ELECTRON-POSITRON CASCADE PROCESS IN A STRONG CONSTANT ELECTRIC FIELD

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A cascade process is considered which occurs in a strong stationary electric field E_0 and which can be self-sustaining if the energy lost by the particles to pair production is replenished as a result of acceleration in a field E_0. The conditions for the appearance of a periodic regime are considered.

As is well known, when a high-energy particle passes through a layer of a dense medium, a large number of cascade electrons and positrons is produced. However, as the particles penetrate deeper into the medium, their energy decreases and the production of electron-positron pairs ceases. At the same time, one can visualize a different picture. If the aforementioned cascade process occurs in a strong electric field (when the energy lost to pair production is replenished by acceleration in the external electric field), then it seems possible for the cascade process to be self-maintaining.

Of course, to realize such an effect it is necessary to have an electric field of appreciable intensity. Therefore the results obtained below can apparently be used for pulsars, where it is assumed that the electric field (as well as the magnetic one) can be very large (see[1]). In addition, the effect investigated below may turn out to be useful also for land-based experiments (see[2] in this connection).

We consider in this article a self-maintaining cascade process within the framework of a very simple model. First, the problem is assumed to be one-dimensional. A uniform electric field E_0 is applied in the region $0 \leq x \leq L$, on the boundaries of which there is a dense medium (say a metal). In the region $0 < x < L$, the charges of opposite polarities move towards each other. The field E_0 is assumed to be strong, and the particle density not very high (the particles can then be regarded as relativistic). To be able to disregard interactions with γ quanta in the analysis that follows, we assume that a magnetic field H_0 is applied parallel to the electric, and that the transverse dimension of the system (relative to E_0) is sufficiently small. Then the γ quanta (unlike the charged particles) will rapidly leave the system[1)]. The character of the interaction of the charged particles is determined by their energy. If the energy is not too high, namely, such that

$$\sigma_p \ll \sigma_a, \tag{1}$$

then it can be assumed that particle annihilation occurs in the region $0 < x < L$. Here

$$\sigma_a \approx \pi r_0^2 \mu^{-1} \ln 2\mu$$

is the annihilation cross section, $r_0 = e^2/mc^2$ is the classical electron radius, $\mu = \epsilon/mc^2 = 1/\sqrt{1-\beta^2}$, ϵ the particle energy, $\beta = v/c$, and v the particle velocity.

We can use for the pair production cross section σ_p the Landau and Lifshitz formula[3], i.e.,

$$\sigma_p \approx r_0^2 \eta^2 \ln^3 \mu / \pi,$$

where $\eta = e^2/\hbar c$ is the fine-structure constant. On the other hand, if the opposite inequality $\sigma_a \ll \sigma_p$ is satisfied (i.e., $\mu^{-1} \ll 10^{-4}$), then particle interaction gives rise to pair production in the space $0 < x < L$.

It is important that the energies ϵ_+ and ϵ_- of the produced positron and electron satisfy the condition[3]

$$mc^2 \ll \varepsilon_+,\ \varepsilon_- \ll \varepsilon = mc^2\mu. \tag{2}$$

Therefore the time of rotation τ_γ of the produced particles in the electric field is much shorter than the time τ between two collisions, i.e., the time necessary to accelerate the particles to an energy on the order of $\epsilon = mc^2\mu$ (see below). We can therefore assume that, in the main, the positrons move along the field and the electrons against the field, while the approximate equations for the concentrations n_1 and n_2 of the positive and negative charges are

$$\frac{\partial n_1}{\partial t} + c\frac{\partial n_1}{\partial x} = 2\sigma_0 c n_1 n_2,$$
$$\frac{\partial n_2}{\partial t} - c\frac{\partial n_2}{\partial x} = 2\sigma_0 c n_1 n_2,$$
$$0 < x < L. \tag{3}$$

Taking into account the statements made above for relatively low energies ($1 \gg \mu^{-1} > 10^{-4}$), when annihilation predominates in the region $0 < x < L$, we assume approximately that $\sigma_0 = -\sigma_a$, and at $\mu^{-1} < 10^{-4}$, when the principal role is played by the pair production process, we assume in (3) $\sigma_0 = \sigma_p$.

