QUANTUM MICROSCOPIC THEORY OF RADIATION BY A CHARGED PARTICLE MOVING UNIFORMLY IN A CRYSTAL

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A quantum microscopic theory of radiation by a uniformly moving charged particle traversing matter is constructed. For wavelengths much greater than the interatomic distances the theory yields a solution identical to the corresponding formula of macroscopic electrodynamics. A solution of the fundamental equation of the theory is obtained and analyzed for wavelengths smaller than the interatomic distances. It is shown that two types of radiation are emitted during the motion of an ultrarelativistic charge along one of the principal crystallographic axes. The first type (dynamic radiation) is emitted at Bragg angles and is the diffracted part of the ultrarelativistic particle field in the crystal. This radiation significantly depends on the energy of the particle. The second type is transition radiation produced by the charged particle in the atomic planes of the crystal and its intensity depends only on the crystal parameters.

1. INTRODUCTION

IT is well known that a uniformly moving charged particle emits transition radiation on crossing the interface between two media.[1] If the particle is ultrarelativistic, then the forward radiation contains, besides optical frequencies, frequencies right up to values $\sim\omega_0 E/\mu c^2$, where ω_0 is the plasma frequency of the material, while E and μc^2 are the total energy and rest energy of the particle.[2,3] It is clear from this that quanta of wavelength much smaller than interatomic distances will be produced. It is obvious that in the last case a description of the processes with the aid of macroscopic electrodynamics is not admissible and it is necessary to take into account the microscopic structure of the material. The problem was solved in [4] in the case when the atoms of the medium are considered as homogeneous spheres described by a dielectric constant of the form $\epsilon(\omega) = 1 - \omega_0^2/\omega^2$ (where ω_0 is determined by the electron density in an atom).

In the present paper this problem is investigated with the aid of a theory based on the quantum microscopic theory developed by Afanas'ev and Kagan[5] for the description of x-ray and neutron diffraction in crystals. We formulate with the aid of this theory a basic equation which defines for the entire frequency spectrum the total field due to the motion of the charged particle in the material medium. It is shown that for wavelengths much greater than the interatomic distances the solution of this equation coincides with the usual expression that is obtained in electrodynamics of continuous media for the field of a charged particle. For wavelengths less than the interatomic distances a solution to the basic integral equation is obtained in the first approximation for the case when the trajectory of the charge coincides with one of the principal crystallographic axes of the crystal. This solution is then analyzed. It is shown that the emitted radiation can, by its nature, be split into two types.

In the first case the mechanism underlying the origin of the radiation is due to the scattering of the hard part of the particle field in the crystal, in analogy with the phenomenon of x-ray diffraction in a crystal. The resulting pattern of radiation, which it is natural to call "dynamic radiation",[4] is a peculiar Laue diffraction pattern, the intensity and angular width of whose spots depend on the particle energy. The radiation obtained at the central spot coincides in order of magnitude with the radiation calculated in [4]. It has recently been experimentally detected.[6]

In the second case there arises the transition radiation of a charged particle in the atomic planes of the crystal. The intensity of this radiation is, because of the close spacing of the atomic planes, independent of the particle energy and is determined solely by the parameters of the crystal. In the case of ultrarelativistic particles the intensity of the dynamic radiation exceeds that of the transition radiation in the atomic planes.

Similar problems have been considered in [7-10]. In [7] the author investigates the interesting possibility of x-ray emission in a single crystal, but by nonrelativistic electrons. Relativistic electrons are considered in [8,9], but in [8] the problem is solved for an infinite amorphous medium while in [9] the author considers the radiation in a single crystal in the range of photon energies higher than the electron rest energy. The paper [10], in which the passage of a particle through a periodic medium is investigated, is of a phenomenological character and dynamic radiation is not obtained since to all intents and purposes a crystal of infinite thickness is considered (see the end of Sec. 3 of the present paper, as well as [11]).

2. BASIC THEORY

Let us for the description of the electromagnetic field in matter use the microscopic Maxwell equations. For the spatial and time Fourier

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{E}(\mathbf{k},\omega) - \mathbf{k}(\mathbf{k}\mathbf{E}(\mathbf{k},\omega)) = i\frac{4\pi\omega}{c^2}\mathbf{j}(\mathbf{k},\omega), \qquad (1)$$

where $\mathbf{j}(\mathbf{k}, \omega)$ is the Fourier transform of the total current density. This current in the general case consists of a free current and a current of bound (or induced) charges:

$$\mathbf{j}(\mathbf{k}, \omega) = \mathbf{j}_{\text{free}}(\mathbf{k}, \omega) + \mathbf{j}_{\text{in}}(\mathbf{k}, \omega).$$

The induced current is the quantum-mechanical average of the induced current density operator.[5] In the general case this operator has the form

$$\hat{\mathbf{j}}_{\text{in}}(\mathbf{r}) = \frac{e}{2m}\sum_i \left[\left(\hat{\mathbf{P}}_i - \frac{e}{c}\hat{\mathbf{A}}(\mathbf{r}_i)\right)\delta(\mathbf{r}-\mathbf{r}_i) + \delta(\mathbf{r}-\mathbf{r}_i)\left(\hat{\mathbf{P}}_i - \frac{e}{c}\hat{\mathbf{A}}(\mathbf{r}_i)\right)\right],$$

where the summation is over all the electrons of the medium, $\hat{\mathbf{P}}_i$ is the momentum operator of the i-th electron, and e and m are the charge and rest mass of the electron.

