INELASTIC ELECTRON COLLISIONS AND NONEQUILIBRIUM STATIONARY STATES IN SUPERCONDUCTORS

G. M. ÉLIASHBERG

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

Submitted April 21, 1971

Zh. Eksp. Teor. Fiz. 61, 1254–1272 (September, 1971)

A derivation of the dynamic equations for superconductors is presented, in which inelastic collisions between electrons or between electrons and phonons are taken into account explicitly. It is assumed that the phonons are in equilibrium (play the role of a thermostat). The conditions for which this assumption is valid are discussed. The equations are written in a form which is analogous to the kinetic equation but their structure is much more complex than for a normal metal. This complication is mainly due to the need for taking into account changes induced in the excitation spectrum density by an alternating external field. For thin films taken as an example, it is shown that under certain conditions the alternating field effect consists mainly of a redistribution of the excitations with respect to energy, the spectrum density remaining constant. In this case the set of equations reduces to an equation for the energy gap, which is identical in form to the BCS equation but has a nonequilibrium distribution function, and to a nonlinear kinetic equation for this function.

1. CHOICE OF MODEL

ONE of the important problems of the dynamic theory of superconductivity is the investigation of the nonequilibrium stationary states of the superconductor that are produced by continuous application of an alternating field. It is clear that the existence of such states is possible only because of the coupling between the electrons and the thermostat, the role of which is played in final analysis by the cooling medium. The mechanism of this coupling can be quite complicated, but it is realized in any case by the inelastic collisions of the electrons with the phonons, with the walls of the sample, and also with one another. Such collisions, however, are not considered in the simplified BCS model[1], on which the greater part of the dynamic-theory investigations is based. This is justified by the relatively low probability of the inelastic electron scattering processes in metals. The corresponding damping of the excitations γ in the normal state is, as is well known, of the order of T^3/Θ_D^2 and T^2/E_F in electron-phonon and electron-electron scattering, and is comparable with the energy scale characteristic of superconductors, namely the value of the parameter Δ, only in a very small neighborhood of the transition temperature:

$$(T_c - T)/T_c \sim (T_c/\Theta_D)^4, \quad (T_c/E_F)^2.$$

In the case of bulky superconductors, the perturbation connected with the action of the alternating field is localized near the surface, and the kinetics of the process is determined mainly by the diffusion relaxation mechanism. Of course, such a mechanism by itself does not lead to a stationary regime. Nonetheless, if the time-averaged deviation from equilibrium is small in the stationary state, then the final results do not contain the characteristics of the inelastic processes. On the other hand, it is precisely by virtue of the slowness of the energy relaxation of the electrons that the dynamic properties of the superconductors are strongly influenced by the deviation from equilibrium in the energy distribution of the electrons; at a small average value, the collision amplitude becomes appreciable. The hydrodynamic approximation is therefore not suitable here, and there is no simple generalization of the Ginzburg-Landau theory for the nonequilibrium case[2-4].

Moreover, the large value of the electron-energy relaxation time gives rise to a noticeable stationary deviation in the energy distribution of the electrons even for relatively weak excitation. A study of the effects that may be connected with this calls for explicit allowance for the inelastic processes. Such an allowance is essential, in particular, for the investigation of the dynamics of superconductors of small dimensions, when the entire volume of the sample is in the nonequilibrium state, and there is no diffusion relaxation.

Before we proceed to derive the equations, we must dwell on the choice of the model. We consider below the simplest case, when the role of the thermostat is played by the lattice, so that the coupling of the electrons with the thermostat is effected by electron-phonon interaction. A definite role can be played here also by electron-electron collisions, which we shall also take into consideration.

Let us discuss now the degree to which this assumption can be regarded as justified in different cases.

The simplest situation in this sense occurs in the case of samples of small dimensions, particularly for thin films, which will be discussed here for concreteness. We are interested in films whose thickness is of the same scale as the correlation radius $\xi_0 \sim \hbar v/T_c$ (v is the electron velocity on the Fermi surface), i.e., much larger than the wavelength of the thermal phonon at $T \sim T_c$. If the ρs (ρ—density, s—speed of sound) in the metal and in the cooling medium are of the same order of magnitude, then the phonon will give up its energy practically in each collision with the wall (see, for example,[5]). On the other hand, the free-path time of a thermal phonon in a bulky sample, connected with

its interaction with the electrons, is of the order of $\hbar v/sT$[6], so that the corresponding free path is $\sim\xi_0$ at $T \sim T_c$. Thus, at a film thickness $d < \xi_0$, the phonon distribution will be determined by the collisions with the walls and its deviation from equilibrium will be small. On the other hand, if ρs is much smaller in the medium than in the metal, then to satisfy this condition it is necessary to have correspondingly thinner films. In the case of relatively thick films, when the phonons relax mainly on electrons, their distribution function will align itself to the distribution of the electrons. Since the phonon specific heat is small compared with that of the electrons, the phonons in this case exert no direct influence at all on the stationary distribution of the electronic excitations, which in this method of excitation is determined by the electron-electron collisions and by the condition that determines the equality of the energy flux absorbed by the electrons to the flux carried away through the walls. In this case, generally speaking, the situation does not reduce at all to simple heating: the electron distribution, and with it the phonon distribution, can differ noticeably from the temperature distribution.

All the foregoing pertains to some intermediate temperature region, when $\Delta \sim T$. The situation can change greatly at low temperatures and in the vicinity of T_c. If $T \ll \Delta$, then at not too intense an irradiation, the number of the excitations is small and they therefore exert a small influence on the phonon system, so that the latter remains at equilibrium also in the case of films thicker than indicated above. The same takes place, albeit for another reason, when $T \gg \Delta$: if the field frequency is comparable with Δ, then it can be shown[7] that the nonequilibrium excitations are concentrated in the energy region $\epsilon \sim \Delta \ll T$, and the bulk of the excitations remains in equilibrium and plays, together with the phonons, the role of the thermostat.

These last considerations, which pertain to extreme temperatures, remain in force also for a bulky superconductor: the region of low temperatures and the vicinity of T_c are here, too, favorable for the use of the chosen model. At the same time, when the problem is formulated in greater detail, it must be borne in mind that in the case of a bulky sample we deal with a picture that is essentially inhomogeneous in space. The nonequilibrium excitations produced in the region δ of field penetration can then proceed from the surface to a greater or a smaller depth, which depends on the ratio of the transport path time τ, which we shall assume to be determined by the impurities, to the energy relaxation time τ_0. If $\tau \ll \tau_0$, then we have diffusion of the nonequilibrium excitation to a depth $L \sim (l v \tau_0)^{1/2}$, which, depending on the impurity concentration, on the temperature, and on the field frequency, can be either larger or smaller than δ. Owing to the large value of τ_0, the inequality $L \gg \delta$ can be satisfied also under the condition $l \ll \xi_0$, and we therefore confine ourselves here to a discussion of only this case. The form of the electron distribution will change with increasing distance from the surface. At distances small compared with L, this change is due mainly to the dependence of the diffusion coefficient of the excitations in the superconductor on the energy: $D_\epsilon = (l v_\epsilon/3 \sim \sqrt{\epsilon^2 - \Delta^2}$, At distances comparable with L, inelastic processes come into play. If the phonon mean free path connected with their interaction with the electrons is larger than L, as can occur at low temperatures, then the phonons remain in equilibrium, and the electrons relax over a length L to the temperature of the phonon thermostat. A similar situation arises, for the reason indicated above, also near T_c. In these cases, the mechanism whereby the heat is carried away does not enter explicitly into the problem.

In the intermediate temperature interval, and also at low temperatures under conditions of sufficiently strong pumping, the number of excitations is large and the equilibrium in their distribution is violated in the entire energy interval. Therefore, as in the foregoing example of a relatively thick film, the phonons have time to align themselves to the electron distribution and drop out of the game. In this case, owing to the electron-electron collisions, thermalization of the excitation gas takes place over distances $\sim L$, but the corresponding temperature may differ from the initial one, and the heat removal must be taken into account explicitly for its determination.

