NONLINEAR MOTION IN AN ANISOTROPIC PLASMA

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Submitted May 10, 1971

Zh. Eksp. Teor. Fiz. 61, 1877–1881 (November, 1971)

Nonlinear Alfvén waves propagating along a magnetic field in an anisotropic plasma are studied under firehose instability conditions.

IN a magnetic field a rarefied plasma is anisotropic, because energy exchange between degrees of freedom is hindered and the pressures parallel and perpendicular to the magnetic field can consequently be unequal. Plasma motion in this case is often investigated on the basis of hydrodynamic equations—the model of Chew, Goldberger, and Low (CGL)[1] or its extended form taking into account the finite Larmor radius of the ions.[2] The model of [1] has been used to study, for example, simple waves[3] and weak shock waves[4] in a stable plasma. When the pressures along and perpendicular to the magnetic field satisfy the inequality $P_{\parallel} > P_{\perp} + H_0^2/4\pi$, Alfvén waves propagating along the field H_0 are unstable—the so-called firehose instability arises. A linear analysis yields the following dispersion law for these waves:

$$\omega_k = \omega_0^{(k)} + i\gamma_k, \quad \omega_0^{(k)} = {}^1/_2\omega_H k^2R^2, \quad \gamma_k = \omega_H kR\sqrt{\Delta p/p_{\parallel} - k^2R^2/4},$$

where $\omega_H = eH_0/m_ic$ is the cyclotron frequency of the ions, $R = \omega_H^{-1}\sqrt{p_{\parallel}/\rho}$ is the Larmor radius of the ions, k is the wave number of small-amplitude waves, γ_k is the growth increment of small perturbations, $\Delta p = p_{\parallel} - p_{\perp} - H_0^2/4\pi$ is the degree of plasma anisotropy. The expression for the increment indicates that the development of a firehose instability depends on the conditions

$$p_{\parallel} > p_{\perp} + H_0^2/4\pi, \; kR < 2\sqrt{\Delta p/p_{\parallel}}, \; \lambda > \lambda^* = \pi R\sqrt{p_{\parallel}/\Delta p},$$

i.e., sufficiently long Alfvén waves are unstable, but short-wave harmonics with $\lambda < \lambda^*$ are stabilized.

We have previously[5, 6] investigated a one-dimensional linear model of the firehose instability. The present paper continues the study of this model of an anisotropic plasma that is unstable with respect to the growth of Alfvén waves.

The initial system of equations has been given in [5]. We assume initially that motion along the magnetic field is absent and that the plasma density is uniform. Then if the square of the transverse field is independent of the z coordinate ($H_{\perp}^2 = H_x^2 + H_y^2$), the gas-kinetic pressures $p_{\parallel}$ and $p_{\perp}$ are also independent of z and longitudinal motion is not excited ($\rho = \rho_0 = \text{const}$, $w \equiv u_z = 0$). We shall henceforth consider only transverse Alfvén waves, assuming now $H_{\perp}^2 \ll H_0^2$. Then the pressures can be written as

$$p_{\parallel} \approx p_{\parallel}^0(1 - H_{\perp}^2/H_0^2), \; p_{\perp} \approx p_{\perp}^0(1 + H_{\perp}^2/2H_0^2),$$

where $p_{\parallel}^0$ and $p_{\perp}^0$ are the unperturbed pressures along and perpendicular to the magnetic field H_0. Inserting these expressions into the initial system, we obtain

$$\rho_0\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\left\{\left(1 - \frac{2p_{\parallel}^0 - p_{\perp}^0/2}{\Delta p}\frac{H_{\perp}^2}{H_0^2}\right)\frac{H_x}{H_0}\Delta p - \frac{p_{\parallel}^0}{\omega_H}\frac{\partial v}{\partial z}\right\} = 0,$$
$$\rho_0\frac{\partial v}{\partial t} + \frac{\partial}{\partial z}\left\{\left(1 - \frac{2p_{\parallel}^0 - p_{\perp}^0/2}{\Delta p}\frac{H_{\perp}^2}{H_0^2}\right)\frac{H_y}{H_0}\Delta p + \frac{p_{\parallel}^0}{\omega_H}\frac{\partial u}{\partial z}\right\} = 0,$$
$$\frac{\partial}{\partial t}\frac{H_x}{H_0} = \frac{\partial u}{\partial z}, \quad \frac{\partial}{\partial t}\frac{H_y}{H_0} = \frac{\partial v}{\partial z}, \tag{1}$$

where

$$\Delta p = p_{\parallel}^0 - p_{\perp}^0 - H_0^2/4\pi.$$

By transforming to new variables:

$$t' = \frac{\omega_H\Delta p}{p^0}t, \quad z' = \frac{\omega_H}{p^0}\sqrt{\rho_0\Delta p}z,$$
$$(u', v') = \frac{1}{\Delta p}\sqrt{\rho_0(2p_{\parallel}^0 - p_{\perp}^0/2)}\,(u, v), \tag{2}$$
$$(H_x', H_y') = \sqrt{(2p_{\parallel}^0 - p_{\perp}^0/2)/\Delta p}\left(\frac{H_x}{H_0}, \frac{H_y}{H_0}\right),$$

we eliminate all coefficients in (1) and obtain the equations (where primes have been omitted)

