# MOTION OF A SUPERFLUID LIQUID IN A NARROW BORE AND AN ANALOG OF THE JOSEPHSON EFFECT

V. P. LUKIN and A. V. TULUB

Leningrad State University

Submitted July 23, 1970; resubmitted April 24, 1971

Zh. Eksp. Teor. Fiz. 61, 1986–1995 (November, 1971)

The equations of motion of superfluid helium are considered in the hydrodynamic approximation on basis of the phenomenological Ginzburg-Pitaevskiĭ theory. An exact solution of the one-dimensional problem in the stationary case is obtained. An investigation of the nonstationary case confirms the general Landau concept of the nature of nonstationary liquid flow. The expressions derived are employed for describing the analog of the Josephson effect in superfluid helium.

1. Observation of the analog of the Josephson effect (AJE) for superfluid helium was reported in[1-4]. In these experiments, in the presence of an external ultrasonic field of frequency $\nu_0$, a binding of the flow of helium in a channel (opening) between two volumes of helium took place when the resonant condition

$$n_1 mgZ = n_2 h\nu_0 \qquad (1.1)$$

was satisfied, where $n_1$ and $n_2$ are integers, m is the mass of the helium atom, g the acceleration of free fall, and Z the difference between the heights of the levels between the two volumes. In its nature, the AJE is similar to the nonstationary Josephson tunnel effect in superconductors[5]. In both cases, the superfluid flow is a periodic function of the phase difference between wave functions, and is connected with a certain energy difference. For the Josephson tunnel effect, this dependence is harmonic, whereas in the case of the AJE the relation is anharmonic, or, more accurately speaking, nonlinear, as is evidenced by the resonance condition (1.1). An intuitive explanation of the AJE is based on the phase-shift model, namely, the difference between the heights of the helium levels in both vessels causes a superfluid flow accompanied by periodic formation of vortices with frequency mgZ/h[6]. When a periodic external field is applied, resonance phenomena are observed. An explanation of this kind is qualitatively correct, but it is nevertheless of interest to present a more complete picture of this phenomenon.

The main condition for observation of the AJE is the existence of a stable oscillating superfluid flow in the channel. By stability of a superfluid flow is meant in this case its hydrodynamic stability. Loss of superfluid-motion stability occurs in accordance with the general picture of the onset of turbulence[7]. The latter proceeds in two stages: in the first stage vortical structures appear in the system, and in the second the superfluid flow becomes dissipated as a result of viscous motion of the vortical structures. During the first stage the flow is stable, but has a spatially inhomogeneous structure. It is under these conditions that the AJE is presumably observed. A detailed examination of the occurrence of an oscillating superfluid flow is possible only by means of simple models. In the present paper we investigate one-dimensional inhomogeneous superfluid flow on the basis of the Ginzburg-Pitaevskiĭ phenomenological theory[8] and its generalization to include the nonstationary case[9,10]. It can be assumed that the density oscillations of the superfluid component (which can be attributed to fluctuations of the ordering parameter) are indeed the cause of formation of quantized vortices in the three-dimensional case. In the one-dimensional case under consideration, the fluctuations of the ordering parameter are not vortices but local inhomogeneities, which can be only arbitrarily called vortical structures. When the indicated theory is employed, it is necessary to take the following into consideration. Experiment shows[1-4] that the normal component remains immobile in the channel and the dissipative effects are negligibly small. In other words, we have here, just as in the case of the Josephson tunnel effect, an equilibrium albeit nonstationary situation.

To elucidate the features of superfluid flow, we have first investigated the stationary case. In the concluding part of the paper we consider the nonstationary problem in the presence, between the two volumes, of a certain energy difference causing the nonstationarity. In this chosen model, oscillating flow can appear because the increase of the phase of the ordering parameter, due to the presence of the energy difference between the two volumes of helium, is lost as a result of fluctuations. The flow is stable if the flow velocity in the stationary state does not exceed a definite value.