The foregoing estimates, of course, are schematic and need refinement as applied to concrete problems. Nonetheless, they show that there exists a sufficiently wide circle of problems the investigation of which can be carried out on the basis of the chosen model and does not require explicit allowance for the heat-removal mechanism.

2. DYNAMIC EQUATIONS IN THE PHONON MODEL

In order to take the electron-phonon collisions into account in a consistent manner, we derive the dynamic equations on the basis of the phonon model. As in the paper of Gor'kov and the author[3], the starting point is the concept of discrete imaginary frequencies, in which the expansion of the Green's function in powers of the external field has a simple diagram structure. The essential complication that arises when working with the phonon model is that it is necessary to deal here with energy-dependent self-energy parts, which will be determined by the same diagram approximation as in the equilibrium case[8] (Fig. 1). ($P = \{\xi, p\}$, $K = \{\omega, k\}$):

$$\Sigma_1(P, P-K) = T\sum_{\varepsilon'}\int \frac{d^3p'}{(2\pi)^3} D(P'-P)\,G(P', P'-K), \qquad (1)$$

$$\Sigma_2(P, P-K) = T\sum_{\varepsilon'}\int \frac{d^3p'}{(2\pi)^3} D(P'-P)\,F(P', P'-K).$$

The functions G and F are assumed to be exact in the sense that they include, besides the external alternating field, also the self-energies Σ_1 and Σ_2. In contrast, the Green's function of the phonons is assumed to be free:

$$D(P'-P) = g^2 \frac{2\omega_{p'-p}}{\omega^2_{p'-p} - (\varepsilon'-\varepsilon)^2}, \qquad (2)$$

corresponding to the assumption that the phonons are

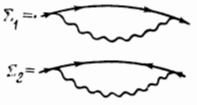

FIG. 1

in equilibrium. (The renormalization of the phonon spectrum can be regarded as taken into account in (2), since it is practically independent of the smearing region near the Fermi surface, which is affected by violation of the equilibrium.)

We shall carry out an analytic continuation of (1) by regarding G and F as sums of diagrams of different order in the alternating field. Since the entire procedure is independent of whether we are dealing with Σ_1 or Σ_2, we shall set the function G in correspondence with the electron line and will not mark the directions of the arrows on the diagrams.

Let us consider a diagram of N-th order in the field. It depends on the frequencies of its extreme electron lines ϵ and $\epsilon - \omega$, and in addition it depends on the frequencies ω_l corresponding to the field vertices ($\sum_{l=1}^{N} \omega_l = \omega$). Since in the approximation (1) = (2) a single electron line passes through any diagram G, the field vertices can be numbered in the sequence in which they follow along this line from left to right, regardless of whether this diagram contains the insert Σ or not (Fig. 2). Let us ascertain the analytic structure of $G^{(N)}$ as a function of the variable ϵ at fixed $\omega_l = 2m_l\pi Ti$. We are interested in the final analysis in the continuation of each of the ω_l from the upper half-plane, and we can therefore assume that all the $m_l > 0$. The diagram of Fig. 1a (i.e., the one not containing Σ), is a product of the zeroth-approximation functions $G(P) = (\xi - \epsilon)^{-1}$, $\xi = v(p - p_0)$:

$$G^{(N)}_{\varepsilon,\varepsilon-\omega} \infty G_\varepsilon G_{\varepsilon-\omega_1} G_{\varepsilon-\omega_1-\omega_2} \dots G_{\varepsilon-\omega}$$

and its singularities as functions of ϵ (in this case, poles), located on lines parallel to the real axis: $\mathrm{Im}\,\epsilon = 0$, $\mathrm{Im}(\epsilon - \omega_1) = 0, \dots, \mathrm{Im}(\epsilon - \omega) = 0$. We shall show now that the same lines are singular also for a diagram $G^{(N)}$ of arbitrary form (for example, Fig. 2b). To this end, obviously, it suffices to verify that $\Sigma^{(N)}$ has the same analytic structure as its internal electron line.

To do so, we transform the sum in (1) into a contour intergral, drawing the cuts in the ϵ' plane along the indicated singular lines $G^{(N)}_{\epsilon',\epsilon'-\omega}$ and along the singular line of the D-function $\mathrm{Im}(\epsilon' - \epsilon) = 0$ ($\epsilon = (2n + 1)\pi Ti$), and choosing the integration contour as shown in Fig. 3. We denote the jump of the $G^{(N)}$ function on its l-th cut by $\delta_l(G^{(N)})$, numbering these cuts in the upward direction. We then obtain the following representation for $\Sigma^{(N)}$:

$$\Sigma^{(N)}_{\varepsilon,\varepsilon-\omega} = \int_{-\infty}^{\infty} \frac{dz}{4\pi i} \Big\{ \mathrm{cth}\frac{z}{2T}(D^R - D^A)_z G^{(N)}_{z+\varepsilon,z+\varepsilon-\omega} + \mathrm{th}\frac{z}{2T}[D(z-\varepsilon)\delta_0(G^{(N)}) + D(z-\varepsilon+\omega_1)\delta_1(G^{(N)}) \dots + D(z-\varepsilon+\omega)\delta_N(G^{(N)})] \Big\}. \quad (3)$$

Since the integration variable z is real, we see that $\Sigma^{(N)}$ contains the same combinations of ϵ and ω_l as $G^{(N)}$, thus proving the statement made above.

Continuing (3) with respect to ϵ from the regions $\mathrm{Im}(\epsilon - \omega) > 0$ and $\mathrm{Im}\,\epsilon < 0$, we obtain the contributions of N-th order in the field to Σ^R and Σ^A, respectively. In these cases, all the combinations of ϵ with ω_l that enter in (3) have imaginary parts of definite sign, and therefore the successive continuation with respect to each of the ω_l does not depend on the order. Shifting then the integration variables in each term of (3) in a simple manner, so as to restore the initial designation of the arguments, and summing over all the N, we arrive at the following formulas:

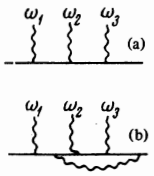

FIG. 2

$$\Sigma_1^{R(A)}(P, P-K) = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \int \frac{d^3p'}{(2\pi)^3} \Big\{ \mathrm{cth}\frac{\varepsilon'-\varepsilon}{2T}(D^R - D^A)_{P'-P} \times G^{R(A)}(P', P'-K) + D^{A(R)}(P'-P)G(P', P'-K) \Big\}, \quad (4)$$

where

$$G(P, P-K) = \sum_{N=0}^{\infty} \sum_{l=0}^{N} \delta_l(G^{(N)}) \mathrm{th}\frac{\varepsilon_l}{2T}, \quad \varepsilon_l = \varepsilon - \omega_1 - \dots - \omega_l. \quad (5)$$

Expressions for $\Sigma_2^{R(A)}$ are obtained from this by replacing $G^{R(A)}$ and G by $F^{R(A)}$ and F. Since all the free lines and the proper spectra forming the diagrams $G^{R(A)}$ are retarded (advanced), it follows that $G^{R(A)}$ and $F^{R(A)}$ can be expressed in the usual manner in terms of $\Sigma_1^{R(A)}$ and $\Sigma_2^{R(A)}$:

$$\left\{ \begin{pmatrix} \xi - \varepsilon - \Sigma_1 + H_1 & -\Sigma_2 \\ \Sigma_2^+ & \xi + \varepsilon - \bar{\Sigma}_1 + \bar{H}_1 \end{pmatrix} \begin{pmatrix} G & F \\ -F^+ & \bar{G} \end{pmatrix} \right\}^{R(A)}_{P,\ P-K} = \hat{1}(2\pi)^4 \delta(\omega)\delta(\mathbf{k}),$$
$$H_1 = -\frac{e}{c}(\mathbf{vA}) + e\varphi, \quad \bar{H}_1 = \frac{e}{c}(\mathbf{vA}) + e\varphi. \quad (6)$$

The function G, introduced by the relation (5), and the analogous function F, play the principal role in the theory of nonequilibrium phenomena. We shall soon show that all the physical quantities, namely the current, Δ, etc., are expressed directly in their terms.