Following [5], we split the vector potential $\hat{\mathbf{A}}(\mathbf{r})$ into two parts $\mathbf{A}_0$ and $\mathbf{A}_1$, where $\mathbf{A}_0$ describes the zero-point oscillations of the electromagnetic field and is discussed in the framework of quantum mechanics, while $\mathbf{A}_1$ describes the external field and is discussed classically. If we assume the scalar potential to be zero, then $\mathbf{A}_1(\mathbf{k}, \omega)$ and the Fourier transform of the external electric field $\mathbf{E}(\mathbf{k}, \omega)$ are coupled through the relation

$$\mathbf{E}(\mathbf{k}, \omega) = (i\omega/c)\mathbf{A}_1(\mathbf{k}, \omega).$$

It is clear that in the problem of the passage of a charged particle through matter the field will consist of two parts: the field directly connected with the charged particle ($\mathbf{E}_{\text{ch}}$), and the field arising as a result of the interaction with the matter ($\mathbf{E}_{\text{sc}}$):

$$\mathbf{E}(\mathbf{k}, \omega) = \mathbf{E}_{\text{ch}}(\mathbf{k}, \omega) + \mathbf{E}_{\text{sc}}(\mathbf{k}, \omega). \tag{2}$$

Correspondingly $\mathbf{A}_1$ will also consist of two parts:

$$\mathbf{A}_1(\mathbf{k}, \omega) = \mathbf{A}_{1\,\text{ch}}(\mathbf{k}, \omega) + \mathbf{A}_{1\,\text{sc}}(\mathbf{k}, \omega).$$

The field of the charge $\mathbf{E}_{\text{ch}}$ is known. It is determined from the equation

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{E}_{\text{ch}}(\mathbf{k}, \omega) - \mathbf{k}(\mathbf{k}\mathbf{E}_{\text{ch}}(\mathbf{k}, \omega)) = i\frac{4\pi\omega}{c^2}\mathbf{j}_{\text{free}}(\mathbf{k}, \omega). \tag{3}$$

In the case of a uniformly moving particle $\mathbf{j}_{\text{free}}(\mathbf{k}, \omega) = 2\pi e\mathbf{v}\delta(\omega - \mathbf{k}\cdot\mathbf{v})$ and the solution of this equation has the form

$$\mathbf{E}_{\text{ch}}(\mathbf{k}, \omega) = 8\pi^2 ei\frac{\omega\mathbf{v} - \mathbf{k}c^2}{k^2c^2 - \omega^2}\delta(\omega - \mathbf{k}\mathbf{v}). \tag{4}$$

Our problem is to find the scattered field $\mathbf{E}_{\text{sc}}$. For this purpose we should use Eq. (1), in which $\mathbf{E}(\mathbf{k}, \omega)$ is defined according to (2) and $\mathbf{j}_{\text{in}}$ depends on both the field of the charge and the scattered field. Since the field of the charge obeys Eq. (3), we obtain from Eq. (1) the following equation for $\mathbf{E}_{\text{sc}}$:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{E}_{\text{sc}}(\mathbf{k}, \omega) - \mathbf{k}(\mathbf{k}\mathbf{E}_{\text{sc}}(\mathbf{k}, \omega)) = i\frac{4\pi\omega}{c^2}\mathbf{j}_{\text{in}}(\mathbf{k}, \omega). \tag{5}$$

Thus, we have a self-consistent formulation of the problem in which the scattered field is determined by means of $\mathbf{j}_{\text{in}}$, while $\mathbf{j}_{\text{in}}$ itself, in its turn, depends besides the field of the charge on the scattered field.

Furthermore, we must find an explicit expression for $\mathbf{j}_{\text{in}}$. Using the method of [5] and assuming that the medium is a crystal whose primitive cell has a volume Ω_0, we obtain

$$j^i_{\text{in}}(\mathbf{k}, \omega) = \frac{\omega}{4\pi i}\frac{\Omega_0}{(2\pi)^3}\int\sum_n e^{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n} g^{ij}(\mathbf{k}, \mathbf{k}')[E^j_{\text{sc}}(\mathbf{k}', \omega) + E^j_{\text{ch}}(\mathbf{k}', \omega)]d\mathbf{k}', \tag{6}$$

where the n-summation is over the locations of all the primitive cells of the crystal, and the tensor $g^{lj}(\mathbf{k}, \mathbf{k}')$ is determined by the formula

$$g^{ij}(\mathbf{k}, \mathbf{k}') = -\frac{\omega_0^2}{\omega^2}\sum_q \exp\{i(\mathbf{k}'-\mathbf{k})\boldsymbol{\rho}_q - M_q(|\mathbf{k}'-\mathbf{k}|)\} \times [\delta^{ij}f_q(|\mathbf{k}'-\mathbf{k}|) + f_{2q}{}^{ij}(\mathbf{k}, \mathbf{k}') + f_{3q}{}^{ij}(\mathbf{k}, \mathbf{k}')]/Z_{\text{cell}} \tag{7}$$

Here $\omega_0^2 = 4\pi Ne^2/m$, N is the number of electrons in a unit volume of the crystal, δ^{lj} is the Kronecker symbol, Z_{cell} is the number of electrons in a primitive cell, $\boldsymbol{\rho}_q$ is the location of the q-th atom in the primitive cell, $M_q(|\mathbf{k}'-\mathbf{k}|)$ determines the Debye-Waller factor, $f_q(|\mathbf{k}'-\mathbf{k}|)$ is the atomic form factor of the q-th atom and the term that corresponds to it in the formula (7) describes the Thomson scattering of photons by atoms, and $f^{lj}_{2q_1}$ and f^{lj}_{3q} give the contributions of the photoelectric and Compton effects and are equal to:

$$f_{2q}{}^{ij}(\mathbf{k}, \mathbf{k}') = \frac{2m}{e^2}\sum_\mu \frac{\langle j_{1q}{}^i(\mathbf{k})\rangle_{0\mu}\langle j_{1q}{}^{j+}(\mathbf{k}')\rangle_{\mu 0}}{\omega^2 - \omega_{0\mu}{}^2 + 2i\omega\delta}\omega_{0\mu}, \tag{8}$$

$$f_{3q}{}^{ij}(\mathbf{k}, \mathbf{k}') = \frac{2e^2}{mc^2}\sum_s{}'\sum_{i,i'}\sum_{\mathbf{k}_1}[\langle\exp\{i(\mathbf{k}_1-\mathbf{k})\mathbf{r}_i\}\hat{A}_0{}^i(\mathbf{k}_1)\rangle_{0s} \cdot\langle\exp\{i(\mathbf{k}'-\mathbf{k}_1)\mathbf{r}_{i'}\}\hat{A}_0{}^{j+}(\mathbf{k}_1)\rangle_{s0}\omega_{s0}/(\omega^2 - \omega_{s0}{}^2 + 2i\omega\delta)]. \tag{9}$$