Of course, Eqs. (3) are valid only for ultrarelativistic particles. This is possible, however, for not very large particle densities, for when the density is increased the mean free path $l \approx 1/n|\sigma_0|$ decreases, and consequently the particle energy stored in the field also decreases. The limitation imposed by this circumstance on the validity of Eqs. (3) can readily be estimated from the condition

$$eE_0/n|\sigma_0| \gg bmc^2, \tag{4}$$

where the parameter b is of the order of 10–10^2 (see[4]). Further, in the system under consideration, the pair production process can lead to separation of charges of opposite signs, i.e., to the appearance of the electric field E of the particles themselves. Of course, this field should be weaker than the external field E_0. If the

[1)]It can also be assumed that the particles are guided by a curvilinear magnetic field $H_0(r)$, with $H_0 \parallel E_0$ as before.

charge is concentrated mainly in the region ΔL (one-dimensional case), then in order for (3) to be valid it is necessary to assume that

$$e(n_1 - n_2)\Delta L \ll E_0. \quad (5)$$

It should be noted that similar systems of equations are encountered in nonlinear optics (see[5,6]).

Equations (3), which are valid for the region $0 < x < L$, should be supplemented by boundary conditions. Since the particles have a high energy as they approach a boundary, say a metal, a shower of the ordinary type (without an electric field) is produced in this dense medium. In this case the particle does not acquire enough energy for pair production during the free-path time. A fraction of the decelerated particles is drawn away by the field back into the region $0 < x < L$. To take this circumstance into account, we shall assume that homogeneous boundary conditions, of the type

$$\begin{aligned} n_1(x=0,\ t) &= (1+\varkappa)n_2(x=0,\ t), \\ n_2(x=L,\ t) &= (1+\varkappa)n_1(x=L,\ t), \end{aligned} \quad (6)$$

are satisfied on the boundary of the region; the constant quantity κ characterizes the relative number of particles that are drawn away. For the sake of caution we note that in the case of particles with relatively low energy, effective production of a shower in a dense medium requires that the medium consist of sufficiently heavy particles. This limitation is lifted for particle energies exceeding approximately 10^{-4} erg.

To find a general solution of the system (3) we proceed as follows (see also[6]). We introduce new variables $\xi_1 = t + x/c$, $\xi_2 = t - x/c$. Then $\partial n_1/\partial\xi_1 - \partial n_2/\partial\xi_2 = 0$, i.e., $n_1 = \partial\varphi/\partial\xi_2$, $n_2 = \partial\varphi/\partial\xi_1$, where $\varphi(\xi_1, \xi_2)$ is an arbitrary function, and it follows from (3) that

$$\psi(\xi_1,\ \xi_2) = \partial\varphi/\partial\xi_2, \quad \partial\psi/\partial\xi^1 = \psi\alpha\partial\varphi/\partial\xi_1, \quad \alpha = \sigma_0 c,$$

from which we obtain after simple transformations

$$n_1(x,t) = -\frac{d\Phi_2(\xi_2)/d\xi_2}{\alpha[\Phi_1(\xi_1)+\Phi_2(\xi_2)]}, \quad n_2(x,t) = -\frac{d\Phi_1(\xi_1)/d\xi_1}{\alpha[\Phi_1(\xi_1)+\Phi_2(\xi_2)]}, \quad (7)$$

where $\Phi_1(\xi_1)$ and $\Phi_2(\xi_2)$ are arbitrary functions of their arguments.

