CYCLOTRON RESONANCE HARMONICS IN A QUANTIZING MAGNETIC FIELD

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The real parts σ'_{xx} and σ'_{zz} of the diagonal components of the conductivity tensor are calculated for a system of nondegenerate and noninteracting electrons in a magnetic field $H = (0, 0, H)$ and an alternating electric field of frequency ω, with scattering by a static potential. In a quantizing magnetic field, σ'_{xx} passes through a maximum at $\omega = n\omega_C$ ($\omega_C = eH/mc$; $n = 2, 3, 4 \ldots$). The oscillations are studied in detail for the case of scattering by a short-range potential. It is found that in this case σ'_{zz} also experiences oscillations. These are much smaller than those of σ'_{xx} and for a given value of ω they exist up to a certain maximal value of n, which depends on the value of H. The possibility of observing the oscillations is discussed and it is suggested that they were observed in[15]. The calculation is performed with the aid of the kinetic equation for the translationally-invariant part of the density matrix in the Wigner representation.

1. In a quantizing magnetic field $H = (0, 0, H)$, the average value of the z-component of the electron wave vector k is small compared with the perpendicular component:

$$|k_z| \sim \hbar^{-1}\sqrt{mT}, \quad k_\perp \sim \sqrt{m\hbar\omega_c}, \quad |k_z|/k_\perp \sim \alpha^{-\frac{1}{2}}.$$

Here $\alpha = \hbar\omega_C/2T$, $\omega_C = eH/mc$, and T is the temperature in energy units; the remaining symbols are standard. The region of uncertainty of the electron coordinates—the wave packet—is stretched out along H. The ratio of the length of the packet to its width is $\sim\alpha$. The center of the packet moves along a classical helix with translational velocity $\sim\hbar|k_z|/m$ along H, completing one turn within a time $2\pi/\omega_C$.

The form of the packet and the character of its motion cause the scattering of the electron to possess an important singularity. For concreteness we shall speak of scattering by a point-like center, and assume that the interaction of the electron with the scattering center is weak.

In first approximation, the motion of the electron in the field of the center will differ little from free motion. The packet moves past the center within a time interval on the order of $m/\hbar k_z^2 \sim \bar{n}/T$, and has time to complete $m\omega_C/\hbar k_z^2 \sim \alpha$ revolutions along the helix during that time. In one revolution, the packet is displaced along H by a fraction α^{-1} of its length. If α is sufficiently large, the displacement during one revolution is negligible. Then the force exerted on the electron by the center will be an almost periodic function of the time.

If, in addition, the field in which the electron is situated is itself periodic in time with frequency ω, then at $\omega = n\omega_C$ the scattering should increase. This resonance can become manifest in the form of periodic oscillations of the kinetic coefficients when ω or ω_C is varied.

An example of such a situation may be an electron-impurity system in an alternating electric field (A). Another case when such a resonance is possible is when electrons in a crystal are scattered by optical phonons (B). The scattering potential varies in time at a frequency equal to the limiting frequency ω_0 of

the optical phonons. Finally, a combination of cases A and B is possible, namely a periodic electric field plus scattering by optical phonons (case C). Now the spectrum of the forces acting on the electron will contain the frequencies $|\omega \pm \omega_0|$ and the resonance will occur when $|\omega \pm \omega_0| = n\omega_C$. In situation B, the resonance leads to an oscillatory dependence of the static conductivity on the magnetic field (the static magneto-phonon resonance). These oscillations were investigated theoretically[1-4] and were observed in experiments[5]. The oscillations of the absorption coefficient in case C constitute the cyclotron-phonon resonances considered in[6]. These oscillations were also observed[7].

We investigate case A in the present paper. We note that the indication that the transverse conductivity can oscillate in a periodic electric field in elastic scattering by acoustic phonons (static potential!) is contained in[8]. However, neither the effect nor the possibility of its observation was investigated.

We consider in this paper a system of noninteracting nondegenerate electrons scattered by a static potential (impurities, elastic scattering by acoustic phonons). The spectrum of the electrons is assumed quadratic and isotropic. When $\omega_C\tau \gg 1$ or $\omega\tau \gg 1$, and also for $E \perp H$ when $|\omega - \omega_C|\tau \gg 1$ (τ is a quantity of the order of the momentum relaxation time) one calculates the real parts of the diagonal components of the conductivity tensor σ'_{xx} and σ'_{zz}, which describe the absorption of the correspondingly polarized radiation. The quantity σ'_{xx} has at $\omega = n\omega_C$ ($n \neq 1$, i.e., at harmonics of the cyclotron resonance) maxima that are relatively large and sharp when $\alpha \gg 1$. The dependences of σ'_{xx} and σ'_{zz} on ω/ω_C were investigated in detail for the case of scattering by a short-range potential. It turned out that σ'_{zz} also experiences oscillations in this case, if α is sufficiently large. These oscillations are much smaller than the oscillations of σ'_{xx} and exist up to a certain maximum value of n, which increases with increasing α. The feasibility of observing the harmonics is discussed. Oscillations of σ'_{xx} should be observed in Ge, Si, and InSb. Oscillations of this type were apparently already observed in

n-InSb. Oscillations of σ'_{zz} can probably also be observed.

We note that, when describing the behavior of a system of electrons in a quantizing magnetic field, one usually speaks of transitions between stationary states corresponding to Landau zones in a magnetic field. In such a description, the appearance of oscillation is related with the presence of singularities in the density of the initial and final states. Such treatment and interpretation, as given here, obviously describe the same phenomenon in the wave and corpuscular languages.