The indices μ and s in these formulas label intermediate states, while the current operator appearing in formula (8) is given by

$$\hat{\mathbf{j}}_{1q}(\mathbf{k}) = \frac{e}{2m}\sum_i (e^{-i\mathbf{k}\mathbf{r}_i}\hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i e^{-i\mathbf{k}\mathbf{r}_i}),$$

the summation in this formula being over all the electrons of the q-th atom of a primitive cell. The quantity f^{lj}_{3q} also contains a contribution due to the scattering of light with absorption and emission of phonons (see [5]).

We have in formula (7) carried out an averaging over the thermal vibrations of the lattice atoms and obtained the Debye-Waller factor, although such an averaging should have been carried out in the expression for the energy flux. The result of such an averaging would have been to obtain, apart from a term with the square of the Debye-Waller factor, a second term giving a diffusion background.[12] The ratio of this last term to the first, as has been shown in [12], decreases with increase in the dimensions of the crystal and, for a crystal with dimensions of the order of 10^{-4} cm, this ratio $\sim 10^{-4}$ at room temperature. Consequently, if we limit ourselves to crystals at not very high temperatures and with dimensions not smaller than the above-cited value, we can neglect diffuse scattering and carry out at once

in (7) the averaging over the thermal vibrations of the lattice atoms.

Using (6), we can rewrite Eq. (5) in a somewhat different form. To this end, we scalar multiply both sides of Eq. (5) by $\mathbf{k}$ and substitute the expression obtained for $\mathbf{k}\cdot\mathbf{E}_{sc}(\mathbf{k},\omega)$ into (5). We obtain as a result a basic equation for the determination of the field of a charged particle in a crystal:

$$\left(k^2-\frac{\omega^2}{c^2}\right)E^l_{sc}(\mathbf{k},\omega) \tag{10}$$
$$=\frac{\Omega_0}{(2\pi)^3}\int\sum_n e^{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n}\left[\frac{\omega^2}{c^2}g^{lj}(\mathbf{k},\mathbf{k}')-k^lk^qg^{qj}(\mathbf{k},\mathbf{k}')\right]$$
$$\times[E_{ch}(\mathbf{k}',\omega)+E_{sc}(\mathbf{k}',\omega)]d\mathbf{k}'.$$

We see that this is a system of linear inhomogeneous integral equations. It is impossible to write down its solution in a general form. Therefore, we shall solve these equations below for the case of interest, namely, for the case of short wavelengths, using an iterative procedure.

However, before proceeding to this case, let us consider the opposite limiting case and verify that when the wavelength of the radiation is much greater than interatomic distances, an exact solution to Eqs. (10) can be found and that this solution coincides with the usual expression which follows from macroscopic electrodynamics.

To this end, let us note that in the case of an infinite crystal

$$\frac{\Omega_0}{(2\pi)^3}\sum_n e^{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n}=\sum_h\delta(\mathbf{k}'-\mathbf{k}_h), \tag{11}$$

where $\mathbf{k}_h=\mathbf{k}+\mathbf{K}_h$, and $\mathbf{K}_h$ is the reciprocal lattice vector multiplied by 2π. In the region of optical frequencies we should restrict ourselves to only the $h=0$ term in the resulting sum on the right hand side of formula (11), since $\mathbf{K}_1\gg\mathbf{k}$ (for velocities of the primary charge much higher than the velocities of the atomic electrons) and, consequently, $k_1\gg k$, i.e., we fall outside the optical limits if we retain $h\neq 0$ terms.

Thus, for optical frequencies Eq. (10) will have the form

$$\left(k^2-\frac{\omega^2}{c^2}\right)E^l_{sc}(\mathbf{k},\omega)=\left(\frac{\omega^2}{c^2}g^{lj}-k^lk^qg^{qj}\right)(E^j_{ch}(\mathbf{k},\omega)+E^j_{sc}(\mathbf{k},\omega)). \tag{12}$$

Let us introduce the dielectric constant $\epsilon^{lj}=\delta^{lj}+g^{lj}$. If, for simplicity, we set $g^{lj}=g\delta^{lj}$, then $\epsilon^{lj}=\epsilon\delta^{lj}$, where

$$\varepsilon=1+g. \tag{13}$$

Using (4), we can solve Eq. (12) for $\mathbf{E}_{sc}$.

The total field in the medium will be equal to the sum of $\mathbf{E}_{sc}$ and the field of the charge (4). After appropriate simple computations we obtain an expression for the total field which coincides with the expression obtained in macroscopic electrodynamics.[13]

Let us now consider the dielectric constant given by formula (13). It is clear that in the optical frequency range we can neglect the term f^{lj}_{3q} in formula (7). Then, assuming, also for simplicity, that there is one atom in a primitive cell, we obtain

$$\varepsilon=1-\frac{\omega_0^2}{\omega^2}\left[1+\sum_\mu\frac{2\omega_{0\mu}|\langle\hat{\mathbf{p}}\rangle_{0\mu}|^2}{Zm(\omega^2-\omega_{0\mu}^2)}\right],\quad \hat{\mathbf{p}}=\frac{m}{e}\hat{\mathbf{j}}, \tag{14}$$

where Z is the number of electrons in an atom. Let us introduce the concept of oscillator strengths $f_{0\mu}=2|\langle\hat{\mathbf{p}}\rangle_{0\mu}|^2/Zm\omega_{0\mu}$, obeying the sum rule $\sum_\mu f_{0\mu}=1$. Then it is not difficult to obtain from (14) the formula

$$\varepsilon(\omega)=1+\omega_0^2\sum_\mu f_{0\mu}/(\omega_{0\mu}^2-\omega^2), \tag{15}$$

which is a well-known formula in optics.