If we now substitute (7) in the first boundary condition (6), then we get after integrating with respect to t

$$\Phi_2(t) = (1+\varkappa)\Phi_1(t) + c_1, \quad (8)$$

where c_1 is an arbitrary constant. It must be emphasized that the relation (8) is valid only when $t > 0$. From the second boundary condition in (6) we obtain with the aid of (8)

$$\frac{d\Phi_2(t+L/c)}{dt} = (1+\varkappa)^2 \frac{d\Phi_2(t-L/c)}{dt}, \quad t > 0. \quad (9)$$

Equation (9) determines the function Φ_2 also at negative arguments ξ in the interval $-L/c < \xi < 0$. The solution of the finite-difference equation (9) can be written in the form

$$\Phi_2(\xi) = e^{\omega_0\xi}\sum_{s=0}^{\infty}[a_s\cos(\chi_s\xi) + b_s\sin(\chi_s\xi)] = e^{\omega_0\xi}Y(\xi), \quad -\frac{L}{c} < \xi < \infty. \quad (10)$$

Here $\omega_0 = L^{-1}c\ln(1+\kappa)$, $\chi_s = \pi cs/L$, and a_s and b_s are the coefficients of the Fourier series $Y(\xi)$ and are determined by the initial conditions. From (8) and (10) we get

$$\begin{aligned} \Phi_1(\xi) &= \frac{1}{1+\xi}\left\{e^{\omega_0\xi}\sum_{s=0}^{\infty}[a_s\cos(\chi_s\xi) + b_s\sin(\chi_s\xi)] - c_1\right\} \\ &= \frac{1}{1+\varkappa}\{e^{\omega_0\xi}Y(\xi) - c_1\}, \quad 0 < \xi < \infty. \end{aligned} \quad (11)$$

We note that the period of the function $Y(\xi)$, represented by the Fourier series in (10) and (11), is equal to $2L/c$.

If the distributions $n_1(x, t = 0) = n_0(x)$ and $n_2(x, t = 0) = n_{00}(x)$, $0 < x < L$ are specified at the instant $t = 0$ ($n_0(x)$ and $n_{00}(x)$ should satisfy the conditions (6)), then it follows from (10) and (11) that at $t = 0$ the functions $\Phi_1(\xi_1)$ and $\Phi_2(\xi_2)$ are specified on different intervals. Namely, $\Phi_1(x, t = 0) = \Phi_{01}(\xi_1)$ is specified in the interval $0 < \xi_1 < L/c$ and $\Phi_2(x, t = 0) = \Phi_{02}(\xi_2)$ in the interval $-L/c < \xi_2 < 0$. Therefore

$$\begin{aligned} Y(\xi) &= \sum_{s=0}^{\infty}[a_s\cos(\chi_s\xi) + b_s\sin(\chi_s\xi)] \\ &= \begin{cases} \Phi_{02}(\xi)e^{-\omega_0\xi}, & -L/c < \xi < 0, \\ e^{-\omega_0\xi}[(1+\varkappa)\Phi_{01}(\xi) + c_1], & 0 < \xi < L/c. \end{cases} \end{aligned} \quad (12)$$

It remains only to connect the values of the functions Φ_{01} and Φ_{02} with the initial charge distributions $n_0(x)$ and $n_{00}(x)$. This connection follows directly from (7), from which we obtain after a number of calculations

$$\Phi^{(1)} = c_2\int_0^x F(x')\,dx' + c_3,$$

$$\Phi^{(2)}(x) = -\frac{c_2F(x)}{n_{00}(x)\sigma_0} - c_2\int_0^x F(x')\,dx' - c_3, \quad (13)$$

where c_2 and c_3 are arbitrary constants,

$$\Phi^{(1)}(x) = \Phi_{01}(x/c), \quad \Phi^{(2)}(x) = \Phi_{02}(-x/c),$$

$$f(x) = n_{00}(x)\sigma_0\left[1 - \frac{n_{00}'(x)}{n_{00}^2(x)\sigma_0} - \frac{n_0(x)}{n_{00}(x)}\right], \quad n_{00}'(x) = \frac{dn_{00}}{dx},$$

$$F(x) = \exp\left\{-\int_0^x f\,dx\right\} = \frac{n_{00}(x)}{n_{00}(0)}\exp\left\{-\sigma_0\int_0^x[n_{00}(x') - n_0(x')]\,dx'\right\}.$$