2. We consider an electron situated in an impurity field $U(\mathbf{r})$ in a constant homogeneous magnetic field $\mathbf{H}$, and in a homogeneous electric field $\mathbf{E}(t)$. The Hamiltonian of the electron in the gauge $A_x = -yH$, $A_y = A_z = 0$ is written in the form

$$H = \frac{1}{2m}\left[\left(\hat{p}_x + \frac{e}{c}yH\right)^2 + \hat{p}_y^2 + \hat{p}_z^2\right] + U(\mathbf{r}) \qquad (1)$$
$$- e\mathbf{r}\mathbf{E}(t) = \hat{H}_0 + U - e\mathbf{r}\mathbf{E}.$$

We shall describe the ensemble of noninteracting electrons with the aid of the density matrix $\rho(\mathbf{r}_1, \mathbf{r}_2, t)$. It satisfies the equation[9]

$$i\hbar\partial\rho/\partial t = (\hat{H}_1 - \hat{H}_2^*)\rho. \qquad (2)$$

Here $\hat{H}_1$ and $\hat{H}_2$ act on the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. With the aid of the well-known eigenfunctions of the Hamiltonian $\hat{H}_0$[9] it is easy to calculate the equilibrium density matrix $\rho^0(\mathbf{r}_1, \mathbf{r}_2)$ $\sim \langle\mathbf{r}_1|\exp(-\hat{H}_0/T)|\mathbf{r}_2\rangle$. The resultant expression takes the form $(l^2 \equiv \hbar c/eH)$

$$\rho^{(0)} = \bar{\rho}^{(0)}(\mathbf{r}_1 - \mathbf{r}_2)\exp[-\tfrac{1}{2}i(y_1 + y_2)(x_1 - x_2)l^{-2}]. \qquad (3)$$

The presence of the exponential factor on the right makes the matrix noninvariant with respect to a shift of the origin. This noninvariance was the result of the fact that $\hat{H}_0$ depends explicitly on the coordinate. Obviously, however, the shift of the origin cannot affect the state of the electron. (This is valid, of course, also in the presence of impurity centers).

Our task is to find the nonequilibrium density matrix $\rho(\mathbf{r}_1, \mathbf{r}_2, t)$. It is clear that the latter is likewise not invariant. We seek it in the form analogous to (3):

$$\rho = \bar{\rho}(\mathbf{r}_1, \mathbf{r}_2, t)\exp[-\tfrac{1}{2}i(y_1 + y_2)(x_1 - x_2)l^{-2}]. \qquad (4)$$

From (2) we readily obtain an equation for $\tilde{\rho}$:

$$i\hbar\frac{\partial}{\partial t}\bar{\rho} = \frac{1}{2m}\left\{\hat{p}_1^2 - \hat{p}_2^2 - \frac{e}{c}[H, \mathbf{r}_1 - \mathbf{r}_2](\hat{p}_1 - \hat{p}_2)\right\}\bar{\rho} \qquad (5)*$$
$$+ \{U(\mathbf{r}_1) - U(\mathbf{r}_2) - e(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{E}\}\bar{\rho}.$$

Equation (5) is invariant relative to a shift of the origin (U depends only on the difference $\mathbf{r} - \mathbf{R}_j$, where $\mathbf{R}_j$ is the radius vector of the j-th center). Consequently, $\tilde{\rho}$ is likewise unchanged by such a shift. We now define the quantum distribution function F (the

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*[H, $\mathbf{r}_1 - \mathbf{r}_2$] = H × ($\mathbf{r}_1 - \mathbf{r}_2$).

density matrix $\tilde{\rho}$ in the Wigner representation) by

$$F(\mathbf{k}, \mathbf{r}, t) = V^{-1}\int d\varkappa e^{i\mathbf{k}\varkappa}\tilde{\rho}(\mathbf{r} - \tfrac{1}{2}\varkappa, \mathbf{r} + \tfrac{1}{2}\varkappa, t) \qquad (6)$$

(V is the normalization volume). From (5) we obtain an equation for F:

$$\left\{\frac{\partial}{\partial t} + \frac{\hbar k}{m}\nabla_r + \left(\frac{e}{\hbar}\mathbf{E} + \frac{e}{mc}[\mathbf{k}\mathbf{H}]\right)\nabla_k\right\}F(\mathbf{k}, \mathbf{r}, t) \qquad (7)$$
$$- \frac{i}{\hbar}\sum_q U_q\left[F\left(\mathbf{k} + \frac{1}{2}\mathbf{q}, \mathbf{r}, t\right) - F\left(\mathbf{k} - \frac{1}{2}\mathbf{q}, \mathbf{r}, t\right)\right]e^{i\mathbf{q}\mathbf{r}} = 0,$$

where U_q is the Fourier component of the impurity potential. The electron density is $\sum_k F(\mathbf{k}, \mathbf{r}, t)$. It is easy to establish that the current density j should be calculated from the formula

$$\mathbf{j}(\mathbf{r}, t) = \frac{e\hbar}{m}\sum_k \mathbf{k}F(\mathbf{k}, \mathbf{r}, t). \qquad (8)$$

F is the Wigner representation of the translation-invariant part of the density matrix. This function can also be introduced in a different manner. For this purpose it is necessary to choose the coordinates and the kinematic momentum as the independent variables in the Wigner representation of the ordinary density matrix. As shown in[10], such a choice of variables follows naturally from the analysis of the translational properties of the density matrix. F depends on the same variables as the classical distribution function, and on no others. This greatly simplifies the calculation[10].

3. Equation (7) is exact. The function F depends on the positions of all the impurities. Since F is unchanged by a shift of the origin, it can be represented in the form

$$F(\mathbf{k}, \mathbf{r}, t) = \sum F_{s_1 \cdots s_N}(\mathbf{k}, t)\exp\left[-i\sum_j \mathbf{s}_j(\mathbf{r} - \mathbf{R}_j)\right]. \qquad (9)$$

Here N is the number of centers, and the summation of the first sum is carried out over all the values of each index. We put $F_{0\ldots 0} \equiv F_0$. For the remaining components we shall only indicate the nonzero indices: $F_{0\ldots s_j\ldots 0} \equiv Fs_j$, $F_{0\ldots s_j'\ldots 0} \equiv F_{s_j s_j'}$ etc. Changing over in (7) to Fourier components and writing out in succession the equations for F_0, F_{s_j}, $F_{s_j s_j'}\ldots$, we obtain an infinite system of coupled equations. F_{s_j} describes the correlation between the electron and the j-th center, $F_{s_j s_j'}$ the correlation between the electron and the centers j and j', etc. In the case of weak interaction $|F_{s_j s_j'}| \ll |F_{s_j}| \ll F_0$. The collision term of the equation for F_0 contains the components F_{s_j} and summation extends over all j $(1 \leq j \leq N)$. The analogous term of the equation for F_{s_j} contains F_0, and also F_{s_j} and $F_{s_j s_j'}$. The latter can be omitted in the case of weak interaction. Now the equation for F_0 and the N equations for F_{s_j} form a closed system of N + 1 equations with N + 1 unknowns. We shall assume all the centers to be identical. Then the F_{s_j} with different j are also identical, the index j can be omitted, the system of equations reduces to two, and

the summation over j in the equation for F_0 can be replaced by multiplication by N.