To establish the connection between G and Σ, we rewrite (5) in a somewhat different form, namely, we separate from the contribution of each diagram the terms corresponding to the upper edge of the upper cut and the lower edge of the lower cut (Fig. 3). The separated terms obviously define the retarded and advanced Green's functions, so that we obtain the following definition of the new quantity $G^{(a)}$:

$$G(P, P-K) = G^R(P, P-K)\mathrm{th}\frac{\varepsilon-\omega}{2T} - \mathrm{th}\frac{\varepsilon}{2T}G^A(P, P-K) + G^{(a)}(P, P-K). \quad (7)$$

It is easy to verify that, in the case of the BCS model, in which Δ enters in the diagrams in analogy to the external field, $G^{(a)}$ coincides with the anomalous part of the Green's function introduced earlier[3]. All the terms making up $G^{(a)}$ have a common property in that they have $G^R(P)$ as the incoming free line and $G^A(P - K)$ as the outgoing line. The difference from the BCS model consists here in the fact that (5) has a part connected with the self-energy cuts. From (5) and

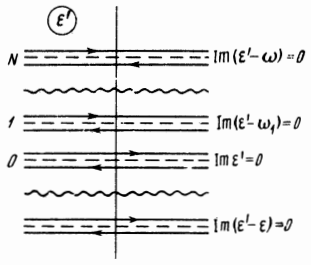

FIG. 3

(7), taking into account the fact that G contains the self-energies Σ_1 and Σ_2 (and with inverted arrow directions), we readily find that

$$G^{(a)}_{\varepsilon,\varepsilon-\omega} = \{G^R(-h+\Sigma_1^{(a)})G^A - F^R(-\bar{h}+\bar{\Sigma}_1^{(a)})F^{+A} - G^R\Sigma_2^{(a)}F^{+A} - F^R\Sigma_2^{+(a)}\bar{G}^A\}_{\varepsilon,\varepsilon-\omega}, \quad (8)$$

where

$$h_{\varepsilon,\varepsilon-\omega} = \left(\operatorname{th}\frac{\varepsilon}{2T} - \operatorname{th}\frac{\varepsilon-\omega}{2T}\right) H_1(\omega),$$

and $\Sigma_1^{(a)}$, $\bar{\Sigma}_1^{(a)}$, $\Sigma_2^{(a)}$, $\bar{\Sigma}_2^{+(a)}$ are expressed in terms of the discontinuities on the cuts of the corresponding self-energies with the aid of relations of the type (5) and (7):

$$\Sigma_{\varepsilon,\varepsilon-\omega} = \Sigma^R_{\varepsilon,\varepsilon-\omega}\operatorname{th}\frac{\varepsilon-\omega}{2T} - \operatorname{th}\frac{\varepsilon}{2T}\Sigma^A_{\varepsilon,\varepsilon-\omega} + \Sigma^{(a)}_{\varepsilon,\varepsilon-\omega}$$

$$= \sum_{N=0}^{\infty}\sum_{l=0}^{N}\delta_l(\Sigma^{(N)})\operatorname{th}\frac{\varepsilon_l}{2T}. \quad (9)$$

The superior bar denotes an inversion of the arrow directions in the diagrams. In a coordinate representation we have identities of the type

$$\bar{G}^A(x,x') = G^R(x',x), \quad \bar{G}^{(a)}(x,x') = G^{(a)}(x',x); \; x = \{t, \mathbf{r}\}.$$

Finally, we note that in (8) we have used the abbreviated notation[1]:

$$\{ABC\}_{\varepsilon,\varepsilon-\omega} = \int\!\!\int_{-\infty}^{\infty}\frac{d\omega_1 d\omega_2}{(2\pi)^2}A_{\varepsilon,\varepsilon-\omega_1}B_{\varepsilon-\omega_1,\varepsilon-\omega_2}C_{\varepsilon-\omega_2,\varepsilon-\omega}. \quad (10)$$

In exactly the same manner, we can obtain expressions for $F^{(a)}$, $F^{+(a)}$, and $\bar{G}^{(a)}$. For example,

$$F^{(a)}_{\varepsilon,\varepsilon-\omega} = \{G^R(-h+\Sigma_1^{(a)})F^A + F^R(-\bar{h}+\bar{\Sigma}_1^{(a)})\bar{G}^A + G^R\Sigma_2^{(a)}\bar{G}^A - F^R\Sigma_2^{+(a)}F^A\}_{\varepsilon,\varepsilon-\omega}. \quad (11)$$

It remains for us to express Σ, defined in accordance with (9), in terms of the Green's functions, as was done above for $\Sigma^{R(A)}$ (see Fig. 4). From (3) we obtain, with the aid of the identity $\coth(x'-x)(\tanh x' - \tanh x) = 1 - \tanh x \tanh x'$,

$$\Sigma_1(P,P-K) = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\left\{\operatorname{cth}\frac{\varepsilon'-\varepsilon}{2T}G(P',P'-K) - (G^R-G^A)_{P',P'-K}\right\}(D^R-D^A)_{P'-P} \quad (12)$$

Formulas for Σ_2 and Σ_2^+ are obtained from this by replacing G with F and F^+. We now have a complete system of equations describing, on the basis of the Fröhlich model, the behavior of a superconductor in an alternating field.

[1] Depending on the representation, integration is implied here with respect to the intermediate coordinates or with respect to the momenta.

FIG. 4

3. KINETIC EQUATIONS

Let us eliminate from the equations of the preceding section the anomalous functions $G^{(a)}$, $\Sigma^{(a)}$, etc. To this end, we note that $G^{(a)}$ (Eq. (8)) satisfies the following Dyson equation, which is obtained by separating on the left the free line $G^R(P) = (\xi - \epsilon - i\delta)^{-1}$:

$$(\xi-\varepsilon)G^{(a)}(P,P-K) = \{(-H_1+\Sigma_1^R)G^{(a)} - \Sigma_2^R F^{+(a)}\}_{P,P-K} + \{(-h+\Sigma_1^{(a)})G^A - \Sigma_2^{(a)}F^{+A}\}_{P,P-K}$$

(the notation is the same as in (8) and (10)). Substituting here in place of the anomalous functions their expressions from (7) and (9) and using Eqs. (6) for $G^{R(A)}$ and $F^{R(A)}$, we obtain an equation for G(5):

$$(\xi-\varepsilon)G(P,P-K) = \{(-H_1+\Sigma_1^R)G - \Sigma_2 R F^+ + \Sigma_1 G^A - \Sigma_2 F^{+A}\} + (2\pi)^4\delta(\omega)\delta(\mathbf{k}). \quad (13)$$

In exactly the same manner, we obtain an equation for F:

$$(\xi-\varepsilon)F(P,P-K) = \{(-H_1+\Sigma_1^R)F + \Sigma_2^R\bar{G} + \Sigma_1 F^A + \Sigma_2\bar{G}^A\} \quad (14)$$

and, analogously, for F^+ and $\bar{G}$. An important property of Eqs. (13) and (14) is the fact that only the Σ of formulas (4) and (12) depend explicitly on the temperature, and we see that T enters in these formulas as the temperature of the phonon thermostat. The situation here is in full correspondence with that demonstrated by Keldysh[9], whose method could be used in our case. The method chosen by us to obtain the dynamic equations makes it possible to retain the direct connection with the previously described diagram technique[3].

We call attention now to the fact that, as noted a number of times[6,8], the self energies Σ are practically independent of the momentum, i.e., of $\xi = v(p - p_0)$. In fact, p enters only in the D function, and since both the external momentum p and the internal momentum p' are close to the Fermi momentum p_0, it follows that $D(p - p')$ depends mainly on the angle between them. This enables us to go over in (4) and (12) from integration with respect to p to integration with respect to ξ and the angles

$$\frac{d^3\mathbf{p}}{(2\pi)^3} \approx \frac{mp_0}{2\pi^2}\frac{dO_p}{4\pi}d\xi$$

and to integrate the electron functions in these formulas with respect to ξ. Introducing the notation

$$g_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \int_{-\infty}^{\infty}d\xi G(P,P-K), f_{\varepsilon,\varepsilon-\omega}(p,k) = \int_{-\infty}^{\infty}d\xi F(P,P-K). \quad (15)$$

we rewrite Σ in the form

$$\Sigma^{R(A)}_{1\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{dO_p}{4\pi}\left\{\operatorname{cth}\frac{\varepsilon'-\varepsilon}{2T}(D^R-D^A)g^{R(A)} + D^{A(R)}g\right\}, \quad (4')$$

$$\Sigma_{1\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{dO_p}{4\pi}\left\{\operatorname{cth}\frac{\varepsilon'-\varepsilon}{2T}g - (g^R-g^A)\right\}(D^R-D^A) \quad (12')$$

and analogously, for Σ_2[2]; $D^{R(A)} \equiv D^{R(A)}_{\epsilon'-\epsilon}(p'-p)$, $g \equiv g_{\epsilon',\epsilon'-\omega}(p, k)$.