3. THE HARD PART OF AN ULTRARELATIVISTIC-PARTICLE FIELD IN A CRYSTAL

Let us now find the solution to Eq. (10) in the case when we are interested in fields of wavelengths smaller than interatomic distances. In other words, we shall be interested in photons whose energies are such that we can approximately neglect their absorption on account of the photoelectric effect, and assume that Compton scattering occurs without a change in frequency, i.e., we assume that only Thomson scattering on atoms determined by the term $\delta^{lj}f_q(|\mathbf{k}-\mathbf{k}'|)$ in formula (7) occurs. It is not difficult to see that for the light elements such a frequency range corresponds to quanta with energies of a few score kilovolts.

If we again assume that there is one atom in a primitive cell, then Eq. (10) takes the form

$$\left(k^2-\frac{\omega^2}{c^2}\right)E^l_{sc}(\mathbf{k},\omega)=-\frac{\omega_0^2}{Z\omega^2}\frac{\Omega_0}{(2\pi)^3}\int\sum_n\exp\{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n-$$
$$-M(|\mathbf{k}'-\mathbf{k}|)\}\cdot f(|\mathbf{k}'-\mathbf{k}|)[\omega^2c^{-2}\delta^{lj}$$
$$-k^lk^j][E_{ch}(\mathbf{k}'\omega)+E^j_{sc}(\mathbf{k}'\omega)]d\mathbf{k}'. \tag{16}$$

We shall solve this equation by iteration. Let us take as the first iteration that field $\mathbf{E}_{sc}$ which will be obtained from Eq. (16) if $\mathbf{E}_{sc}$ is set equal to zero on the right hand side of this equation. In the event the correction computed in the next iteration turns out to be much smaller than the value for $\mathbf{E}_{sc}$ obtained in the first iteration, we can limit ourselves to this iteration. Otherwise, it is necessary to consider the entire iteration series.

We shall assume that the particle moves along the z-axis which is at the same time one of the principal crystallographic axes of the crystal, while the crystal itself is of infinite extension in the x- and y-directions and occupies the space between the planes $z=-z_0N_3/2$ and $z=z_0N_3/2$, where z_0 is the spacing of the atomic planes in the z-direction. Integrating over $\mathbf{k}'$, we obtain in the first iteration

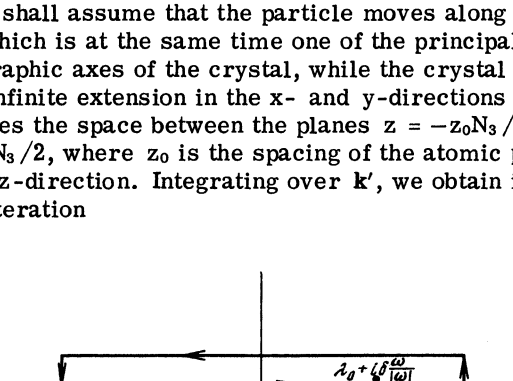

FIG. 1. Contour integration in the complex k_z plane.

$$\mathbf{E}_{sc}^{(1)}(\mathbf{k},\omega)=-\frac{\omega_0^2}{Zc^2}\frac{4\pi iez_0}{v}\frac{\varphi(\omega/v-k_z)}{\varkappa^2+k_z^2-\omega^2/c^2}$$
$$\times\sum_h\frac{\omega v/c^2-\varkappa_h-\omega n_z/v+c^2\mathbf{k}[\varkappa\varkappa_h+k_z\omega(1-\beta^2)/v]/\omega^2}{\varkappa_h^2+\omega^2/v^2-\omega^2/c^2}\cdot e^{-M(L_h(k_z))}f(L_h(k_z)), \quad (17)$$

where

$$\varphi(x)=[\sin(xz_0(N_3+1)/2)]/\sin(xz_0/2), \quad (18)$$

$L_h(x)=[\varkappa_h^2+(\omega/v-x)^2]^{1/2}$; $\varkappa_h$ is the projection of the reciprocal lattice vector (multiplied by 2π) onto the (x, y)-plane; $\kappa_h=\kappa+\varkappa_h$, κ being the component of the vector $\mathbf{k}$ in the (x, y)-plane; and $\mathbf{n}_z$ is the unit vector in the direction of the z-axis.

The field itself will be determined by the following formula:

$$\mathbf{E}_{ch}^{(1)}(\mathbf{r},t)=\frac{1}{(2\pi)^4}\int\mathbf{E}_{ch}^{(1)}(\mathbf{k},\omega)e^{i(\varkappa\rho+k_zz-\omega t)}d\varkappa dk_zd\omega. \quad (19)$$

Let us calculate the asymptotic expression for this field as $z\to\infty$ (far from the crystal in the direction of motion of the charged particle) which is of the greatest physical interest.