The arbitrary constants c_2 and c_3, together with c_1, are determined from the condition for the continuity of the function $Y(\xi)$, i.e., from the condition that $n_1(x, t)$ and $n_2(x, t)$ be finite (see (7)). Thus, to determine the connection between c_1, c_2, and c_3 we obtain the relations

$$Y(-0) = Y(+0),\ Y(-L/c) = Y(L/c), \quad (14)$$

where $Y(-0)$ and $Y(+0)$ are the values of the function $Y(\xi)$ to the left and to the right of the point $\xi = 0$. From (14) we have

$$\frac{c_2}{\sigma_0 n_{00}(0)} + (2+\varkappa)c_3 + c_1 = 0,$$

$$c_2\left\{\frac{F(L)}{n_{00}(L)\sigma_0} + \frac{2+\varkappa}{1+\varkappa}\int_0^L F(x')\,dx'\right\} + c_3\frac{2+\varkappa}{1+\varkappa} + \frac{c_1}{(1+\varkappa)^2} = 0. \quad (15)$$

It is obvious that one of the quantities c_1, c_2, or c_3 can be set equal to unity (if the corresponding determinant of the system (15) is not equal to zero; see (7). Formula (12) can be rewritten in the form

$$Y(\xi)=\begin{cases} e^{-\omega_0\xi}\left[-\dfrac{c_2F'(-c\xi)}{n_{00}(-c\xi)\sigma_0}-c_2\displaystyle\int_0^{-c\xi}F(x)\,dx-c_3\right] \\ \text{for}\quad -\dfrac{L}{c}<\xi<0, \\ \\ e^{-\omega_0\xi}\left\{(1+\varkappa)\left[c_2\displaystyle\int_0^{c\xi}F(x)\,dx+c_3\right]+c_1\right\} \\ \text{for}\quad 0<\xi<\dfrac{L}{c}, \end{cases} \tag{16}$$

and c_1, c_2, and c_3 are determined by (15).

If we use (7), (10), and (11), then we obtain for $n_1(x, t)$ and $n_2(x, t)$ the expressions

$$n_1(x,t)=-\frac{(1+\varkappa)\left[\omega_0Y(\xi_2)+Y'(\xi_2)\right]}{\alpha\left[(1+\varkappa)Y(\xi_2)+Y(\xi_1)e^{2\omega_0x/c}-c_1e^{-\omega_0\xi_2}\right]}, \tag{17}$$

$$n_2(x,t)=-\frac{\omega_0Y(\xi_1)+Y'(\xi_1)}{\alpha\left[Y(\xi_1)+(1+\varkappa)Y(\xi_2)e^{-2\omega_0x/c}-c_1e^{-\omega_0\xi_1}\right]}, \tag{18}$$

where $Y'(\xi_i) = dY(\xi_i)/d\xi_i$, $i = 1, 2$, and the periodic function $Y(\xi)$ is determined by (15) and (16).

It follows from (17) and (18) that the character of the time behavior of the electron and positron densities n_1 and n_2 depends essentially on the signs of α and ω_0. Thus, if $\omega_0 > 0$ ($\kappa > 0$), $\alpha < 0$ (corresponding to particle annihilation in the region $0 < x < L$), then the system executes periodic motion asymptotically (as $t \to \infty$). On the other hand, if $\omega_0 < 0$ and $\alpha > 0$ (corresponding to pair production on particle interaction in the space $0 < x < L$), then the situation is more complicated. At sufficiently low initial concentration, the system executes damped oscillations. On the other hand, if the initial concentration is sufficiently large (see below), then the system executes oscillations that increase with time. The boundary between these two regimes is the unstable periodic motion characterized by the value $c_1 = 0$, corresponding to vanishing of the corresponding determinant of the system (15), i.e.,

$$\frac{1}{(1+\varkappa)n_{00}(0)}-\frac{F(L)}{n_{00}(L)}-\frac{(2+\varkappa)\sigma_0}{1+\varkappa}\int_0^L F(x)\,dx=0, \tag{19}$$
$$\sigma_0>0,\qquad \varkappa<0.$$