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\left[(1 - H_{\perp}^2)H_x - \frac{\partial v}{\partial z}\right] = 0,$$
$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial z}\left[(1 - H_{\perp}^2)H_y + \frac{\partial u}{\partial z}\right] = 0,$$
$$\frac{\partial H_x}{\partial t} = \frac{\partial u}{\partial z}, \quad \frac{\partial H_y}{\partial t} = \frac{\partial v}{\partial z}. \tag{3}$$

Thus the problem of firehose instability in an anisotropic rarefied plasma for different parameters will have similar solutions in the absence of longitudinal motion and in the case of small but finite degrees of plasma anisotropy (Δp).

The system (3) is a convenient model for studying unstable transverse Alfvén waves, because it is considerably simpler than the initial system and retains all features of the considered phenomenon—the growth of small perturbations, limitation of the amplitude as a result of nonlinearity, and stabilization of short-wave perturbations as a result of magnetic viscosity, which is described by the last terms in the first two equations.

A linear analysis yields

$$\omega_k = \omega_0^{(k)} + i\gamma_k, \quad \omega_0^{(k)} = {}^1/_2k^2, \quad \gamma_k = k\sqrt{1 - k^2/4}.$$

The harmonics with $k = \sqrt{2}$ and $\lambda = \pi\sqrt{2}$ possess the maximum rate of growth; for these $\gamma_k = 1$. Harmonics with wave numbers $k < k^* = 2$ are unstable; those with $k > k^*$ are stable.

As in [5], we shall seek a solution of (3) in the form of a monochromatic wave with circular polarization:

$$H_x(z, t) = B(t)\sin(kz + \varphi(t)), \; H_y(z, t) = B(t)\cos(kz + \varphi(t)). \tag{4}$$

Inserting these expressions into (3), we easily determine that the rate of change of the phase (or frequency) of the wave is $\varphi = -k^2/2$, and that the wave amplitude B(t) is determined by the simple equation of a nonlinear oscillator:

$$\ddot{B} - \gamma_k^2 B + k^2 B^3 = 0, \quad (5a)$$

or

$$\dot{B}^2 + U(B) = E = \text{const}, \quad (5b)$$

where the "potential energy" is $U(B) = \frac{1}{2}k^2B^2 \times (B^2 + k^2/2 - 2)$, and the "total energy" E depends on the amplitude B_0 of the initial perturbation: $E = k^2B_0^4/2$.

The change of amplitude of a nonlinear monochromatic wave can be represented qualitatively as follows. Let a wave of small amplitude $B_0 \ll 1$ be given at initial time. During a relatively small time interval the amplitude of this wave will grow exponentially because of instability, until the nonlinear terms become large. The amplitude then grows more slowly, reaches a maximum, falls off to zero, grows again, etc., thus oscillating periodically in time. The maximum square of the transverse magnetic field in the wave, $H^2_{\perp\,max} = B^2_{max} = 2(1 - k^2/4)$, diminishes as the wavelength decreases and vanishes, of course, at the boundary of the instability for $k = k^* = 2$.

The kinetic energy of particles in the wave is

$$K = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}H_\perp^2(1 - \frac{1}{2}H_\perp^2).$$

When $H^2_{\perp\,max} \leq 1$, which occurs for waves with $k \geq \sqrt{2}$, the behavior of K(t) duplicates that of $H^2_\perp(t)$, with monotonic growth to $K(H^2_{\perp\,max}) = \gamma_k^2/4$ followed by a decline to zero. When $H^2_{\perp\,max} > 1$ (for waves with $k < \sqrt{2}$) the time dependence of particle kinetic energy is changed: K(t) increases to its maximum $K_{max} = \frac{1}{4}$ at the moment when $H^2_\perp(t) = 1$, then drops to an intermediate minimum $K(H^2_{\perp\,max}) = \gamma_k^2/4$, increases to K_{max}, and again drops to zero together with $H^2_\perp$.

The period of the magnetic field oscillations and of the particle velocity in the wave is calculated from

$$T_k \approx (2/\gamma_k) \ln (8\gamma_k^2/k^2B_0^2),$$

which shows that the period increases as the initial amplitude is reduced. Transforming to dimensional variables, we obtain the following equations for the physical quantities in a circularly polarized nonlinear monochromatic wave:

$$H_\perp^2/H_0^2 = \frac{\Delta p}{2p_\parallel^0 - p_\perp^0/2}B^2(t),$$

$$(H_\perp^2/H_0^2)_{max} = (1 - k^2/4)\Delta p/(p_\parallel^0 - p_\perp^0/4),$$

$$K(t) = \frac{(\Delta p)^2}{2p_\parallel^0 - p_\perp^0/2}\cdot\frac{1}{2}B^2(t)\left(1 - \frac{1}{2}B^2(t)\right),$$