In the approximation linear in **E** we put $F_0 = \Phi_0 + f_0$, $F_S = \Phi_S + f_S$, where the functions f are of the order of **E**. From the indicated two equations we obtain in the zeroth approximation in **E** two equations for Φ_0 and Φ_S, and in the first approximation we obtain two equations for f_0 and f_S. The latter takes the form

$$\left(\frac{\partial}{\partial t} + \frac{e}{mc}[\mathbf{kH}]\nabla_k\right)f_0(\mathbf{k}) - \frac{i}{\hbar}N\sum_s u_s\left[f_s\left(\mathbf{k} + \frac{1}{2}\mathbf{s}\right) - f_s\left(\mathbf{k} - \frac{1}{2}\mathbf{s}\right)\right] = -\frac{e}{\hbar}\mathbf{E}\nabla_k\Phi_0(\mathbf{k}), \quad (10a)$$

$$\left(\frac{\partial}{\partial t} + \frac{e}{mc}[\mathbf{kH}]\nabla_k - \frac{t\hbar}{m}\mathbf{ks}\right)f_s(\mathbf{k}) = -\frac{i}{\hbar}u_{-s}\left[f_0\left(\mathbf{k} + \frac{1}{2}\mathbf{s}\right) - f_0\left(\mathbf{k} - \frac{1}{2}\mathbf{s}\right)\right] - \frac{e}{\hbar}\mathbf{E}\nabla_k\Phi_s(\mathbf{k}). \quad (10b)$$

Here u_S is the Fourier component of the potential $u(\mathbf{r})$ of the individual center:

$$U(\mathbf{r}) = \sum_j u(\mathbf{r} - \mathbf{R}_j).$$

Let us formulate the conditions under which the interaction can be regarded as weak. In order to be able to neglect the terms containing $F_{s_j s'_j}$ in the collision term of the equation for F_{s_j}, it is necessary to satisfy one of the inequalities $\hbar/\tau \ll \hbar^2 k^2/2m \equiv \epsilon_k$, $\omega_c\tau \gg 1$ or $\omega\tau \gg 1$. Further, omitting in the same equation the terms with F_{s_j}, we confine ourselves to the Born approximation for scattering by the impurity center. In a magnetic field this is legitimate if one satisfies the condition for applicability of perturbation theory in the one-dimensional case[9], namely $|u| \ll \hbar^2|k_z|/ma$, where a is the effective radius of $u(\mathbf{r})$. Thus, our analysis is meaningful if the relative number of "slow" electrons γ is small: $\gamma = k_0/k_{av} \ll 1$, where $k_0 \sim ma|u|\hbar^{-2}$ and k_{av} is the average value of the z component of the wave vector of the electron system. The foregoing derivation is also suitable for an inhomogeneous field **E**, if the latter varies little over microscopic distances (such as the thermal length of the electron wave $\lambda = (\hbar^2/2mT)^{1/2}$, the impurity radius a, etc.). In this case it is necessary to regard Φ_0, Φ_S, f_0, and f_S as slow functions of **r**, and it is necessary to add to the left sides of the equations in (10) the terms $m^{-1}\hbar\mathbf{k}\nabla_r f_0$ and $m^{-1}\hbar\mathbf{k}\nabla_r f_S$, respectively.

Equations (10) were derived by us for a system comprising an electron plus impurity centers. A similar derivation can be obtained for any static scattering potential, for example, for elastic scattering by lattice vibrations. In the latter case it is necessary to omit N from (10a) and to replace u_S by the Fourier component of the corresponding interaction potential.

Let us assume that the electric field is turned on adiabatically: $\mathbf{E} \sim e^{(\delta - i\omega)t}$, $\delta \to +0$. We then obtain from (10b)

$$f_s(\mathbf{k}, t) = \frac{1}{\hbar\omega_c}\int_\infty d\varphi' G_\omega(\varphi, \varphi')\left\{iu_{-s}\left[f_0\left(\mathbf{k}' + \frac{1}{2}\mathbf{s}, t\right)\right.\right. \quad (11)$$
$$\left.\left. - f_0\left(\mathbf{k}' - \frac{1}{2}\mathbf{s}, t\right)\right] + e\mathbf{E}(t)\nabla_{k'}\Phi_s(\mathbf{k}')\right\}.$$

We have introduced the cylindrical coordinates $k_\perp$, φ, and k_z in **k**-space, with the axis along **H**, and the notation

$$\mathbf{k}' = (k_\perp, \varphi', k_z), \quad G_\omega(\varphi, \varphi') = G(\varphi, \varphi')\exp\left[-i(\omega/\omega_c)(\varphi - \varphi')\right],$$
$$G(\varphi, \varphi') = \exp\left\{(\delta - il^2 k_z s_z)(\varphi - \varphi') - il^2[\mathbf{s}_\perp, \mathbf{k}_\perp - \mathbf{k}_\perp']\right\}. \quad (12)$$

The lower limit in (11) is chosen to make $f_S(\mathbf{k})$ a periodic function of φ.