2. In the stationary equilibrium case, the ordering parameter $\Psi$ is determined from the condition that the total thermodynamic potential be a minimum. In the hydrodynamic representation $\Psi$ is written in the form

$$\Psi(\mathbf{r}) = f(\mathbf{r}) \exp(i\varphi(\mathbf{r})), \quad \rho_s = m|\Psi|^2, \quad \mathbf{V} = (\hbar/m)\nabla\varphi. \qquad (2.1)$$

Here $f(\mathbf{r})$ is the amplitude, $\varphi(\mathbf{r})$ the phase, and $\rho_s$ the density of the superfluid component, and $\mathbf{V}$ its velocity. The thermodynamic potential per unit volume is[8]

$$F = (\hbar^2/2m)|\nabla\Psi|^2 + F_0(p, T, |\Psi|^2), \qquad (2.2)$$

where $F_0$, in accordance with the general theory of second-order phase transitions, can be set equal to[11]

$$F_0 = F_I(p, T) - \alpha|\Psi|^2 + {}^1/_2\,\beta|\Psi|^4. \qquad (2.3)$$

This expansion of $F_0$ is used in the present paper.

The following values are cited in the literature[8,9] for the coefficients $\alpha$ and $\beta$: $\alpha \approx 4.5 \times 10^{-17}(T_\lambda - T)/T_\lambda$ erg and $\beta \approx 4 \times 10^{-40}$ erg-cm$^3$. Recognizing that the normal component is immobile, the equilibrium value of $\Psi$ is given by[8]

$$-\frac{\hbar^2}{2m}\Delta\Psi - \alpha\Psi + \beta|\Psi|^2\Psi = 0, \tag{2.4}$$

where the parameter $\alpha$ is fixed.

In the one-dimensional case of interest to us, the equations for the functions $f(x)$ and $\varphi(x)$ become

$$f\frac{d^2\varphi}{dx^2} + 2\frac{df}{dx}\frac{d\varphi}{dx} = 0, \tag{2.5}$$

$$\frac{d^2 f}{dx^2} - \left(\frac{d\varphi}{dx}\right)^2 f + f - f^3 = 0. \tag{2.6}$$

We shall henceforth use the dimensionless variables

$$x = (2m\alpha/\hbar^2)^{1/2}x_i, \qquad f = (\beta/\alpha)^{1/2}f_i, \tag{2.7}$$

where $x_i$ and $f_i$ are dimensional quantities. The coordinate system is rigidly connected to the walls of the channel, and the origin is in the middle of the channel, whose length is $L$.

The last term in (2.5) describes the change of phase as a result of the density oscillations, and since $df/dx$ is large at interatomic distances, the phase should vary jumpwise in the region of such local density oscillations. We shall henceforth regard these singularities as inhomogeneities. From (2.5) follows the well known expression for the flow velocity:

$$d\varphi/dx = J/f^2, \tag{2.8}$$

where $J = \text{const}$ is the specified liquid flux. Substitution of (2.8) in (2.6) allows us to rewrite the latter in the form

$$\frac{d^2 f}{dx^2} + f - f^3 - \frac{J^2}{f^3} = 0. \tag{2.9}$$

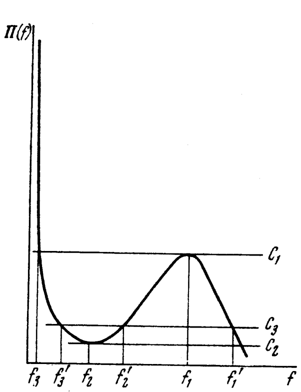
The first integral in (2.9) can be written in the form

$$\frac{1}{2}\left(\frac{df}{dx}\right)^2 + \Pi(f) = C, \quad \Pi(f) = \frac{1}{2}f^2 - \frac{1}{4}f^4 + \frac{J^2}{2f^2}. \tag{2.10}$$

Here $C$ is a constant energy and determines at fixed $J$ the initial flow velocity. The stable solutions of (2.9) are determined by the extrema of the function $\Pi(f)$, which can be found from the equation

$$z^3 - z^2 + J^2 = 0, \tag{2.11}$$

where $z = f^2$.