We see now that ξ enters explicitly only in the left-hand sides of (13) and (14), and we can therefore obtain equations directly for g and f. To do so, we write out for G and F equations analogous to (13) and (14), but obtain from (8) and (11) by separating the free right-hand line: $G^A(P-K) = (\xi - \mathbf{vk} - \epsilon + \omega + i\delta)^{-1}$ in the case of G and $\bar{G}^A(P-K) = (\xi - \mathbf{vk} + \epsilon - \omega - i\delta)^{-1}$ in the case of F:

$$(\xi - \mathbf{vk} - \varepsilon + \omega)G(P, P-K) = \{G(-H_1 + \Sigma_1^A) - F\Sigma_2^{+A} + G^R\Sigma_1 - F^R\Sigma_2^+\} + (2\pi)^4\delta(\omega)\delta(\mathbf{k}),$$
$$(\xi - \mathbf{vk} + \varepsilon - \omega)F(P, P-K) = \{F(-\bar{H}_1 + \bar{\Sigma}_1^A) + G\Sigma_2^A + F^R\bar{\Sigma}_1 + G^R\Sigma_2\}.$$

Combining these equations with (13) and (14), so as to exclude ξ from the left-hand side, and integrating with respect to ξ, we obtain

$$(\omega - \mathbf{vk})g_{\epsilon,\epsilon-\omega} = \{H_1 g - gH_1 + g\Sigma_1^A - \Sigma_1^R g - f\Sigma_2^{+A} + \Sigma_2^R f^+ + g^R\Sigma_{\cdot} - \Sigma_1 g^A - f^R\Sigma_2^+ + \Sigma_2 f^{+A}\}_{\epsilon,\epsilon-\omega}, \quad (16)$$

$$(2\varepsilon - \omega - \mathbf{vk})f_{\epsilon,\epsilon-\omega} = \{H_1 f - f\bar{H}_1 + f\bar{\Sigma}_1^A - \Sigma_1^R f + g\Sigma_2^A - \Sigma_2^R \bar{g} + f^R\bar{\Sigma}_1 - \Sigma_1 f^A + g^R\Sigma_2 - \Sigma_2\bar{g}^A\}_{\epsilon,\epsilon-\omega}. \quad (17)$$

Besides the processes of scattering and decay of excitations, Eqs. (16) and (17) take into account also the renormalization effects. The latter are determined by the quantity $\Sigma_1^R + \Sigma_1^A$ and are practically insensitive to violation of the equilibrium[3]. We shall disregard them and retain in the equations only the dissipative part Σ_1^R:

$$2i\gamma_{\epsilon,\epsilon-\omega} \equiv \Sigma_1^R - \Sigma_1^A = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{dO_{p'}}{4\pi}\left[\operatorname{cth}\frac{\varepsilon'-\varepsilon}{2T}(g^R - g^A) - g\right](D^R - D^A) \quad (18)$$

With the same accuracy, the role of Δ is assumed by the quantity

$$\Delta = {}^1/_2(\Sigma_2^R + \Sigma_2^A), \quad (19)$$

which varies as a function of ϵ in an interval $\sim\omega_D$, so that when $\epsilon \sim \Delta$ and $T \ll \omega_D$ this dependence is immaterial. The first term under the integral sign in (4') for Σ_2 introduces in Δ a contribution $\sim\Delta(T/\omega_D)^2$. We can therefore use for the determination of Δ the simplified equation

$$\Delta_\omega(k) = \lambda\int_{-\omega_D}^{\omega_D}\frac{d\varepsilon}{4\pi i}\int\frac{dO_p}{4\pi}f_{\epsilon,\epsilon-\omega}(p, k). \quad (20)$$

At the same time, for $\Sigma_2^R - \Sigma_2^A$ it is necessary to retain the exact expression:

$$2i\delta_{\epsilon,\epsilon-\omega} \equiv \Sigma_2^R - \Sigma_2^A = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{dO_p}{4\pi}\left[\operatorname{cth}\frac{\varepsilon'-\varepsilon}{2T}(f^R - f^A) - f\right](D^R - D^A). \quad (21)$$

[2]We have included the factor $mp_0/2\pi^2$ in the D function (2), where now $g^2 mp_0/2\pi^2 \equiv \lambda \sim 1$.

[3]As is well known[6,8] in the equilibrium case $\operatorname{Re}\Sigma^R_{|\epsilon} \approx a\epsilon$, where a ~ 1 has the same value in the normal and in the superconducting states, accurate to within $(\Delta/\omega_D)^2$.

After these simplifications, Eq. (16) takes the form

$$(\omega - \mathbf{vk})g_{\epsilon,\epsilon-\omega} = \{H_1 g - gH_1 - f\Delta^* + \Delta f^+\}_{\epsilon,\epsilon-\omega} + I_{\epsilon,\epsilon-\omega}, \quad (22)$$

where

$$I = \{-i(g\gamma + \gamma g) + i(f\delta^+ + \delta f^+) + g^R\Sigma_1 - \Sigma_1 g^A - f^R\Sigma_2^+ + \Sigma_2 f^{+A}\}. \quad (23)$$

It is easy to verify that the collision integral (23) vanishes when the equilibrium expressions are substituted for $g_{\epsilon,\epsilon-\omega} = 2\pi\delta(\omega)g_\epsilon$ and $f_{\epsilon,\epsilon-\omega} = 2\pi\delta(\omega)f_\epsilon$:

$$g_\epsilon = (g^R - g^A)_\epsilon \operatorname{th}\frac{\varepsilon}{2T}, \qquad f_\epsilon = (f^R - f^A)_\epsilon \operatorname{th}\frac{\varepsilon}{2T}, \quad (24)$$

i.e., Eq. (16) describes relaxation to the equilibrium.

Even more significant simplifications can be made in Eq. (17) for f. The point is that here the right-hand side contains the variable $\epsilon \sim \Delta, T$, whereas the dissipative parts γ and δ ((18) and (21)) are much smaller in the entire significant temperature interval, with the exception of a very small vicinity of T_c, an estimate for which was given in the beginning of the article. Their role here, as in the expressions for $g^{R(A)}$ and $f^{R(A)}$ consists, roughly speaking, in the fact that they lead to a certain smearing of the singularity in the density of the excitation spectrum near $\epsilon = \Delta$, and are, as a rule, by no means the principal source of such smearing. Therefore such quantities should be neglected in (17) (as also in $g^{R(A)}$ and $f^{R(A)}$, which enter in (23)). As a result we obtain for f the equation

$$(2\varepsilon - \omega - \mathbf{vk})f_{\epsilon,\epsilon-\omega}(p, k) = \{H_1 f - f\bar{H}_1 + g\Delta - \Delta\bar{g}\}_{\epsilon,\epsilon-\omega}. \quad (25)$$

The system (22)--(25) plays the role of the kinetic equation in the case of the superconducting state. In the variant linearized with respect to the external field, equations of this type have already been used by a number of authors (see, in particular,[10]). It should be noted however, that it was possible to obtain nonlinear equations only because of the procedure of integration with respect to ξ,[4], whereas the true kinetic equation would correspond to equations for the Green's functions integrated with respect to ϵ. But it is precisely for this reason that the equations given here remain valid also when the concept of the excitation spectrum becomes meaningless. Such a situation will occur, for example, if Δ is a sufficiently rapid function of the coordinates and of the time.