$$K_{max} = \begin{cases} \dfrac{k^2(1 - k^2/4)(\Delta p)^2}{4(2p_\parallel^0 - p_\perp^0/2)} & \text{for } B_{max} < 1, \\ \dfrac{(\Delta p)^2}{4(2p_\parallel^0 - p_\perp^0/2)} & \text{for } B_{max} \geqslant 1, \end{cases}$$

$$p_\parallel(t) = p_\parallel^0\left(1 - \frac{\Delta p}{2p_\parallel^0 - p_\perp^0/2}B^2(t)\right),$$

$$p_\perp(t) = p_\perp^0\left(1 + \frac{\Delta p}{4p_\parallel^0 - p_\perp^0}B^2(t)\right),$$

$$\Delta p(t) = p_\parallel(t) - p_\perp(t) - \frac{H_0^2}{4\pi} = \left(1 - \frac{p_\parallel^0 + p_\perp^0/2}{2p_\parallel^0 - p_\perp^0/2}B^2(t)\right)\Delta p.$$

As the amplitude of the nonlinear monochromatic wave grows the plasma becomes less anisotropic. With the transition from the initial state to the state where the magnetic field has reached its maximum the change in the anisotropy is

$$\delta = \frac{p_\parallel^0 + p_\perp^0/2}{2p_\parallel^0 - p_\perp^0/2}B^2_{max}\Delta p = \left(p_\parallel^0 + \frac{p_\perp^0}{2}\right)\frac{H^2_{\perp max}}{H_0^2}.$$

The quasilinear theory (QLT) of firehose instability was considered in [7,8]. When we compared the average plasma properties derived from the QLT and the CGL model for small anisotropy we obtained the following results. According to the QLT the relation between the changes of longitudinal and transverse pressure is given by

$$dp_\parallel/dp_\perp \approx -4(p_\parallel^0 - p_\perp^0/2)/p_\parallel^0,$$

whereas from the CGL theory we obtain

$$dp_\parallel/dp_\perp \approx -2p_\parallel^0/p_\perp^0.$$

Since applicability of the QLT requires fulfillment of the conditions

$$p_\parallel^0,\ p_\perp^0 \gg H_0^2/4\pi,\ (p_\parallel^0 - p_\perp)^0/p_\parallel^0 \ll 1, \quad (6)$$

both equations reduce to

$$dp_\parallel/dp_\perp \approx -2.$$

A similar result is obtained for the temporal change in the degree of plasma anisotropy (the quantity determining the increment of instability):

$$\Delta p(t) = p_\parallel(t) - p_\perp(t) - H_0^2/4\pi.$$

The quasilinear theory gives

$$\Delta p(t) \approx \Delta p - (5p_\parallel^0 - 2p_\perp^0)h(t),$$

while the CGL model gives

$$\Delta p(t) \approx \Delta p - (2p_\parallel^0 + p_\perp^0)h(t),$$

where $h(t) = \frac{1}{2}\sum_k H^2_{\perp k}/H_0^2$. Under the conditions (6) for the QLT the two equations yield

$$\Delta p(t) \approx \Delta p - 3p_\parallel^0 h(t).$$

We shall now consider how the investigated nonlinear monochromatic wave is affected by other waves for which we assume $k \ll k_0$ (where k_0 is the wave number of the "principal" wave). For these additional waves the increment is $\gamma_k \ll \gamma_{k_0} \equiv \gamma_0$; therefore during a time of the order of the period T_0 of the principal wave these waves can be considered in a linear approximation:

$$dh/dt = \sum_k 2\gamma_k h_k.$$

In this equation the increment γ_k will be a function of time, because the pressures $p_\parallel$, $p_\perp$ in the expression for the increment are related to the change of the magnetic field in the principal wave:

$$\gamma_k = \gamma_k(t) = k\sqrt{1 - k^2/4 - (4 + 3p_\perp^0)H_\perp^2/(2H_0^2\Delta p)}.$$

Taking one additional wave, for simplicity, and assuming $k_0 = \sqrt{2}$ ($\gamma_0 = 1$), we obtain the following: At the end of the period T_0 of the principal wave the square of the transverse magnetic field, due to the growth of the "small" wave, becomes

$$H_\perp^2(T_0) \approx (1 + \alpha k)H_{\perp k}^2(0),$$

where α is a numerical coefficient of the order 10.

Consequently, the changes of wave amplitude and plasma properties are irreversible, the degree of anisotropy is reduced, and the longitudinal and transverse pressures tend to be equalized, in qualitative agreement with the conclusion in [7,8] regarding quasilinear stabilization of the firehose instability. A more exact comparison of the CGL model and the quasilinear theory will be possible following a numerical solution of (3) for a set of initial waves with different wave numbers.

The author wishes to thank Academician R. Z. Sagdeev for advice and discussions.

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Translated by I. Emin
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