Plot of the function $\Pi(f)$ for $J < J_{cr}$.

It follows from (2.11) that if $J^2 < 4/27$, then $\Pi(f)$ has a local minimum at the stability point $f_2$ and a local maximum at the critical point $f_1$ (see the figure). If $J^2 > 4/27$, then $\Pi(f)$ has no extremum. Thus, the character of the flow is determined by two parameters: the value of the flux $J$ and the value of the constant energy $C$. We shall call a flux equal to $\frac{2}{3}\sqrt{3}$ critical. At $J > J_{cr}$, Eq. (2.9) has no positive solutions with respect to $f$ at all. Such a case can be interpreted as disruption of the superfluid flow by intense formation of inhomogeneities. At fixed $J < J_{cr}$, the character of the motion is determined by the constant $C$. Stable flow corresponds to values of $C$ in the interval $C_2 \le C \le C_1$ (see the figure). If it is assumed that the fluctuation is located at the origin and that the flow is homogeneous at the entrance to the channel, then the solution (2.9) in this interval can be written in the form

$$f(x) = \left[z_2' - a_1 \,\mathrm{cn}^2\left(\frac{1}{\sqrt{2}}a^{1/2}x;\ k\right)\right]^{1/2}, \quad k = \left(\frac{z_2' - z_3'}{z_1' - z_3'}\right)^{1/2}. \tag{2.12}$$

Here $\mathrm{cn}(u; k)$ is the elliptic cosine, $a_1 = z_2' - z_3'$, $a = z_1' - z_3'$ and $z_1'$, $z_2'$, and $z_3'$ are the roots of the equation

$$C - \Pi(z) = 0. \tag{2.13}$$

The solution (2.13) is periodic in the coordinate $u$ with period $2K(k)$, where $K(k)$ is a complete elliptic integral of the first kind. It follows from the behavior of the solution (2.12) that the density oscillations produce regions of anomalously rapid change of amplitude and accordingly of the phase. Such singularities can be interpreted as nuclei of inhomogeneities. For a more complete explanation of the physical meaning of the obtained solution, let us consider the change of the phase during the period $T$. According to (2.8), the change of phase can be written in the form

$$\Delta\varphi = J\int_0^T \frac{d\varphi}{dx}dx = \frac{JT}{z_2'} + J\int_0^T\left(\frac{1}{z} - \frac{1}{z_2'}\right)dx,$$

or using (2.10), this expression can be rewritten in the form

$$\Delta\varphi = \frac{JT}{z_2'} - \frac{2J}{\sqrt{2}\,z_2'}\int_{z_3'}^{z_2'}\frac{(z - z_2')^{1/2}\,dz}{z[(z - z_1')(z - z_3')]^{1/2}}. \tag{2.14}$$

Integrating, we obtain

$$\Delta\varphi = \frac{JT}{z_2'} + \frac{4J}{\sqrt{2}\,z_2' z_3'}\frac{1}{\sqrt{z_1' - z_3'}}\left[z_2'\Pi_1(n;k) - z_3'K(k)\right], \tag{2.15}$$

where $\Pi_1(n; k)$ is a complete elliptic integral of the third kind and $n = (z_2' - z_3')/z_3'$. The first term in (2.15) corresponds to the slow phase change connected with the motion of the flow as a whole. The second describes the jumplike change of phase in the region of the nucleus. Let us analyze individual cases at $J < J_{cr}$.