In the next sections we shall obtain for the electron-impurity and electron-electron collision integrals expressions that should be added to (23).

4. SCATTERING BY IMPURITIES

The interaction with impurities is included in the general scheme of the theory, as well as the interaction with the external field[3]. In the imaginary-frequency representation, self-energy parts σ_1, σ_2, and σ_2^+ appear in the Green's-function diagrams after averaging over the impurity positions. As a result of analytic continuation, each of them, as in the case of phonons, generates three quantities $\sigma^{R(A)}$ and σ, expressions for which will be derived here under the assumption that the scattering is isotropic:

[4]Equations for the functions integrated with respect to ξ were obtained for the equilibrium case by Eilenberger [11].

$$\sigma_1^{R(A)} = \frac{1}{2\pi\tau}\langle g^{R(A)}\rangle, \quad \sigma_1 = \frac{1}{2\pi\tau}\langle g\rangle,$$
$$\sigma_2^{R(A)} = \frac{1}{2\pi\tau}\langle f^{R(A)}\rangle, \quad \sigma_2 = \frac{1}{2\pi\tau}\langle f\rangle,$$
$$\langle\ldots\rangle = \int \frac{dO_p}{4\pi}, \tag{26}$$

where τ is the transport free-path time of the electron[5], and g and f are the functions defined above, calculated with allowance for the impurities. Adding (26) to the corresponding phonon self-energies in (16) and (17), we obtain the integrals for collisions with impurities:

$$I_p^{(imp)} = \int \frac{dO_{p'}}{4\pi}\{[g^R g]_{pp'} + [gg^A]_{pp'} - [f^R f^+]_{pp'} - [ff^{+A}]_{pp'}\}, \tag{27}$$

$$K_p^{(imp)} = \int \frac{dO_{p'}}{4\pi}\{[g^R f]_{pp'} + [gf^A]_{pp'} + [f\bar{g}^A]_{pp'} + [f^R\bar{g}]_{pp'}\},$$
$$[gf]_{pp'} \equiv g_p f_{p'} - g_{p'} f_p, \tag{28}$$

which must be included respectively in the right-hand sides of (22) and (25). These expressions possess the property that their averages over the directions of p vanish, as a consequence of the elasticity of the scattering by impurities.

5. ELECTRON-ELECTRON INTERACTION

We are interested in the interaction between electrons only to the extent that it leads to collision processes that are of importance for the kinetics. In addition, it makes its own contribution to the kernel of the equation for Δ. However, within the framework of the model equation (20) we can simply assume that λ is defined with allowance for all the interactions. Finally, the electron-electron interaction necessitates, generally speaking, that the electronic system be regarded as a Fermi liquid. We note in this connection that the renormalized quantities that arise in the theory of Fermi liquids are practically insensitive to the transition of the metal into the superconducting state, and it can therefore be assumed that they are accounted for in the employed characteristics of the normal metal, such as the spectrum density, the vertex of the interaction with the external field, etc. On the other hand, we shall not consider effects connected with the possible existence of excitations of the zero-sound type.

In the derivation of the equation for the electron-electron collision integral, we shall start from the fact that in a superconductor, as in a normal metal, only paired collisions are significant: processes in which a large number of excitations take part have a low probability because of degeneracy. Just as in the calculation of the damping of the excitations in a normal Fermi liquid[13], we should pick out in each diagram Σ in succession all the cross sections with three lines (Fig. 4), and take into account the dependence on the discrete imaginary frequencies (and by the same token on the temperature), on the external field, and in the given case on the parameter Δ of the superconducting state only in these cross sections. This procedure makes it necessary for us to regard both vertices on the diagram of Fig. 4 as the true amplitudes for the scattering of two excitations on the Fermi surface, containing no dependences on the aforementioned parameters which characterize the smearing region.

Since superconductivity is taken into account in the separated lines of the diagram of Fig. 4, Σ_1 and Σ_2 are actually determined by several diagrams of this type, which differ in the disposition of the arrows. Each such diagram corresponds to a different combination of amplitudes. In the imaginary-frequency representation, Σ_1 and Σ_2 can be expressed as follows:

$$\Sigma_1(P, P-K) = T^2 \sum_{\epsilon_1,\epsilon_2} \int\int \frac{d^3p_1 d^3p_2}{(2\pi)^6}\{AG_1G_2\bar{G}_3 - BF_1F_2^+G_3\},$$
$$\Sigma_2(P, P-K) = T^2 \sum_{\epsilon_1,\epsilon_2} \int\int \frac{d^3p_1 d^3p_2}{(2\pi)^6}\{BG_1\bar{G}_2F_3 - AF_1F_2F_3^+\}, \tag{29}$$

where A and B are two different quadratic forms of the amplitudes, the explicit form of which we do not need. For clarity, we present here expressions for A and B in perturbation theory, when the amplitudes can be replaced by a certain effective potential V_q:

$$A = -2|V_{p-p_2}|^2 + V_{p-p_2}V_{p-p_1},$$
$$B = -2|V_{p_1+p_2}|^2 + V_{p-p_1}V_{p-p_2} + V_{p_1+p_2}V_{p-p_2} + V_{p-p_1}V_{p_1+p_2}. \tag{30}$$

We have taken into account the fact that, depending on the method of joining the vertices in Σ, an additional electron loop may or may not arise. The sign of the diagram also changes when a pair of lines FF^+ is added. The internal lines correspond to the complete Green's functions including the alternating field, $G_1 = G(P_1, P_1 - K_1)$, etc., and in order to write down the succeeding formulas in symmetrical form, we have determined the internal variables by starting from the "decay" conservation law $P = P_1 + P_2 + P_3$ and $K = K_1 + K_2 + K_3$. This is the reason for the appearance of $\bar{G}$ in (29).

Regarding the diagram Σ (Σ_1 or Σ_2) of definite order in the external field as a function of the complex variable ϵ at fixed imaginary frequencies of the field vertices, we can verify, just as in the case of the phonons, that it corresponds to cuts on the lines $\mathrm{Im}(\epsilon - \Omega_k) = 0$, located between the extreme lowest cut $\mathrm{Im}\,\epsilon = 0$ and the highest cut $\mathrm{Im}(\epsilon - \omega) = 0$. The values of Ω_k represent here, too, certain combinations of the frequencies of the field vertices, but the set of these combinations and their total number depend in this case on the distribution of the field vertices over the internal lines of the diagram of Fig. 4. Let us assume that the G functions of these lines correspond to the sets of cuts $\mathrm{Im}(\epsilon_1 - \omega_{1i}) = 0$, $\mathrm{Im}(\epsilon_2 - \omega_{2k}) = 0$, $\mathrm{Im}(\epsilon_3 - \omega_{3l}) = 0$, and let us transform the sum over the frequencies in (29) into a contour integral (a double integral in this case). The transformation is carried out here in two steps: we first replace with an integral the sum over one of the variables, say ϵ_2, at a fixed imaginary value of ϵ_1, and then go over to the integral in the sum over ϵ_1. The external variable ϵ and the field frequencies remain imaginary to the end. The result takes the form

[5] If paramagnetic impurities are also present, then σ_1 and σ_2 contain respectively the relaxation times τ_1 and τ_2 [12].

$$\Sigma \infty T^2 \sum_{\varepsilon_1,\varepsilon_2} G_{\varepsilon_1} G_{\varepsilon_2} G_{\varepsilon-\varepsilon_1-\varepsilon_2}$$

$$= \int\int_{-\infty}^{\infty} \frac{dz_1\,dz_2}{(4\pi i)^2} \Big\{ \sum_{i,k} \delta_i(G_{z_1+\omega_{1i}})\,\delta_k(G_{z_2+\omega_{2k}})\,G_{\varepsilon-\omega_{1i}-\omega_{2k}-z_1-z_2} \operatorname{th}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T}$$
$$- \sum_{k,l} G_{z_1+\varepsilon-\omega_{2k}-\omega_{3l}}\,\delta_k(G_{z_2+\omega_{2k}})\,\delta_l(G_{-z_1-z_2+\omega_{3l}}) \operatorname{cth}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T}$$
$$+ \sum_{k\,l} G_{z_1+\varepsilon-\omega_{2k}-\omega_{3l}}\,\delta_k(G_{z_2-z_1+\omega_{2k}})\,\delta_l(G_{-z_2+\omega_{3l}}) \operatorname{cth}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T}$$
$$- \sum_{i,l} \delta_i(G_{z_1+\omega_{1i}})\,G_{z_2-z_1+\varepsilon-\omega_{1i}-\omega_{3l}}\,\delta_l(G_{-z_2+\omega_{3l}}) \operatorname{th}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T} \Big\}. \quad (31)$$