Let us explain the noted singularities by means of simple examples. We assume that at the initial instant of time, electrons are injected with uniform density into the region $0 < x < L$, i.e., $n_0(x) = 0$ and $n_0(x) = n_{00} = \text{const}$. The initial electron density is assumed to be small: $\gamma = \sigma_0 n_{00} L$, $\delta = \sigma_0 n_{00} c$, $|\gamma| \ll 1$. Then, for example for the electron density on the plane $x = 0$, we have from (15), (16), and (18)

$$n_2(0,t)=\begin{cases} 0 \quad \text{for}\quad (2k-1)L/c<t<2kL/c, \\ -\varkappa n_{00}\{\gamma+\delta\varkappa(\varkappa+2)t_1-(\varkappa+\gamma)e^{-\omega_0(t-t_1)}\}^{-1} \\ \text{for}\quad 2kL/c<t<(2k+1)L/c, \end{cases} \tag{20}$$

with $k = 0, 1, 2, \ldots$, $t_1 = t - 2kL/c$, $0 < t_1 < L/c$.
$= (1 + \kappa)n_2(0, t)$. We consider first the case $\sigma_0 < 0$,
The positron density at the boundary $x = 0$ is $n_1(0, t)$
$\kappa > 0$, i.e., $\omega_0 > 0$. It is easy to see that at $t = +0$ we have in (20) $n_2(0, +0) = n_{00}$, and as $t \to \infty$ the system goes over into a regime with periodic oscillations having an amplitude independent of the value of the arbitrarily small initial concentration n_{00}. In this sense, the system in question behaves like a generator. Thus,

$$n_2(0,t)=\begin{cases} 0 \quad \text{for}\quad (2k-1)L/c<t<2kL/c, \\ \varkappa\{L|\sigma_0|[1+\varkappa(\varkappa+2)ct_1/L]\}^{-1} \\ \text{for}\quad 2kL/c<t<(2k+1)L/c, \end{cases} \tag{21}$$
$$\sigma_0<0,\ \varkappa>0,\ k\gg1.$$

The steady-state concentration is of the order of $n_m \approx \kappa/L|\sigma_0|$ (in particular, $n_m \ll 1/L|\sigma_0|$ when $\kappa \ll 1$).

Let us proceed to explain the singularities of the second kind, when $\sigma_0 > 0$ and $\kappa < 0$. In accordance with the statements made above, it follows from (20) that in this case when $\gamma < -\kappa$ the system executes damped oscillations ($\omega_0 < 0$), and when $\gamma > -\kappa$ it executes oscillations that increase in amplitude. Of course, the solution of Eqs. (3) in the form (20) describes the system satisfactorily so long as the concentration is not very large (see (4) and (5)). The limit of the aforementioned regimes is the unstable periodic motion realized when condition (19) is satisfied, or in our case ($\gamma \ll 1$) when $\gamma = -\kappa$. Thus, when $\kappa < 0$ and $\gamma > 0$ there are no periodic regimes in the employed approximate approach. It should be borne in mind, however, that it suffices to change somewhat the formulation of the problem (for example, to change the form of the boundary conditions) to get periodic regimes also for the case $\gamma > 0$.

In addition to the two cases indicated above, we note that at $\sigma_0 < 0$ and $\kappa < 0$ the system executes damped oscillations for all initial conditions, and when $\sigma_0 > 0$ and $\kappa > 0$ the oscillations of this system increase in time (see (20)).

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180