Notice that the integrand in (19) $\mathbf{E}_{sc}(\mathbf{k},\omega)$ has on the real axis of the complex k_z plane two poles $k_z=\pm\lambda_0$, where $\lambda_0=(\omega^2/c^2-\kappa^2)^{1/2}$. To evaluate the integral (19) over k_z from $-\infty$ to ∞ along the real axis, we must move these poles by an infinitesimally small amount into the complex plane. In order to obtain the correct asymptotic behavior of the field (19) as $z\to\infty$ (so that the field does not grow without limit) we must for $\omega>0$ move the pole $k_z=+\lambda_0$ into the upper half-plane, and the pole $k_z=-\lambda_0$ into the lower half-plane. Taking the integral along the contour shown in Fig. 1, and noting that the integral along the upper part of the contour vanishes for $z\to\infty$, we obtain, after allowing the lateral sections of the contour to approach infinity,

$$\mathbf{E}_{sc}^{(1)}(\mathbf{r},t)=\frac{ez_0}{4\pi^2Zv}\int\frac{\omega_0^2}{\lambda_0c^2}\varphi(\omega/v-\lambda_0)e^{i(\varkappa\rho+\lambda_0z-\omega t)}d\varkappa d\omega$$
$$\times\sum_h\frac{\omega v/c^2-\varkappa_h-\omega n_z/v+c^2(\varkappa+\lambda_0n_z)[\varkappa\varkappa_h+\lambda_0\omega(1-\beta^2)/v]/\omega^2}{\varkappa_h^2+\omega^2/v^2-\omega^2/c^2}\times e^{-M(L_h(\lambda_0))}f(L_h(\lambda_0)). \quad (20)$$

When $N_3\gg1$ the function $|\varphi(x)|$ has sharp peaks at values of the argument

$$x_n=2\pi n/z_0 \quad (21)$$

(n is an integer), i.e., this function has a delta-like character. However, passage from the function $\varphi(x)$ to a sum of δ-functions requires special care.

Let us for this purpose consider which of the peaks $\omega/v-\lambda_0=x_n$ of the function $|\varphi(\omega/v-\lambda_0)|$ in (20) can be attained for the real values of κ which describe the freely propagating wave. Equating $\omega/v-\lambda_0$ to the values of x_n given by (21), we obtain $\kappa=[\omega^2/c^2-(\omega/v-2\pi n/z_0)^2]^{1/2}$ (n is an integer). For the quantity κ to be real, it is necessary that n be positive when $\omega>0$ and negative when $\omega<0$ and satisfy the conditions

$$|\omega|(1-\beta)/v<2\pi|n|/z_0<|\omega|(1+\beta)/v. \quad (22)$$

We see from this that since $|\omega|(1-\beta)/v<2\pi/z_0$, the peaks $\omega/v-\lambda_0=x_1, x_2, \ldots$ can be attained at appropriate values of κ (for $\omega>0$). At the same time we can also see from (22) that the peak $\omega/v-\lambda_0=0$ cannot be attained at any value of κ.

However, in the case of ultrarelativistic particles the value of $\omega/v-\lambda_0$ can get very close to zero and it is at precisely these values of λ_0 (or κ) that the field (20) is strong, owing to the smallness of the denominator $\kappa_h^2+\omega^2(1-\beta^2)/v^2$. This will happen if

$$(\varkappa+\mathrm{K}_h)^2\sim\omega^2(1-\beta^2)/v^2. \quad (23)$$

We obtain from this that $\kappa=-\mathrm{K}_h+\delta\kappa$, where $|\delta\kappa|\sim\omega(1-\beta^2)^{1/2}/v$, and that the emission angle will be given by the relation

$$\sin\vartheta=|\varkappa|/(\omega/c)\approx|\mathrm{K}_h|c/\omega,$$

i.e., it coincides with the corresponding Bragg angle. The angular spread of the radiation is given by the ratio $|\delta\kappa|c/\omega$, i.e., of the order of $(1-\beta^2)^{1/2}$. We have at $h=0$, $\mathrm{K}_0=0$ the central spot of the diffraction pattern of the dynamic radiation whose middle coincides with the trajectory of the particle.

Thus, for values of $\omega/v-\lambda_0$ close to zero a replacement of $\varphi(\omega/v-\lambda_0)$ by $\delta(\omega/v-\lambda_0)$ is inadmissible since that will lead to the loss of the dynamic radiation which, as will be shown below, is the dominant radiation for ultrarelativistic particles.

Let us now consider the values of $\omega/v-\lambda_0$ close to $x_1, x_2, \ldots$. Since we assume that

$$\omega(1-\beta^2)/v\ll1/z_0,$$

the peaks $|\varphi(\omega/v-\lambda_0)|$ are attained at

$$\varkappa\approx2[\pi n\omega(1-nv\pi/\omega z_0)/z_0v]^{1/2}, \quad (24)$$

where $n=1, 2, \ldots$. λ_0 has then the values

$$\mathrm{tg}\,\vartheta=\frac{\varkappa}{\lambda_0}\approx2\left[\frac{\pi nv}{\omega z_0}\left(1-\frac{\pi nv}{\omega z_0}\right)\right]^{1/2}\Big/\left(1-\frac{2\pi nv}{\omega z_0}\right).$$

The corresponding emission angle ϑ is given by the formula

$$\lambda_0=\omega/v-2\pi n/z_0.$$

When $2\pi nv/\omega z_0\ll1$ we have

$$\mathrm{tg}\,\vartheta\approx2(\pi nv/\omega z_0)^{1/2}. \quad (25)$$

These angles coincide with the emission angles of the transition radiation in a layered medium of period z_0.[14]

Thus, we must consider separately the range of values of $\omega/v-\lambda_0$ in the vicinity of zero where we cannot replace $\varphi(\omega/v-\lambda_0)$ by the corresponding δ-function and the remaining range when $\omega/v-\lambda_0$ can coincide with one of the x_n ($n\geq1$) and a replacement of $\varphi(\omega/v-\lambda_0)$ by a sum of δ-functions is justified, allowing for the fact that we are solving the problem by iteration.

4. DYNAMIC RADIATION

In accord with what we have said at the end of the preceding section, let us consider the range of the values of $\omega/v-\lambda_0$ in the vicinity of zero without replacing $\varphi(\omega/v-\lambda_0)$ by a δ-function. To begin with let us convince ourselves that in the case being considered we can limit ourselves to the first iteration when certain additional conditions are fulfilled.