In order not to clutter the formulas, we have omitted the second index of the G functions. The combinations obtained here of ϵ with ω_{1i}, ω_{2k}, and ω_{3l} form the set of cuts of the given diagram Σ. Continuing (31) with respect to ϵ from the regions $\operatorname{Im}(\epsilon - \omega) > 0$ and $\operatorname{Im}\epsilon < 0$, and then with respect to all the field frequencies from the upper half-plane we obtain without difficulty Σ^R and Σ^A. We are interested only in the dissipative part $\Sigma^R - \Sigma^A$, for which we obtain the expression

$$\Sigma^R - \Sigma^A \infty \int\int_{-\infty}^{\infty} \frac{d\varepsilon_1\,d\varepsilon_2}{(4\pi i)^2} \{G_1G_2(G_3^R - G_3^A) + G_1(G_2^R - G_2^A)G_3$$
$$+ (G_1^R - G_1^A)G_2G_3 + (G_1^R - G_1^A)(G_2^R - G_2^A)(G_3^R - G_3^A)\}. \quad (32)$$

Using a definition of the type (9)

$$\Sigma_{\varepsilon,\varepsilon-\omega} = \sum_{N=0}^{\infty} \sum_{k} \delta_k(\Sigma^{(N)}_{\varepsilon,\varepsilon-\omega}) \operatorname{th}\frac{\varepsilon - \Omega_k}{2T},$$

we obtain

$$\Sigma \infty \int\int_{-\infty}^{\infty} \frac{d\varepsilon_1\,d\varepsilon_2}{(4\pi i)^2} \{G_1G_2G_3 + G_1(G_2^R - G_2^A)(G_3^R - G_3^A)$$
$$+ (G_1^R - G_1^A)G_2(G_3^R - G_3^A) + (G_1^R - G_1^A)(G_2^R - G_2^A)G_3\}. \quad (33)$$

Then applying these formulas to (29), we obtain expressions for the quantities $\Sigma_1^R - \Sigma_1^A$, Σ_1 and $\Sigma_2^R - \Sigma_2^A$, Σ_2 of interest to us. Before we present the results, we note that all these quantities are expressed in fact in terms of the Green's functions integrated with respect to ξ, and do not depend on ξ themselves. The point is that owing to the short-range character of the effective interaction, the amplitudes in (30) can be regarded as dependent on the angles only. We can therefore write, for example,

$$\int\int \frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{(2\pi)^6} A G_1G_2G_3$$
$$= \left(\frac{mp_0}{2\pi^2}\right)^2 \int\int \frac{dO_{\mathbf{p}_1}\,dO_{\mathbf{p}_2}}{(4\pi)^2} A \int\int\int d\xi_1\,d\xi_2\,d\xi_3\,\delta(\xi_3 - v(p_3 - p_0))\,G_1G_2G_3.$$

The δ function limits here, in the main, the integration with respect to the angle, requiring that $p_3 \equiv |\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2| \approx p_0$, and thus

$$\int\int \frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{(2\pi)^6} A G_1G_2G_3 = \left(\frac{mp_0}{2\pi^2}\right)^2 \frac{1}{p_0 v} \int\int \frac{dO_{\mathbf{p}_1}\,dO_{\mathbf{p}_2}}{(4\pi)^2}\,\delta\left(\frac{p_3}{p_0} - 1\right) A g_1g_2g_3. \quad (34)$$

Taking all these formulas into account, we can now write

$$2i\gamma = \hat{L}[A\{g_1g_2\bar{g}^3\}_1 - B\{f_1f_2{}^+g_3\}_1],$$
$$\Sigma_1 = \hat{L}[A\{g_1g_2\bar{g}_3\}_2 - B\{f_1f_2{}^+g_3\}_2],$$
$$2i\delta = \hat{L}[B\{g_1\bar{g}_2f_3\}_1 - A\{f_1f_2f_3{}^+\}_1],$$
$$\Sigma_2 = \hat{L}[B\{g_1\bar{g}_2f_3\}_2 - A\{f_1f_2f_3{}^+\}_2],$$

$$\hat{L} = \left(\frac{mp_0}{2\pi^2}\right)^2 \frac{1}{p_0 v} \int\int_{-\infty}^{\infty} \frac{d\varepsilon_1\,d\varepsilon_2}{(4\pi i)^2} \int\int \frac{dO_{\mathbf{p}_1}\,dO_{\mathbf{p}_2}}{(4\pi)^2}\,\delta\left(\frac{p_3}{p_0} - 1\right), \quad (35)$$

where $\{\ldots\}_1$ and $\{\ldots\}_2$ are forms patterned after the curly brackets in (32) and (33), respectively. Substitution of these formulas in (23) leads to an expression for $\Gamma^{(e)}$.

6. CHARGE AND CURRENT DENSITIES. ELECTRONEUTRALITY CONDITION

In the preceding sections we obtained equations for the Green's functions integrated with respect to ξ. It is clear that they can be used to calculate only quantities that are determined completely by the electrons near the Fermi surface. It is therefore necessary to exercise a certain caution when writing down the expressions for the current densities and the electron numbers. If we start from the expressions for j and N in the imaginary-frequency representation, then a direct application of the continuation procedure employed above, with allowance for the definition (5) of the function G, leads to the formulas

$$\mathbf{j}_\omega(k) = -\frac{2e}{m} \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi i} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathbf{p} G_{\varepsilon,\varepsilon-\omega}(\mathbf{p}_+, \mathbf{p}_-) - \frac{e^2}{mc}(NA)_{\omega,k}, \quad (36)$$

$$N_\omega(k) = -2 \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi i} \int \frac{d^3\mathbf{p}}{(2\pi)^3} G_{\varepsilon,\varepsilon-\omega}(\mathbf{p}_+, \mathbf{p}_-); \quad \mathbf{p}_\pm = \mathbf{p} \pm \frac{\mathbf{k}}{2}. \quad (37)$$

We shall assume that in the absence of an external field the medium is homogeneous. We stipulate that in the presence of a field the density of the number of electrons remain the same as before. The electroneutrality requirement frees us of the need for taking into account the Coulomb character of the interaction under conditions when the frequencies, gradients, and field amplitudes are small compared with quantities of atomic scale. To formulate this condition explicitly, we separate that part of the function G which is connected with the contribution made to (36) and (37) by the remote regions, and therefore cannot be calculated by integration with respect to ξ. This part, obviously, should not depend on Δ. It is determined by the regular terms (see (7)) of the function of the normal state, since all of them, with the exception of the free line, vanish upon integration with respect to ξ. Correct allowance for the contribution of these terms to N in first order in the electrochemical potential $\varphi_\omega(k)$[6] leads to the following expression:

$$N_\omega(k) - N_0(2\pi)^3\,\delta(\omega)\,\delta(\mathbf{k})$$
$$= -\frac{mp_0}{\pi^2} \Big[e\varphi_\omega(k) + \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi i} \int \frac{dO_{\mathbf{p}}}{4\pi}\, g'_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) \Big], \quad N_0 = \frac{p_0^3}{3\pi^2},$$

where the prime denotes that it is necessary to subtract from g the contribution of the free line $2\pi i \tanh(\epsilon/2T)$. By virtue of the neutrality condition we have

[6] It is obviously necessary to include in φ both the scalar potential of the field and the variation of the chemical potential. Allowance of terms of higher order in φ lead to corrections $\sim e\varphi/E_F$. The latter pertains also to terms with the vector potential.

$$e\varphi_\omega(k) = -\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\int\frac{dO_p}{4\pi}g'_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}). \quad (38)$$

A similar separation of the contribution made to $\mathbf{j}$ by the regions far from the Fermi surface leads to a cancellation of the second term in (36) (see also[3]):

$$\mathbf{j}_\omega(k) = -\frac{p_0 e}{\pi^2}\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\int\frac{dO_p}{4\pi}\mathbf{p}g_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}). \quad (39)$$

Using the equations obtained above for g and f, we can easily verify that (38) coincides with the equation div j = 0.