4. Putting **E** = 0 in (10b) and making the substitution $f_S \to \Phi_S$ and $f_0 \to \Phi_0$, we obtain an equation for Φ_S. (An equation for Φ_S is obtained in the same manner from (10a)). A solution of this equation, corresponding to adiabatic turning on of the interaction, is

$$\Phi_s(\mathbf{k}) = \frac{iu_{-s}}{\hbar\omega_c}\int_\infty d\varphi' G(\varphi, \varphi')\left[\Phi_0\left(\mathbf{k}' + \frac{1}{2}\mathbf{s}\right) - \Phi_0\left(\mathbf{k}' - \frac{1}{2}\mathbf{s}\right)\right]. \quad (13)$$

The function Φ_S can be obtained directly by expanding the equilibrium density in powers of the interaction. The zeroth term of the expansion is

$$\Phi_0(\mathbf{k}) = 8\pi^{3/2}\lambda\lambda_\perp^2 n_e V^{-1}\exp\left(-\lambda_\perp^2 k_\perp^2 - \lambda^2 k_z^2\right). \quad (14)$$

Here n_e is the electron concentration and $\lambda_\perp^2 = l^2\tanh\alpha$. The function (14) was obtained in[10]. The first-order term in the expansion of the equilibrium density can be reduced to the form (13). In such a derivation there is no need to assume that the interaction is turned on adiabatically. It can also be shown that Φ_S satisfies the equation

$$\sum_s u_s\left[\Phi_s\left(\mathbf{k} + \frac{1}{2}\mathbf{s}\right) - \Phi_s\left(\mathbf{k} - \frac{1}{2}\mathbf{s}\right)\right] = 0. \quad (15)$$

Thus, (13) causes the collision integral in the equation for Φ_0 to vanish. This means that Φ_0 is stationary with respect to transitions between states with large $|k_z|$: (At small $|k_z|$, perturbation theory is not applicable and expression (13) cannot be used.) Since the relative number of states with small $|k_z|$ is of the order of γ, the function (14) is stationary in the assumed approximation ($\gamma \ll 1$) and its use as the equilibrium function is valid.

5. We proceed to calculate the absorption of electromagnetic radiation. We assume that the following conditions are satisfied: at $\mathbf{E} \perp \mathbf{H}$ the condition $(\omega_c\tau)^{-1} \ll 1$ or $(\omega\tau)^{-1} \ll 1$ and also $(\omega\tau)^{-1} \ll 1$ (by the same token, we exclude cyclotron resonance from consideration), and at $\mathbf{E} \parallel \mathbf{H}$ the condition $(\omega\tau)^{-1} \ll 1$. Then the first term in the left-hand side of (10a) is large compared with the collision term, and the solution can be sought in the form of iterations in terms of an appropriate small parameter. The terms containing f_0 and Φ_S in (11) are then of the same order.

From (10a), omitting the interaction term, we find f_0 in the zeroth approximation (accurate to an arbitrary function of $k_\perp$ and k_z; the latter, however, can be readily shown to make no contribution to the current) and substitute it in (11). It is then necessary to substitute Φ_S from (13) in (11). This results in a double integral with respect to the angle. By reversing the order of integration, this integral can be transformed in such a way that the expression under the sign of the internal integral does not contain Φ_0. The internal integral can then be calculated. (In the transformations it is necessary to use the following property

of the function G: $G(\varphi, \varphi')G(\varphi', \varphi'') = G(\varphi, \varphi''))$.
Ultimately, the corresponding part of f_s can be
written as a set of three terms. The first term is
cancelled out by that part of f_s which arises when the
zeroth-approximation f_0 is substituted in (11). The
second term contains terms proportional to the com-
ponents of the vector $\nabla_k \Phi_s(k)$, which, by virtue of (15),
gives zero after substitution in (10a). We shall there-
fore not write them out. The third term is

$$f_s(k) = \frac{e}{\hbar^2} l^2 u_{-s} \left[(\omega^2 - \omega_c^2)^{-1} \left(\mathbf{E}_\perp \mathbf{s}_\perp + i \frac{\omega_c}{\omega} [\mathbf{E}_\perp \mathbf{s}_\perp] \right) \right. \quad (16)$$

$$+ \omega^{-2} E_z s_z \Big] \int_\infty^\varphi d\varphi' [G(\varphi, \varphi') - G_\omega(\varphi, \varphi')] \left[\Phi_0 \left(k' + \frac{1}{2} s \right) \right.$$
$$\left. - \Phi_0 \left(k' - \frac{1}{2} s \right) \right].$$

By virtue of the statement made above, expression (16)
for the function f_s is not complete, but the omitted
terms are of no interest.

We replace $\partial f_0 / \partial t$ by $-i\omega f_0$ in (10a), multiply this
equation by $e\hbar k_i/m$, and sum over k. After obvious
transformations we obtain

$$-i\omega j_x - \omega_c j_y = \frac{e^2 n_e}{m} E_x - \frac{ieN}{m} \sum_{k,s} u_s s_x f_s(k), \quad (17a)$$

$$\omega_c j_x - i\omega j_y = \frac{e^2 n_e}{m} E_y - \frac{ieN}{m} \sum_{k,s} u_s s_y f_s(k), \quad (17b)$$

$$-i\omega j_z = \frac{e^2 n_e}{m} E_z - \frac{ieN}{m} \sum_{k,s} u_s s_z f_s(k). \quad (17c)$$

Here

$$j_i(t) = \frac{e\hbar}{m} \sum_k f_0(k,t) k_i \quad (18)$$

is the i-th component of the current density. In (17) we
took into account the normalization condition $\sum_k \Phi_0(k) = n_e$.