Let us for this purpose compare the h = 0 addend, which is the largest in the formula (20) of the first iteration, with the field obtained in the following iteration. An estimate shows that on the fulfillment of the condition

$$\omega_0^2 l / 4\omega c \ll 1, \tag{26}$$

where

$$l = N_3 z_0 \tag{27}$$

is the thickness of the crystal, the correction to the field in the first iteration will be small in comparison with the field of the first iteration itself. If we take $\omega_0 \sim 2 \times 10^{-2}$ keV, then for $\omega \sim 30$ keV, the condition (26) is fulfilled for $l < 6 \times 10^{-3}$ cm.

Thus, restricting ourselves to the first iteration and the central spot of the diffraction pattern (h = 0), we obtain from (20) for the transverse component of the electric field ($e^{-M} \approx 1$, $f \approx Z$)

$$\mathbf{E}_{sc}^{(1)\perp}(\mathbf{r}, t) = -\frac{ez_0}{4\pi^2 v} \times \int \frac{\omega_0^2 \varphi(\omega/v - \lambda_0)\varkappa \exp i(\varkappa\rho + \lambda_0 z - \omega t)}{\lambda_0 c^2[\varkappa^2 + \omega^2(1-\beta^2)/v^2]} d\varkappa d\omega. \tag{28}$$

Let us now calculate the flux of the Poynting vector through a plane z = const, located sufficiently far from the crystal. The calculation yields

$$W = \frac{8e^2\omega_0^4}{\pi c}\int \frac{d\omega}{\omega^4}\frac{\sin^2[(1-\beta^2\cos^2\vartheta)\, l\omega/4v]}{(1-\beta^2\cos^2\vartheta)^4}\sin^3\vartheta d\vartheta, \tag{29}$$

where the angle ϑ is found from the formula $\kappa = (\omega/c)\sin\vartheta$.

It is not difficult to obtain from this the number of emitted quanta:

$$N = \frac{8\omega_0^2}{137\pi}\int \frac{d\omega}{\omega^5}\frac{\sin^2[(1-\beta^2\cos^2\vartheta)\, l\omega/4v]}{(1-\beta^2\cos^2\vartheta)^4}\vartheta^3 d\vartheta. \tag{30}$$

Figures 2 and 3 show the results of the numerical computation of the spectrum of the dynamic radiation $dN/d\omega$, as well as the total number of quanta as a function of the crystal thickness and the charged particle energy. As can be seen from Fig. 3, the total number of quanta of the dynamic radiation grows with the charged particle energy.

However, the curves are only of a tentative nature since, in the first place, we must bear in mind that at those sections of the curves where the condition (26) is not fully met, it is necessary to take into account the correction introduced by subsequent iterations and, secondly, that no allowance was made in formula (30) for the absorption of quanta from the dynamic radiation in the crystal itself. Allowance for absorption changes the relative contribution of quanta of different energies to the total number of quanta and may influence the dependence of the total number of quanta on the particle energy.

We have thus far considered the h = 0 addend in (20). Let us now consider the contributions of the h ≠ 0 addends. For the values of κ given by (23), the corresponding addends will also make contributions to the dynamic radiation. The directions of these emissions were discussed at the end of Sec. 3, where we saw that the charged particle field yields for the crystal a distinctive "Laue diffraction pattern," consisting of a central

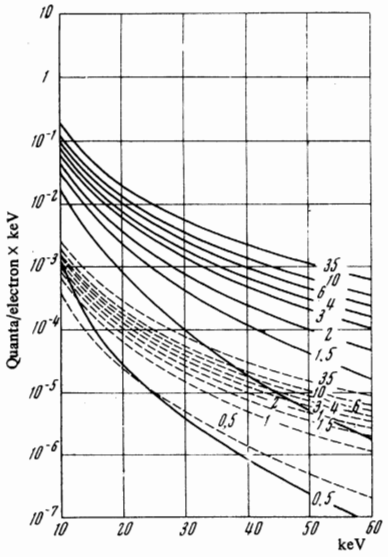

FIG. 2. Spectrum of the dynamic radiation as a function of the primary electron energy and crystal thickness. The energy of the quanta of the radiation is plotted along the abscissa while the number of quanta is plotted along the ordinate. The continuous curves pertain to a crystal of thickness 100μ, the dashed curves—to a crystal of thickness 10μ. The numbers on the curves indicate the primary electron energy in GeV.

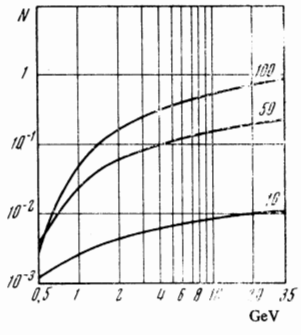

FIG. 3. Dependence of the total number of quanta of the dynamic radiation on the primary electron energy for different crystal thicknesses. The primary electron energy is plotted along the abscissa and the number of emitted quanta in the energy range from 10 to 60 kev—along the ordinate. The numbers on the curves indicate the crystal thickness in microns.

spot (h = 0) and lateral spots (h ≠ 0). The angular spread of both the central and lateral spots is of the order of $(1-\beta^2)^{1/2}$ (see (23)). However, the intensity of the lateral spots decreases with distance from the central spot owing to the presence in (20) of the form factor $f(\kappa_h)$. And moreover in the case of $c\kappa_h/\omega > (1-\beta^2)^{1/2}$ these intensities may contain still another small factor.

Notice that the formula (30) for the number of quanta of the central spot of the dynamic radiation does not contain any parameters connected with the crystal structure. This circumstance points, apparently, to the fact that crystallinity of matter is not important for the production of the central spot of the dynamic radiation.[1)] As to the lateral spots of the Laue diffraction pattern of the dynamic radiation, crystallinity of matter is, of course, vital for their production.

If we compare formula (30) with the corresponding formula in [4], we easily find that they coincide in respect of the particle energy dependence and in order of magnitude.

[1)]The authors are indebted to B. M. Bolotovskiĭ for this observation.

5. TRANSITION RADIATION IN THE ATOMIC PLANES

Let us now consider formula (20) in that region of the values of κ, determined by formula (24), when the peaks of the function $|\varphi(\omega/v - \lambda_0)|$ can be attained.