7. THIN FILMS IN A HIGH-FREQUENCY FIELD

We apply here the obtained equations to description of a thin film in the field of a wave incident normally onto its surface. We choose a gauge with a vector potential in the plane of the film, and confine ourselves to the case when the film is so thin that A_ω, Δ, and the other quantities with them are constant over its cross section. We assume at the same time that the electron mean free path determined by scattering from impurities is small compared with the film thickness, making it possible to disregard the singularities of electron reflection from the walls. This condition also ensures satisfaction of the inequalities $\Delta\tau \ll 1$, $T\tau \ll 1$, $\omega\tau \ll 1$ and leads to a weak dependence of g and f on the angle ϑ between p and $\mathbf{A}_\omega$:

$$g = g^{(0)} + g^{(1)}\cos\vartheta, \quad f = f^{(0)} + f^{(1)}\cos\vartheta. \quad (40)$$

From (22) and (25), in which in this case it is necessary to put k = 0, we readily obtain equations for the individual components. The equations averaged over the angles do not contain terms corresponding to collisions with impurities, whereas in the equations for $g^{(1)}$ and $f^{(1)}$ these collisions play the principal role, and therefore the inelastic processes can be neglected here. In spite of these simplifications, the equations still retain a rather complicated structure, in accord with the complicated character of the action of the alternating field on the superconducting state.

Let us discuss first the case of the normal state, when $\Delta = 0$ and $g^{R(A)} = \pm\pi i$. The last equation means that, within the framework of the approximation connected with the integration with respect to ξ, the field has no influence on the level density of the normal electrons. We write out, assuming the field to be monochromatic, an equation for the element $g^{(0)}_{\epsilon,\epsilon} \equiv g_\epsilon$, which is diagonal in the energy:

$$0 = \frac{ev}{3c}\{A_\omega(g^{(1)}_{\varepsilon,\varepsilon+\omega} - g^{(1)}_{\varepsilon-\omega,\varepsilon}) + A_{-\omega}(g^{(1)}_{\varepsilon,\varepsilon-\omega} - g^{(1)}_{\varepsilon+\omega,\varepsilon})\} + I_\varepsilon. \quad (41)$$

Here I_ϵ takes the inelastic collisions into account, and its order of magnitude is $I_\epsilon \sim i\tau_0^{-1}(g_\epsilon - g_{\epsilon 0})$, where τ_0 is a certain effective energy relaxation time, and $g_{\epsilon_0} = 2\pi i\tanh(\epsilon/2T)$. If the field frequency is sufficiently high so that $\omega\tau_0 \gg 1$, then the nondiagonal elements $g^{(0)}_{\epsilon,\epsilon-2\omega}$ etc. are small compared with g_ϵ relative to the parameter $(\omega\tau_0)^{-1}$. Therefore $g^{(1)}$ can be expressed only in terms of g_ϵ. Since it follows for a normal metal from (27) that $I^{(imp)} = -(i/\tau)g^{(1)}$, we obtain, inasmuch as $\omega\tau \ll 1$,

$$g^{(1)}_{\varepsilon,\varepsilon-\omega} = i\tau\frac{ev}{c}A_\omega(g_{\varepsilon-\omega} - g_\varepsilon).$$

Substituting in (41), we obtain an equation for g_ϵ:

$$0 = 2i\left(\frac{e}{c}\right)^2\frac{vl}{3}A_\omega A_{-\omega}(g_{\varepsilon-\omega} - 2g_\varepsilon + g_{\varepsilon+\omega}) + I_\varepsilon. \quad (42)$$

Introducing the notation

$$g_\varepsilon = 2\pi i(1 - 2n_\varepsilon), \quad (43)$$

we can easily verify that the electron-phonon and electron-electron collision integrals obtained above have a canonical form with n_ϵ as the distribution function.

As seen from (42), when $\omega \lesssim T$ the correction to n_ϵ linear in the field intensity is of the order of magnitude

$$\delta n_\varepsilon \sim \frac{\alpha\omega^2\tau_0}{T^2}, \quad \alpha = D\left(\frac{e}{c}\right)^2 A_\omega A_{-\omega}, \quad D = \frac{lv}{3}, \quad (44)$$

which indeed determines the field intensity at which a noticeable deviation from equilibrium takes place in the energy distribution of the electrons[7]: $\alpha \sim T^2/\omega^2\tau_0$. We note that although (42) contains effects that are nonlinear in the field intensity, they are limited to those corresponding to successive single-quantum transitions. The probability of simultaneous absorption of two quanta has a small factor $(\omega\tau)^{-2}$ compared with cascade absorption.

In the case of the superconducting state at not too low temperatures, when there are enough excitations already in equilibrium ($T \sim \Delta$), it is natural to expect the field intensity at which an appreciable change in the energy distribution of the excitations takes place to be determined, as before, from (44). The situation is made more complicated, however, by the fact that here the field also influences the density of the spectrum. Formally this is taken into account by the functions $g^{R(A)}$ and $f^{R(A)}$, which enter in the equations, and are altered by the turning on of the field in the case of the superconducting state. The situation here is analogous to that obtaining in the case of a constant magnetic field, although it is made more complicated by retardation effects. Consequently, substantial changes in the spectrum should be expected in fields whose amplitude is comparable with the critical magnetic field of the film, meaning $\alpha \sim \Delta$, T in the notation (44)[8]. Since τ_0 is quite large (for example, $\tau_0 T_c \sim (T_c/\Theta_D)^2$), it follows that $\omega^2\tau_0 > T$ even at not too high frequencies, and the changes in the excitation distribution become significant in fields which are still small compared with H_c. Under such conditions, when calculating the energy distribution of the excitations, we can regard the field in the first approximation only as the source of transition at a constant spectrum density. We shall simplify the equations further as applied to this situation, confining ourselves, as in the example with the normal metal, to single-photon transitions ($\omega\tau_0 \gg 1$).

The diagonal element g_ϵ is determined here, too, from (41), and in Eq. (25) for f_ϵ we should leave out

[7] Since we are dealing with a thin film in which, as noted in Sec. 1, the phonons are close to equilibrium, the effect under consideration does not reduce to simple heating, although it has the same scale.

[8] Actually, the superconductivity of the films is destroyed by the current [14], but the parameters of the superconducting state nevertheless change in the scale of H_c: the destruction by the current occurs abruptly.

the field term, since the field enters here alongside with $\epsilon \sim \Delta$, unlike in (41), where it enters in combination with τ_0^{-1}. We thus have simply[9)]

$$f_\epsilon = (\Delta/\epsilon) g_\epsilon. \tag{45}$$

In the equations for $g^{(1)}$ and $f^{(1)}$, which follow from (22) and (25) with allowance for (27) and (28), it is necessary to retain only the diagonal elements g_ϵ and f_ϵ. As to $g^{R(A)}$ and $f^{R(A)}$, their averages over the angles should be replaced by the unperturbed values

$$g_\epsilon^{R(A)} = \frac{\epsilon}{\Delta} f_\epsilon^{R(A)} = \pi i \frac{\epsilon}{\xi_\epsilon^{R(A)}},$$

$$\xi_\epsilon^R = -(\xi_\epsilon^A)^* = \begin{cases} \sqrt{\epsilon^2-\Delta^2}\,\mathrm{sign}\,\epsilon + i\delta, & \epsilon^2 > \Delta^2, \\ i\sqrt{\Delta^2-\epsilon^2}, & \epsilon^2 < \Delta^2, \end{cases}$$

and the angle parts should be calculated in first order in the field:

$$g_{\epsilon,\epsilon-\omega}^{(1)R(A)} = \frac{ev}{c} A_\omega \frac{\pi i}{(\xi_\epsilon + \xi_{\epsilon-\omega})^{R(A)} + i/\tau}\left[1 - \frac{\epsilon(\epsilon-\omega)+\Delta^2}{(\xi_\epsilon \xi_{\epsilon-\omega})^{R(A)}}\right],$$

$$f_{\epsilon,\epsilon-\omega}^{(1)R(A)} = -\frac{ev}{c} \frac{\pi i}{(\xi_\epsilon + \xi_{\epsilon-\omega})^{R(A)} + i/\tau} \frac{2\epsilon-\omega}{(\xi_\epsilon \xi_{\epsilon-\omega})^{R(A)}}.$$