Let us calculate the sum $\sum_k f_s(k)$. It will be con-
venient below to introduce the dimensionless vectors
$l\mathbf{k}$ and $l\mathbf{s}$, which we shall designate simply by $\mathbf{k}$ and
$\mathbf{s}$. (No dimensional wave vectors will be involved in
what follows.) In addition, we introduce the symbol
$\alpha_\perp = \lambda_\perp^2 / l^2$. The integrand in (16) is a product of ex-
ponentials, the arguments of which are polynomials of
second degree of the components of the vectors $\mathbf{k}$ and
$\mathbf{k}'$. It is easy to transform these polynomials in such
a way that the angle φ' enters only in the form of the
difference $\chi = \varphi - \varphi'$, after which the summation over
$\mathbf{k}$ and the integration over χ can be carried out inde-
pendently. Carrying out first the summation over $\mathbf{k}$,
we obtain

$$\sum_k f_s(k) = \frac{el}{\hbar^2} n_e u_{-s} \left[(\omega^2 - \omega_c^2)^{-1} \left(\mathbf{E}_\perp \mathbf{s}_\perp + i \frac{\omega_c}{\omega} [\mathbf{E}_\perp \mathbf{s}_\perp] \right) \right. \quad (19)$$

$$+ \omega^{-2} E_z s_z \Big] \exp \left(-\frac{1}{4} a s_z^2 - \frac{1}{2} a_\perp^{-1} s_\perp^2 \right) L(s_\perp, s_z),$$

where

$$L(s_\perp, s_z) = L' + iL'' = \int_{-\infty}^0 d\chi \, (e^{-i\omega\chi/\omega_c} - 1) \quad (20)$$

$$\times \left\{ \exp \left[\frac{1}{2} s_\perp^2 \mathrm{sh}^{-1} a \cos(\chi - ia) - \frac{1}{4} a^{-1} s_z^2 (\chi - ia)^2 \right] - \text{c.c.} \right\}.$$

Substituting (19) in (17), we obtain the components of
the conductivity tensor $\sigma_{ik} = \sigma_{ik}^{(0)} + \sigma_{ik}^{(1)}$, where $\sigma_{ik}^{(0)}$ and
$\sigma_{ik}^{(1)}$ are the zeroth and first terms of the expansion of
σ_{ik} in $(\omega_c \tau)^{-1}$ (or in $(\omega\tau)^{-1}$), corresponding to the first
and second terms in (17) on the right. The diagonal
components of the tensor $\sigma_{ik}^{(1)}$ are equal to

$$\sigma_{xx}^{(1)} = \sigma_{yy}^{(1)} = \frac{e^2 n_e N (\omega^2 + \omega_c^2)}{2m\hbar^2 \omega (\omega^2 - \omega_c^2)^2} \sum_s |u_s|^2 s_\perp^2 L(s_\perp s_z)$$

$$\times \exp \left\{ \left(-\frac{1}{4} a s_z^2 - \frac{1}{2a_\perp} s_\perp^2 \right) \right\}, \quad (21a)$$

$$\sigma_{zz}^{(1)} = \frac{e^2 n_e N}{m\hbar^2 \omega^3} \sum_s |u_s|^2 s_z^2 L(s_\perp, s_z) \exp \left\{ \left(-\frac{1}{4} a s_z^2 - \frac{1}{2a_\perp} s_\perp^2 \right) \right\}. \quad (21b)$$

In the derivation of (21) it was assumed that $|u_s|^2$ is
independent of the direction of $s_\perp$.

The absorption of the electromagnetic radiation is
determined by the real part of the diagonal component
of the conductivity tensor $\sigma'_{ii} = \mathrm{Re}\, \sigma_{ii}$. In the absence
of scattering, there is no absorption: $\sigma_{ii}^{(0)\prime} = 0$. To
obtain $\sigma'_{ii} = \sigma_{ii}^{(1)\prime}$ it is necessary to separate L' in (20).
After simple transformations we obtain

$$L'(s_\perp, s_z) = \mathrm{sh} \left(a \frac{\omega}{\omega_c} \right) \int_{-\infty}^\infty d\chi \, e^{-i\omega\chi/\omega_c} \exp \left(\frac{1}{2} s_\perp^2 \frac{\cos\chi}{\mathrm{sh}\, a} - \frac{s_z^2}{4a} \chi^2 \right). \quad (22)$$

In the classical limit $a \ll 1$ the essential values of
s_z in (21) are such that only small angles are of im-
portance in (22). Representing $\cos\chi$ in the form
$1 - \chi^2/2$ and integrating with respect to χ, we can
obtain expressions for σ'_{xx} and σ'_{zz} in the absence of
quantization.

In the quantum case $a \gtrsim 1$ it is convenient to trans-
form the formula in a different manner. To this end,
we expand the exponential containing $\cos\chi$ in the
argument in a Fourier series in χ and integrate with
respect to χ. We obtain

$$L'(s_\perp, s_z) = \frac{2\pi^{1/2} a^{1/2}}{|s_z|} \mathrm{sh} \left(a \frac{\omega}{\omega_c} \right) \sum_{n=-\infty}^\infty I_n \left(\frac{s_\perp^2}{2\,\mathrm{sh}\, a} \right)$$

$$\times \exp \left[-\frac{a}{s_z^2} \left(n - \frac{\omega}{\omega_c} \right)^2 \right], \quad (23)$$

where $I_n(x)$ is a Bessel function. Substituting this in
(21), we obtain σ'_{ii}.

6. Let us analyze the transverse conductivity. If
ω/ω_c tends to an integer n_0 ($n_0 \neq 1$), then the contri-
bution of the term with $n = n_0$ in the sum (23) to σ'_{xx}
increases and becomes infinite at $\omega/\omega_c = n_0$, since
the sum over s_z in (21a) turns out to be logarithmically
divergent in the region of small $|s_z|$. Thus, the de-
pendence of σ'_{xx} on ω (or on ω_c) has an oscillating
character with maxima (infinite in our approximation)
at $\omega = n_0 \omega_c$. Gurevich and Firsov[1] obtained an ex-
pression for the transverse static conductivity σ_{GF} at
$\omega_c \tau \gg 1$ for scattering by phonons. In the case of
scattering by optical phonons, σ_{GF} also goes logarith-
mically to infinity as $\omega_c \to \omega_0/n_0$ (magneto-phonon
oscillations). The similarity in the physical nature of
the oscillations of σ'_{xx} and σ_{GF} is reflected in the
fact that the sum contained in (23) is present also in
σ_{GF}, the only difference being that ω is replaced by

the phonon frequency. In addition to the indicated sums, σ'_{XX} and σ_{GF} also contain factors that depend in smooth fashion on ω_C. These factors are naturally different and therefore the dependences on $\omega_C \sigma'_{XX}$ (at $\omega = \text{const}$) and σ_{GF} are generally speaking different. When $\alpha \gg 1$, however, the width of the oscillations is small compared with ω_C, and the smooth functions ω_C, near the oscillations can be regarded as constant. Then the dependences of σ'_{XX} and σ_{GF} near the resonances are identical. As $\omega \to 0$ we find that σ'_{XX} coincides with σ_{GF}, if we set the phonon frequency equal to zero in the latter.