As has already been discussed in Sec. 3, for a sufficiently large N_3 (strictly speaking, for an infinitely thick crystal) we may, in the case being considered, replace the function $\varphi(x)$ by a sum of δ-functions

$$\varphi(x) = (2\pi/z_0)\sum_n \delta(x - x_n). \tag{31}$$

Let us consider the contribution to the scattered field (20) of the first iteration made by the peak of $|\varphi(\omega/v - \lambda_0)|$ at $\omega/v - \lambda_0 = x_n$ ($n = 1, 2, \ldots$). Taking (31) into account and retaining only the $h = 0$ addend in (17), we have for the transverse Fourier component of the field

$$\mathbf{E}_{sc}^{(1)\perp}(\mathbf{k}, \omega) = A_n \frac{\delta(k_z + 2\pi n\omega/z_0|\omega| - \omega/v)\varkappa(1 - c^2\varkappa^2/\omega^2)}{(\varkappa^2 - \varkappa_n^2)(\varkappa^2 + \tilde{\varkappa}_0^2)}, \tag{32}$$

where we have introduced the notations:

$$\begin{aligned} A_n &= \frac{8\pi^2 ie\omega_0^2}{vZc^2}\exp[-M(2\pi n/z_0)]f(2\pi n/z_0), \\ \varkappa_n^2 &= \omega^2/c^2 - (\omega/v - 2\pi n\omega/z_0|\omega|)^2, \\ \tilde{\varkappa}_0^2 &= \omega^2/v^2 - \omega^2/c^2. \end{aligned} \tag{33}$$

We have in these formulas inserted only positive n, taking its sign into account with the help of the factor $\omega/|\omega|$.

By computing the scattered field in the next iteration, we easily see that there appears in $\mathbf{E}_{sc}^{(2)\perp}(\mathbf{k}, \omega)$ a series of additional addends, one addend of which has a pole of second order while the rest have simple poles. We can continue the iteration procedure. Evidently, $\mathbf{E}_{sc}^{(p+1)\perp}(\mathbf{k}, \omega)$ will consist of $\mathbf{E}_{sc}^{(p)\perp}(\mathbf{k}, \omega)$ and a series of addends among which one addend will have a pole of order $p + 1$.

$$A_n \frac{(-\omega_0^2/c^2)^p \delta(k_z + 2\pi n\omega/z_0|\omega| - \omega/v)\varkappa(1 - c^2\varkappa^2/\omega^2)}{(\varkappa^2 - \varkappa_n^2)^{p+1}(\varkappa^2 + \tilde{\varkappa}_0^2)}, \tag{34}$$

while the rest will have poles of lower order.

Following the concept of partial summation of the most divergent terms in all the orders of perturbation theory (see, for example, [15]), let us carry out such a partial summation of terms like (34) in the entire iteration series. We obtain

$$\mathbf{E}_{sc}^{\perp}(\mathbf{k}, \omega) = A_n \frac{\delta(k_z + 2\pi n\omega/z_0|\omega| - \omega/v)\varkappa(1 - c^2\varkappa^2/\omega^2)}{(\varkappa^2 - \varkappa_n^2 + \omega_0^2/c^2)(\varkappa^2 + \tilde{\varkappa}_0^2)}. \tag{35}$$

Comparing the expression (35) with the formula (32), we see that, as a result of the summation of the series, the pole of the Fourier transform of the scattered field has shifted, and this shift may be interpreted as a change in the phase velocity of propagation of the scattered field due to some effective "dielectric constant" of the crystal. Indeed, it is not difficult to obtain from (35) that

$$\varepsilon_{eff} = 1 - \omega_0^2/\omega^2.$$

In addition to the above-considered most divergent terms in each iteration order, we could also, in similar fashion, take the next most divergent terms into account. We would then obtain an ϵ_{eff} which would contain an atomic form factor and a temperature-dependent Debye–Waller factor. Furthermore, if we allow for damping processes of different kinds (due to the photoelectric effect, Compton effect, phonon processes, etc.), then the effective dielectric constant should also contain a small imaginary part. However, for an estimate of the order of magnitude of the scattered field we restrict ourselves to the expression (35) only.

Now if we compute the flux of the Poynting vector through the plane $z = \text{const}$, we find it is infinite owing to the fact that we assumed, when computing the field (35), the crystal to be of infinite thickness and neglected absorption in the crystal. This happens because a transition radiation is emitted in each atomic plane and therefore it is expedient to calculate the energy emitted from a unit length of the path of the charge. Let us calculate for this purpose the longitudinal Fourier component of the scattered field due to the n-th peak of $|\varphi(x_n)|$. We obtain in complete analogy with the foregoing

$$E_{sc}^{z}(\mathbf{k}, \omega) = -A_n \frac{\delta(k_z + 2\pi n\omega/z_0|\omega| - \omega/v)c^2 k_z \varkappa^2/\omega^2}{(\varkappa^2 - \varkappa_n^2 + \omega_0^2/c^2)(\varkappa^2 + \tilde{\varkappa}_0^2)} \tag{36}$$

And the field itself in explicit form is equal to

$$\begin{aligned} E_{sc}^{z}(\mathbf{r}, t) = &-A_n(2\pi)^{-4}\int d\omega \exp i[(\omega/v - 2\pi n\omega/z_0|\omega|)z - \omega t] \\ &\times \frac{c^2}{\omega^2}\left(\frac{\omega}{v} - \frac{2\pi n\omega}{z_0|\omega|}\right)\left[\left(\varkappa_n^2 - \frac{\omega_0^2}{c^2}\right)K_0\left(-i\frac{\omega}{|\omega|}\sqrt{\varkappa_n^2 - \frac{\omega_0^2}{c^2}}\,\rho\right)\right. \\ &\left. + \tilde{\varkappa}_0^2 K_0(\tilde{\varkappa}_0\rho)\right]/(\tilde{\varkappa}_0^2 + \varkappa_n^2 - \omega_0^2/c^2), \end{aligned} \tag{37}$$

where $K_0(x)$ is the modified Hankel function.