Taking all the foregoing into account, we obtain after simple manipulations

$$g_{\epsilon,\epsilon-\omega}^{(1)} = i\tau \frac{ev}{c} A_\omega [\epsilon(\epsilon-\omega)+\Delta^2]\left[\frac{g_{\epsilon-\omega}}{(\epsilon-\omega)\xi_\epsilon^R} + \frac{g_\epsilon}{\epsilon \xi_{\epsilon-\omega}^A}\right], \quad l \ll \xi_0 \tag{46}$$

and, substituting (46) in (41), we arrive at an equation for g_ϵ. We write down the result, introducing the notation

$$g_\epsilon = 2\pi i \frac{\epsilon}{\sqrt{\epsilon^2-\Delta^2}} \theta(\epsilon^2-\Delta^2)(1-2n_\epsilon), \tag{47}$$

where n_ϵ is defined as an even function of the energy, the equation for which is ($\epsilon \geq \Delta$)

$$0 = 2\alpha[U_-(n_{\epsilon-\omega}-n_\epsilon) - U_+(n_\epsilon - n_{\epsilon+\omega}) + V(1-n_\epsilon-n_{\omega-\epsilon})] + J_\epsilon,$$

$$U_\pm = \frac{[\epsilon(\epsilon\pm\omega)+\Delta^2]\theta(\epsilon\pm\omega-\Delta)}{\sqrt{(\epsilon^2-\Delta^2)[(\epsilon\pm\omega)^2-\Delta^2]}},$$

$$V = \frac{[\epsilon(\omega-\epsilon)-\Delta^2]\theta(\omega-\epsilon-\Delta)}{\sqrt{[(\omega-\epsilon)^2-\Delta^2](\epsilon^2-\Delta^2)}}. \tag{48}$$

The expression for $J_\epsilon = I_\epsilon/4\pi$ follows directly from (23), (18), (21), and (35) with allowance for the simplifications listed here:

$$J = J^{(ph)} + J^{(e)},$$

$$J_\epsilon^{(ph)} = \frac{\pi\lambda}{2(sp_0)^2} \int_\Delta^\infty d\epsilon' \int_0^\infty d\omega \frac{\omega^2 \epsilon\epsilon'}{[(\epsilon^2-\Delta^2)(\epsilon'^2-\Delta^2)]^{1/2}}$$

$$\times \left\{[n'(1-n)(1+N_\omega) - n(1-n')N_\omega]\left(1-\frac{\Delta^2}{\epsilon\epsilon'}\right)\delta(\epsilon'-\epsilon-\omega)\right.$$

$$+[n'(1-n)N_\omega - n(1-n')(1+N_\omega)]\left(1-\frac{\Delta^2}{\epsilon\epsilon'}\right)\delta(\epsilon-\epsilon'-\omega)$$

$$\left.+[(1-n)(1-n')N_\omega - nn'(1+N_\omega)]\left(1+\frac{\Delta^2}{\epsilon\epsilon'}\right)\delta(\epsilon+\epsilon'-\omega)\right\};$$

$$N_\omega = (e^{\omega/T}-1)^{-1},$$

$$J_\epsilon^{(e)} = \frac{1}{p_0 v} \int_\Delta^\infty\int\int \frac{d\epsilon_1\, d\epsilon_2\, d\epsilon_3}{[(\epsilon^2-\Delta^2)(\epsilon_1^2-\Delta^2)(\epsilon_2^2-\Delta^2)(\epsilon_3^2-\Delta^2)]^{1/2}} \tag{49}$$

$$\times \{M_1[(1-n)n_1n_2n_3 - n(1-n_1)(1-n_2)(1-n_3)]\delta(\epsilon-\epsilon_1-\epsilon_2-\epsilon_3)$$
$$+ 3M_2[n_1n_2(1-n)(1-n_3) - nn_3(1-n_1)(1-n_2)]\delta(\epsilon+\epsilon_3-\epsilon_1-\epsilon_2)$$
$$+ 3M_3[n_1(1-n)(1-n_2)(1-n_3) - (1-n_1)nn_2n_3]\delta(\epsilon+\epsilon_2+\epsilon_3-\epsilon_1)\};$$

$$M_1 = a(\epsilon\epsilon_1\epsilon_2\epsilon_3 - \Delta^4) - \frac{b}{3}(-\epsilon^2+\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3)\Delta^2,$$

$$M_2 = -M_1(-\epsilon_3); \qquad M_3 = M_1(-\epsilon_2, -\epsilon_3), \tag{50}$$

where a and b are numbers of the order of unity, obtained from (30) by averaging over the angles:

$$a = -2\pi\left(\frac{mp_0}{2\pi^2}\right)^2 \iiint \frac{dO_p\, dO_{p_1}\, dO_{p_2}}{(4\pi)^3}\delta\left(\frac{|\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2|}{p_0}-1\right)A,$$

and b is similarly connected with B. We emphasize that the equations contain the nonequilibrium value of Δ, which should be determined in accordance with (20) and (45) from the BCS equation:

$$1 = \lambda \int_\Delta^{\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2-\Delta^2}}(1-2n_\epsilon) \tag{51}$$

with the nonequilibrium distribution function n_ϵ.

In conclusion, we note that the estimates given at the beginning of the section pertained to the temperature region $T \sim \Delta$. Using the results of the direct calculation of the correction to Δ, linear in the field intensity[15,17], we can show that in the vicinity of T_c the condition for the applicability of (48)—(51) is $\omega^2\tau_0 \approx \Delta$, which coincides with the condition given above for $\Delta \sim T$. The case of low temperatures calls for a special investigation.

I am grateful to L. P. Gor'kov and A. I. Larkin for numerous discussions.

[1] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev., 108, 1175 (1957).
[2] M. P. Kemoklidze and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 52, 1556 (1967) [Sov. Phys.-JETP 25, 1036 (1967)].
[3] L. P. Gor'kov and G. M. Éliashberg, ibid. 54, 612 (1968) [27, 328 (1968)].
[4] G. M. Éliashberg, ibid. 55, 2443 (1968) [28, 1298 (1969)].
[5] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953, Sec. 65.
[6] A. B. Migdal, Zh. Eksp. Teor. Fiz. 34, 1438 (1958) [Sov. Phys.-JETP 7, 996 (1958)].
[7] G. M. Éliashberg, ZhETF Pis. Red. 11, 186 (1970) [JETP Lett. 11, 114 (1970)].
[8] G. M. Éliashberg, Zh. Eksp. Teor. Fiz. 39, 1437 (1960) [Sov. Phys.-JETP 12, 1000 (1961)].
[9] L. V. Keldysh, ibid. 47, 1515 (1964) [20, 1018 (1965)].
[10] V. L. Pokrovskiĭ and S. K. Savvinykh, ibid. 43, 564 (1962) [16, 404 (1963)].
[11] G. Eilenberger, Zs. Phys., 214, 195 (1968).
[12] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. 39, 1781 (1960) [Sov. Phys.-JETP 12, 1243 (1961)].
[13] G. M. Éliashberg, ibid. 42, 1658 (1962) [15, 1151 (1962)].
[14] I. O. Kulik, ibid. 57, 600 (1969) [30, 329 (1970)].
[15] L. P. Gor'kov and G. M. Éliashberg, ibid. 56, 1297 (1969) [29, 698 (1969)].

Translated by J. G. Adashko
129

[9)] In this case $\Delta = \Delta^*$, $f = f^+$, $\bar{g} = -g$.