It is clear that the existence of oscillations of the transverse conductivity is not connected with the details of the interaction of the electron with the scatterers, i.e., with the concrete form of u_s. We shall consider below in detail only the case when $|u_s|^2$ does not depend on s (pointlike centers, elastic interaction with acoustic phonons). We change over from summation over s to integration. The latter can be carried out[11]. The result takes the form

$$\sigma_{xx}' = \frac{e^2 n_e \hbar \omega_c^2 (\omega^2 + \omega_c^2)}{2\pi^{1/2} mT\omega\tau_0 (\omega^2 - \omega_c^2)^2} \,\text{sh}\left(\alpha\frac{\omega}{\omega_c}\right)$$

$$\times \left\{ \text{cth}\,\alpha K_0\left(\alpha\frac{\omega}{\omega_c}\right) + \sum_{n=1}^{\infty} e^{-\alpha n}(\text{cth}\,\alpha + n)\left[K_0\left(\alpha\left|n - \frac{\omega}{\omega_c}\right|\right) \right. \right.$$
$$\left. \left. + K_0\left(\alpha\left(n + \frac{\omega}{\omega_c}\right)\right)\right]\right\}. \quad (24)$$

We have introduced here the momentum relaxation time τ_0 of an electron with energy T at $H = 0$. For pointlike defects it is equal to

$$\tau_0^{-1} = \frac{NVm|u|^2}{\pi\hbar^4}\sqrt{2mT}, \quad (25)$$

for acoustic phonons it is necessary to replace $NV|u|^2$ in (25) by $C^2 T/dv_0^2$, where C is the constant of the deformation potential, d is the crystal density, v_0 is the speed of sound, and $K_0(x)$ is the Macdonald function.

Let us analyze (24) in the quantum case $e^{-\alpha} \ll 1$, bearing in mind that[11]

$$K_0(x) = \begin{cases} -\ln(x/2), & x \ll 1, \\ \sqrt{\pi/2x}\,e^{-x}, & x \gg 1. \end{cases} \quad (26)$$

We first discard the term containing $K_0(\alpha(n + \omega/\omega_C))$ (it is always exponentially small compared with the first term in the curly bracket) and replace $\text{coth}\,\alpha$ by unity. We consider first σ'_{XX} far from the resonances (the "background," $\sigma'_{XX} = \sigma'_{xxb}$):

$$\alpha|n - \omega/\omega_c| \gg 1 \quad (27)$$

for any n, including zero. Then for $n < \omega/\omega_C$ we can take the asymptotic expression for k_0, and discard the terms with $n > \omega/\omega_C$. We obtain

$$\sigma_{xxb}' = \frac{e^2 n_e \hbar \omega_c^2 (\omega^2 + \omega_c^2)}{4\sqrt{2mT}\omega\tau_0\alpha^{1/2}(\omega^2 - \omega_c^2)^2} \sum_{n=0}^{\infty} (n+1)\left(\frac{\omega}{\omega_c} - n\right)^{-1/2}\theta\left(\frac{\omega}{\omega_c} - n\right), \quad (28)$$

where $\theta(x)$ is the Heaviside function.

Let now for a certain $n_0 \neq 0, 1$

$$\alpha|\omega/\omega_c - n_0| \ll 1. \quad (29)$$

In the terms with $n < n_0$ we replace $K_0(x)$ by the asymptotic expression, and in the term with $n = n_0$ we

replace it by the expression for small x, and we discard the terms with $n > n_0$. In addition, where possible, we write ω/n_0 in place of ω_C. In this case $\sigma'_{XX} = \sigma'_{ssres}$, where

$$\sigma_{xx\,res}' = \frac{e^2 n_e \hbar (n_0^2 + 1)}{4\sqrt{2mT}\alpha^{1/2}\omega\tau_0 (n_0^2 - 1)^2} \quad (30)$$

$$\times \left[\sum_{n=0}^{n_0-1} (n+1)(n_0 - n)^{-1/2} - \sqrt{\frac{2\alpha}{\pi}}(n_0 + 1)\ln\left(\frac{\alpha}{2}\left|n_0 - \frac{\omega}{\omega_c}\right|\right)\right]$$

The first term in the square brackets describes the absorption of the background and the second the resonant oscillations. If α is sufficiently large, this term can be large compared with the first. The oscillating part of the conductivity contains the logarithm usually obtained in this situation, which diverges as $\omega \to n_0 \omega_C$. At small $|\omega - n_0\omega_C|$, it is necessary to introduce cutoff because the assumed approximations are not valid near resonance (the Born approximation, elasticity of scattering, etc.; the question of the most effective cutoff mechanism is beyond the scope of the present article).

Let $\alpha(\omega/\omega_C) - n_0 \gg 1$, We can then use formula (28). Assume, on the other hand, that α is large enough to make $\omega/\omega_C - n_0 \ll 1$. Then the term $n = n_0$ in (28) is larger than all the remaining terms. It can be regarded as the wing of the resonant oscillation. Thus, the absorption on the high-frequency side of the oscillation varies like $(\omega/\omega_C - n_0)^{-1/2}$. The dependence of the absorption on the frequency on the low-frequency side is logarithmic. With increasing ω ($\omega_C = \text{const}$) the absolute intensity of the oscillations decreases in proportion to n_0^{-2}, and the intensity relative to the background decreases like $n_0^{-1/2}$ ($n_0 \gg 1$). On the other hand, if $\omega = \text{const}$, then the absolute and relative intensities are proportional to $n_0^{-1/2}$ with decreasing ω_C. (It is assumed that the logarithm depends little on n_0.)