The energy carried off by the radiation from a unit length of the trajectory of the particle is equal to

$$\frac{dW}{dz} = \frac{1}{2}\int_{-\infty}^{\infty} dtc\rho[E_{sc}^{z}(\rho, z, t)]^2. \tag{38}$$

For large values of $|x|$ we have

$$K_0(x) \approx (\pi/2x)^{1/2}\exp(-x) + \ldots$$

Therefore we can at sufficiently large values of ρ neglect the second addend in the numerator of (37). The first addend has the exponential factor

$$\exp\{i[\sqrt{\varkappa_n^2 - \omega_0^2/c^2}\,\rho + (\omega/v - 2\pi n\omega/z_0|\omega|)z - \omega t]\}. \tag{39}$$

Substituting (37) in (38), we obtain for the number of quanta

$$\frac{dN}{dz} = \frac{\omega_0^4 \exp[-2M(2\pi n/z_0)]f^2(2\pi n/z_0)}{137 \cdot 8\pi^{5/2}c^2Z^2}\int_0^{\infty}\frac{d\omega}{\omega^3(\omega n/z_0 v - n^2\pi/z_0^2)^{1/2}}. \tag{40}$$

We can see from (39) that this radiation is emitted at an angle $\vartheta \sim 2(n\pi v/z_0\omega)^{1/2}$ (see also (25)).

We can see from (40) that the intensity of radiation depends only on the number n (i.e., on the angle ϑ), decreasing with decrease in n owing to the presence of the square of the form factor $f^2(2\pi n/z_0)$. For a given value of ϑ the intensity does not depend on the azimuthal angle, i.e., we have a ring of radiation of uniform intensity with its center lying on the trajectory of the charged particle. However, if we also take into account the con-

tributions of the $h \neq 0$ addends in (20), then the intensity of the radiation, in contrast to the transition radiation in a stack of macroscopic plates,[14] ceases to be uniform along the ring and will have maxima and minima determined by κ_h, i.e., by the structure of the atomic planes of the crystal.

Let us compare the number of quanta of the dynamic (30) and transition (40) radiations. It is expedient for this purpose to take a crystal of thickness $\sim 10^{-2}$ cm since, for example, for Si the absorption thickness for quanta of energy ~ 10 keV is equal to this thickness. An estimate with the aid of formula (40) shows that $N_{tr} \sim 10^{-10}$. On the other hand, we have from Fig. 2 that $N_{dyn} \sim 10^{-2}$. Thus, in the considered case of ultrarelativistic particles the dynamic radiation is the dominant radiation.

6. PHYSICAL INTERPRETATION OF THE RESULTS

Let us approach the above-obtained results with a somewhat different, physically more intuitive point of view.

First, let us show why the function $\varphi(\omega/v - \lambda_0)$ appears in the expression (20) for the scattered field. A charged particle passing through atomic planes produces radiation at the points 1, 2, 3, ... (Fig. 4). Let us find the phases of the rays arriving from these points at the observation plane z = const. It is important to note that the z-component of the wave vector of the field of the charge is then ω/v while that of the wave vector of the radiation field is λ_0. Therefore, the radiation arriving from the point 1 will have a phase factor $\exp(i\lambda_0 z)$, from the point 2—a factor $\exp i[\omega z_0/v + \lambda_0(z - z_0)]$, and so on. The total field in the plane z = const will have a phase factor (we shall not, as in the entire paper, take the absorption in the crystal into account):

$$e^{i\lambda_0 z}(1 + e^{i(\omega/v-\lambda_0)z_0} + \ldots + e^{i(\omega/v-\lambda_0)(N_3-1)z_0}) = e^{i\lambda_0 z}\varphi(\omega/v - \lambda_0).$$

It is not also difficult to understand why the intensity of the transition radiation in the atomic planes is proportional to the thickness of the crystal, while there is no such direct proportionality for the dynamic radiation. This is connected with the fact that for $(\omega/v - \lambda_0)z_0 = 2\pi n$ (n is an integer) the fields of the radiations from the points 1, 2, ... are in phase and combine additively (Fig. 5 ,. But to this corresponds that case when $\omega/v - \lambda_0 = x_n$ $(n \geq 1)$ (see formula (21)). When, however, a peak of the function $|\varphi(\omega/v - \lambda_0)|$ is not attainable, i.e., for values of $\omega/v - \lambda_0$ close to zero, the scattered fields from the points 1, 2, ... are never in phase and we have the case shown in Fig. 5, b. In consequence, there occurs a partial compensation of the radiations produced in the various atomic planes of the crystal and they will not additively combine. This circumstance is characteristic of the phenomenon of diffraction. However, owing to the smallness of the denominator $\kappa_h^2 + \omega^2(1 - \beta^2)/v^2$ in the formula (20) for ultrarelativistic particles the intensity of the dynamic radiation, in spite of the partial compensation, nevertheless exceeds by several orders the intensity of the transition radiation in the atomic planes.

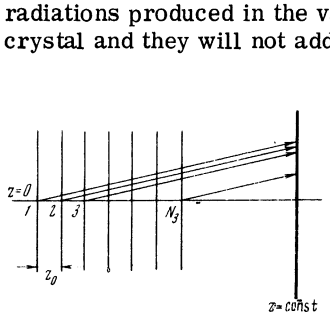

FIG. 4. Diagram illustrating the appearance of phase differences for the fields of radiations produced in different atomic planes.

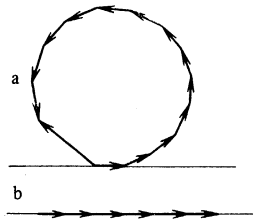

FIG. 5. Addition of the radiation fields in the cases: a) when the fields are in phase, b) when they are shifted in phase.

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