta^2\gamma^4 \quad [\text{эрг/сек}]. \qquad (3)$$

The total scattering cross section is defined as

$$\sigma = \frac{Q}{cE^2/4\pi}. \qquad (4)$$

Making use of Eq. (1) one can find that

$$\sigma = \sigma_t\gamma^2(\gamma^2 - 1)/b^2. \qquad (5)$$

We recall that

$$b = \frac{eE}{mc\omega}, \quad k = \frac{3}{2}\frac{\lambda}{r_0}, \quad s = \frac{\sigma}{\sigma_t}.$$

In these notations the equations (2) and (3) can be rewritten as:

$$\gamma\beta = \pm\frac{\omega_H}{\omega}\beta + b\cos\varphi, \quad kb\beta\sin\varphi = \beta^2\gamma^4. \qquad (6)$$

Eliminating the angle φ from the system (6) one finds

$$b^2 = (\gamma^2 - 1)\left[\frac{\gamma^6}{k^2} + \left(1 - \frac{1}{\gamma}\frac{\omega_H}{\omega}\right)^2\right]. \qquad (7)$$

from where it can be seen that for $\omega_H/\omega = \gamma$ a resonance occurs. Now the equations (5) and (7) yield

$$\frac{\omega_H}{\omega} = \gamma\left[1 \pm \left(\frac{b^2}{\gamma^2 - 1} - \frac{\gamma^6}{k^2}\right)^{1/2}\right], \quad \gamma = \left[\frac{1 + (1 + 4sb^2)^{1/2}}{2}\right]^{1/2}, \qquad (8)$$

where the minus sign corresponds to the region to the left of the resonance, and the plus sign to that to the right.

An investigation of the dependence of ω_H/ω on sb^2

gives characteristic resonance curves $s = s(\omega_H/\omega)$ for different values of the amplitude parameter of the wave.

THE PROPAGATION OF A STRONG WAVE IN A PLASMA

Consider a plasma with electron concentration N $[\text{cm}^{-3}]$. The position vector of the electron will be shifted in phase relative to the electric field of the wave

$$r = \frac{v}{\omega}e^{i\varphi}\frac{E}{|E|}.$$

One can define the one-electron contribution to the index of refraction for $N \to 0$:

$$\frac{n^2 - 1}{N} = -\frac{4\pi er}{E} = -\frac{4\pi e(v/\omega)e^{i\varphi}}{E}. \qquad (9)$$

Making use of the equations of the preceding section one can derive the form of the imaginary part of (9). It corresponds to the form of the scattering cross section

$$\text{Im}\frac{n^2 - 1}{N} = -\frac{4\pi ev\sin\varphi}{\omega E} = -\sigma\lambda. \qquad (10)$$

For the real part we obtain

$$\text{Re}\frac{n^2 - 1}{N} = -\frac{4\pi ev\cos\varphi}{\omega E} \approx \begin{cases} \mp\lambda^2(6\pi\sigma - \sigma^2/\lambda^2)^{1/2} \\ \mp\lambda^2\frac{4\pi r_0}{b}\left[1 - \left(\frac{\sigma}{\sigma_t k/b}\right)^2\right]^{1/2}. \end{cases} \qquad (11)$$

Here the upper line gives the expression for the case of a nonrelativistic electron and the lower line—for the relativistic electron; the minus sign corresponds to the region left of the resonance, and plus to the region to the right of it.

In the region where the scattering cross section has two different values, the imaginary part of the index of refraction also becomes ambiguous. Therefore two regimes of wave propagations become possible, depending on whether the wave propagates along a magnetic field which decreases or increases in magnitude.

In a wide region $-b < \omega_H/\omega < b$, $(kb)^{1/4}$ we have for a strong wave

$$\text{Re}(n^2 - 1) = -\frac{1}{b}\frac{4\pi Ne^2}{m\omega^2};$$

this means that with increasing b the index of refraction of a plasma of given density decreases. In other words, this corresponds to a widening of the frequency interval of waves penetrating into the plasma. The condition for propagation of a strong extraordinary wave of frequency ω is more stringent than for a weak wave: $\omega < \omega_H/b$ for propagation along a magnetic field which decreases in magnitude and $\omega < \omega_H/(kb)^{1/4}$ for propagation along a magnetic field which increases (i.e., respectively for the two branches of the resonance curve).

In the absence of the constant magnetic field the formula for the real part of the index of refraction has a simpler form:

$$\text{Re}\frac{n^2 - 1}{N} = \begin{cases} -\lambda^2 4\pi r_0/\sqrt{b^2 + 1} & \text{for } b < k^{1/3} \\ -\lambda^2 4\pi r_0 k^{1/4}b^{-1/4} & \text{for } b > k^{1/3} \end{cases} \qquad (12)$$

In all this we have neglected the contribution of the ions to the index of refraction.

We now discuss the problem of containment of electrons by the plasma field. If a constant electric field is

formed in the plasma, balancing the force of radiation pressure, then the electron will not be accelerated by that pressure and its motion will be stationary. The force acting on the electron is obviously equal to eE_z. Considering the Lorentz force makes it quite obvious that the following identity holds: $F_z = eE_z = Q/c$, where Q is the power scattered by the electron. This is understandable, since the radiation is scattered symmetrically and does not carry away momentum. At the same time the incident wave from which the energy is absorbed has a definite direction so that in taking away energy from the wave the electron must also take away the appropriate quantity of momentum. For a constant electric field we have the equation

$$\frac{dE_z}{dz} = \frac{d(\sigma E^2/4\pi)}{e\,dz} = 4\pi\rho e, \qquad (13)$$

where ρ is the charge density in the plasma. The plasma must also contain positively charged particles, and therefore the following inequality should be obvious: $\rho < Ne$, where N is the electron centration. This yields a limiting condition which guarantees containment of the electron by the plasma field

$$\sigma E^2 d(\sigma E^2)\,/\,dE^2 < (4\pi e)^2, \qquad (14)$$

or, in relative units,

$$sb^2 d(sb^2)\,/\,db^2 < k^2. \qquad (15)$$

We note that near the resonance $sb^2 d(sb^2)/db^2 \lesssim k^2$, but in the region far from the resonance the criterion for electron containment by the plasma field is well satisfied.

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