When $0 < \omega/\omega_C < 1$ it is necessary to retain in (24) only the first term in the curly bracket. Letting ω approach zero and cutting off the argument of the function K_0 in suitable fashion, we obtain the well-known formula for the transverse static conductivity in a quantizing magnetic field[12] in the particular case corresponding to scattering by a short-range potential (formula (30) at $n_0 = 0$ with replacement of the argument of the logarithm by the cutoff parameter). With increasing ω, the conductivity first depends little on the frequency (at $\alpha\omega/\omega_C \lesssim 1$). On the other hand, in the region $\alpha\omega/\omega_C \gg 1$ at $\omega \ll \omega_C$, the conductivity is proportional to $\omega^{-3/2}$ (formula (28)).

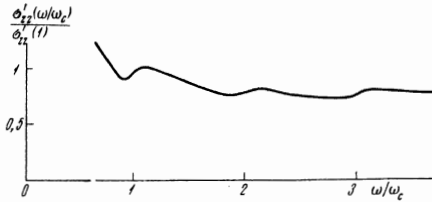Let us investigate absorption in parallel fields. We obtain in place of (24)

$$\sigma_{zz}' = \frac{e^2 n_e \hbar \omega_c^2}{\pi^{1/2} mT\omega^3\tau_0}\,\text{sh}\left(\alpha\frac{\omega}{\omega_c}\right)\left\{\frac{\omega}{\omega_c}K_1\left(\alpha\frac{\omega}{\omega_c}\right)\right. \quad (31)$$

$$\left. + \sum_{n=1}^{\infty} e^{-\alpha n}\left[\left|n - \frac{\omega}{\omega_c}\right|K_1\left(\alpha\left|n - \frac{\omega}{\omega_c}\right|\right) + \left(n + \frac{\omega}{\omega_c}\right)K_1\left(\alpha\left(n + \frac{\omega}{\omega_c}\right)\right)\right]\right\}.$$

Recognizing that[11]

$$K_1(x) = \begin{cases} x^{-1}, & x \ll 1, \\ \sqrt{\pi/2x}\,e^{-x}, & x \gg 1, \end{cases} \quad (32)$$

we have in place of (28)

$$\sigma_{zzb}' = \frac{e^2 n_e \hbar \omega_c^2}{2\sqrt{2\alpha}mT\omega^3\tau_0} \sum_{n=0}^{\infty}\left(\frac{\omega}{\omega_c} - n\right)^{1/2}\theta\left(\frac{\omega}{\omega_c} - n\right), \quad (33)$$

Dependence of the function $\sigma'_{zz}(\omega/\omega_c)/\sigma'_{zz}(1)$ on ω/ω_c at $\omega =$ const and $\alpha' = 10$.

and in place of (30)

$$\sigma'_{zz\,res} = \frac{e^2 n_e \hbar}{2\sqrt{2}\,\alpha m T \omega \tau_0 n_0^2}\left\{ \sum_{n=0}^{n_0-1}(n_0-n)^{1/2} + \left(\frac{2}{\alpha\pi}\right)^{1/2}\left[1 + \alpha\left(\frac{\omega}{\omega_c}-n_0\right)\right]\right\}.$$

(34)

From (34) we see that the point $\omega = n_0\omega_c$ is not an extremum of $\sigma'_{zz\,res}$.

Let us consider in greater detail the dependence of σ'_{zz} on $\nu = \omega/\omega_c$ at $\omega =$ const and $\alpha' \equiv \alpha\omega/\omega_c$ $= \hbar\omega/2T > 1$. The plot of $\sigma'_{zz}(\nu)/\sigma'_{zz}(1)$ at $\omega =$ const and $\alpha' = 10$ is shown in the figure. The main features of the function $\sigma'_{zz}(\nu)$ are as follows: far from the resonances $d\sigma'_{zz}/d\nu < 0$, near $\nu = n_0$ and at $\nu > n_0$ the quantity $d\sigma'_{zz}/d\nu$ has a maximum. If α' is sufficiently large, then it can turn out that $d\sigma'_{zz}/d\nu$ is positive at the maximum. If this is so, then σ'_{zz} near the resonance has a minimum on the left (at $\nu = \nu_{min}$ $< \eta_0$) and a maximum on the right (at $\nu = \nu_{max} > \eta_0$), i.e., σ'_{zz} experiences oscillations near $\nu = n_0$. These oscillations differ from the oscillations of σ'_{xx} in some respects.

1) The oscillation with serial number n_0 exists only when α' is larger than a certain value that depends on n_0: $\alpha' > \alpha'_{n_0}$. (In the transverse case, only the relative magnitude of the oscillations depends on α'.) 2) The maximum of σ'_{zz} lies to the right of the point $\nu = n_0$. 3) The magnitude of the oscillations is much smaller. Thus, for $n_0 = 1, 2, 3$, the amplitude of the σ'_{zz} oscillations is smaller by an approximate factor α' than for σ'_{xx}. (We assume that the logarithm in (30) amounts to several units.) 4) The value of σ'_{zz} at the maximum is finite already in the first approximation of perturbation theory.

The main conclusion that can be drawn from this is as follows: the influence of the resonance on σ'_{zz} is much smaller than on σ'_{xx}. This is due to the fact that the resonance affects most strongly the electrons with small k_z, the contribution of which to σ'_{zz} is small.

To determine α'_{n_0} it would be necessary to determine the values of α' at which $d\sigma'_{zz}/d\nu$ vanishes near $\nu = n_0$. This can apparently be done only by numerical means. However, α'_{n_0} can be estimated by the following method. Using the fact that, as shown by numerical calculation, the maxima of $d\sigma'_{zz}/d\nu$ lie at points close to n_0, we determine the conditions under which the derivative $d\sigma'_{zz}/d\nu$ vanishes at the point $\nu = n_0$. A simple calculation shows that α' should then equal

$$(\alpha')^{1/2} = \tfrac{1}{2}C_{n_0} + \sqrt{\tfrac{1}{4}C^2_{n_0} + 1}, \tag{35}$$

where

$$C_{n_0} = \left(\frac{\pi}{8}\right)^{1/2}\sum_{n=0}^{n_0-1}\frac{2n_0-3n}{\sqrt{n_0^2-n_0 n}}. \tag{36}$$

It can be assumed, roughly speaking, that α' as determined from (35) coincides with α'_{n_0}. With increasing n_0, C_{n_0} increases. Therefore oscillations of higher order appear at larger α'. The first few values of α'_{n_0} are $\alpha'_1 = 3.8$, $\alpha'_2 = 4.7$, and $\alpha'_3 = 6.4$.

As noted earlier, σ'_{zz} reaches a maximum to the right of integer values of ν. Oscillations of parallel conductivity also arise in magneto-phonon resonance[4]. The question of the exact position of the extrema was not investigated in[4]. It was observed, however, in the experimental study of magnetophonon oscillations[5] that the extrema of the parallel conductivity also lie to the right of integer values of ω_0/ω_c. The shift of the extrema of σ'_{zz} towards larger values of ω/ω_c can be explained as follows. Absorption at $\alpha \gtrsim 1$ arises as a result of transitions of the electrons from the states $(0, k_z)$ to the states (n, k'_z) (n—number of the Landau zone), owing to the joint action of the scattering potential and of the electric field. The energy increases in the transition by $\hbar\omega$. The value of σ'_{zz} is proportional to the intensity of the transitions (to the number of transitions per second). This intensity can be obtained by integrating the transition probability over the final states and over the electron distribution in the initial states. At $\omega = n_0\omega_c$, transitions between states $(0, k_z)$ and $(n_0 k_z)$ without a change of k_z become possible. The product of the state densities for such transitions is proportional to k_z^{-2}, which gives rise to a logarithmic divergence in σ'_{xx}. But the transitions without change of k_z make no contribution to σ'_{zz} (they do not lead to relaxation of the longitudinal current): the probability of the transition with absorption of a quantum $\hbar\omega$ turns out to be proportional to $(k'_z - k_z)^2$. The intensity of such transitions, as can easily be verified, reaches a maximum at ω somewhat larger than $n_0\omega_c$.

We point out that formula (31), apart from the factor in front of the curly bracket and apart from exponentially small terms, can be obtained if one calculates the intensity of the transitions in the manner indicated above, using a probability proportional to $(k'_z - k_z)^2\delta(\epsilon' - \epsilon - \hbar\omega)$, where ϵ and ϵ' are the energies of the initial and final states. It should be noted that when account is taken of the cutoff mechanism, it may turn out that the maxima of σ'_{xx} likewise do not coincide with the points $\omega/\omega_c = n_0$.

8. The requirements imposed on the magnetic fields and on the temperatures, which are needed for observation of the harmonics of cyclotron resonance $\omega_c\tau$ $\gg 1$ and $\alpha > 1$, are easily satisfied in typical semiconductors (Ge, Si, InSb) at helium temperatures for a wavelength $\sim 100\,\mu$. These are in essence the same conditions which are needed for cyclotron-phonon resonance. As to the choice of the material, there is a difference here. The conditions $\alpha > 1$ and $\omega_c\tau \gg 1$, which are necessary for the observation of resonances of all three types (A, B, and C; see Sec. 1) denote in practice the use of temperatures such that scattering by optical phonons is secondary. Therefore in cases B and C the background is connected with the main scattering mechanism, and the oscillations with the secondary one. To observe the resonances B and C it is necessary to choose materials in which the interaction of the electrons with the optical phonons is

relatively large (for example, ionic semiconductors). To the contrary, in the situation considered here, both the background and the oscillations are connected with the same (main) mechanism. Therefore to observe cyclotron-harmonic resonance both ionic and covalent semiconductors—Ge and Si—are equally suitable. To obtain the necessary electron density in Ge and Si at helium temperatures we can use the method that has become already standard in experiments on cyclotron resonance, namely illumination of the semiconductor with white light. (We note that at sufficient illumination intensity the scattering by acoustic phonons in these semiconductors becomes predominant[13].) An estimate using the characteristics of an n-InSb submillimeter-band receiver[14] shows that the nonresonant conductivity signal is much larger under these conditions than the noise level. Therefore the oscillations of σ'_{xx}, which are not small compared with the background, should be observed. The amplitude of the σ'_{zz} oscillations depends on α' and n_0. In a particular case ($\alpha' = 10$ and $n_0 = 1.2$) it amounts to $\sim 10\%$ of the background. However, since the signal-noise ratio is large, such changes of the absorption should also be observed.

Cyclotron-resonance harmonics were observed many times. Let us stop to discuss one of the latest papers[15]; in which the first serious attempt has apparently been made to interpret the harmonics. In[15], the harmonics up to the fifth were observed in n-InSb at 20°K and $E \perp H$. Radiation with wavelength $\sim 100\ \mu$ was used. The authors show that such factors as the nonparabolicity of the conduction band and the finite value of the radiation wave vector, which, in principle, can cause the appearance of harmonics, cannot explain the results. It is indicated further that scattering by acoustic phonons cannot produce harmonics either. This explanation is also excluded by the authors for two reasons. 1) A frequency shift by an amount equal to the frequency should be observed of a certain characteristic phonon that participates in the scattering. 2) The interaction with the acoustic phonons is much weaker than with the optical ones. The intensity of the harmonics should therefore be lower than the intensity of the peaks of the cyclotron-phonon resonance (the latter was also observed in[15], but with the opposite intensity ratio). Thus[15], the presence of harmonics is not explained.

It follows from the present exposition that the harmonics are produced by elastic scattering from

acoustic phonons; no characteristic energy of the phonon appears at all. Therefore the first objection is eliminated. The second objection is substantial, but the following explanation is possible. In samples similar to those used in[15] (impurity concentration difference $\sim 10^{15}\ \text{cm}^{-3}$) there takes place at 20°K a strong scattering by ionized impurities. According to our results, at $E \perp H$ the scattering by all static defects (including impurities) should lead to the appearance of harmonics.

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