As $C \to C_2$, the second term in (2.15) is minimal, i.e., the value of $C_2$ corresponds to the minimum initial velocity at which nuclei of inhomogeneities begin to appear. In this case the nuclei form a one-dimensional lattice with a period equal to $\pi$. With increasing $C$ (increase of initial velocity), the second term in (2.15) increases, indicating an increase in the nuclei, and the period of their lattice increases like $2K(k)$. At the critical point $C = C_1$, expressions (2.12) and (2.15) can be transformed, using the equations for $\Pi_1(n; k)$ and $\mathrm{cn}(u; k)$ as $k \to 1$[12] and $z_2' = z_1' = z_1$, as follows:

$$f(x) = [z_1 - a_1 \operatorname{ch}^{-2}(2^{-1/2}a^{1/2}x)]^{1/2}, \qquad (2.16)$$

$$\Delta\varphi = \frac{JT}{z_1} + 2\operatorname{arctg}\left(\frac{z_1 - z_3}{z_3}\right)^{1/2}. \qquad (2.17)$$

We see that the second term in (2.17) is almost equal to $\pi$, since $z_1 > z_3$. Such a change of phase can be interpreted as the formation of inhomogeneity having a critical dimension or of a critical inhomogeneity after the critical point is reached. The solution (2.16) describes an isolated critical inhomogeneity, since the lattice period is infinite. It can thus be assumed that the state of the moving superfluid liquid is metastable, namely, as the initial flow velocity increases, inhomogeneity nuclei are produced and reach a critical dimension after a definite velocity is reached. The quantity $C_1 - C_2$ represents the energy necessary for the formation of the critical inhomogeneity. At $J = J_{cr}$, this energy is equal to zero and, as already noted, the critical inhomogeneities are easily produced.

Let us calculate the flow velocity of the critical point, as determined by the first term in (2.17),

$$V_{cr} = J/z_1; \qquad (2.18)$$

we shall henceforth call it the critical velocity. The most stable state of motion corresponds to the maximum energy $C_1 - C_2$ necessary for the formation of inhomogeneities with critical dimensions. The constants $C_1$ and $C_2$ can be expressed in terms of J from the condition that the roots of (2.13) be equal. An investigation of this equation shows that the difference $C_1 - C_2$ is maximal at $J \approx J_{cr}/6$. The value of $z_1$ is given by

$$z_1 = {}^2/_3 + ({}^4/_9 - {}^4/_3 C_1)^{1/2}.$$

The constant $C_1$ can be determined from the approximate formula

$$C_1 \approx {}^1/_3 - ({}^1/_{27} - {}^1/_4 J^2)^{1/2}.$$

For $J = J_{cr}/6$ we obtain $V_{cr} \approx 0.02$. Recognizing that $V = 1$ in the units of (2.7) corresponds to $10^3$ cm/sec in ordinary units, we get $V_{cr} \approx 20$ cm/sec; we have assumed $T_\lambda - T \sim 0.1°K$.

It is interesting to trace the character of the flow at $J < J_{cr}$ and $C > C_{cr}$. In this case Eq. (2.13) has one real root, $z_3' = c$, and two complex conjugate roots $z_1' = a + ib$ and $z_2' = a - ib$. The solution of (2.9) can now be written in the form

$$f(x) = [c - p\,\mathrm{sc}^2(2^{-1/2}p^{1/2}x; k)]^{1/2}, \qquad (2.19)$$

where

$$p = [(a-c)^2 + b^2]^{1/2}, \quad k = \left(\frac{p - a + c}{2p}\right)^{1/2}, \quad \mathrm{cs}^{-1}(u; k) = \frac{\mathrm{sn}(u; k)}{\mathrm{cn}(u; k)}$$

and $\mathrm{sn}(u'; k)$ is the elliptic sine.

The solution (2.19) describes a linear chain of inhomogeneities of critical dimension with period 2K(k). At $J = J_{cr}$, $z_2' = a$, $z_1' = z_3' = c$, $k = 0$ and (2.19) describes a linear chain of the indicated inhomogeneities with the minimum possible period, equal to $\pi$.

3. We proceed to consider the nonstationary problem arising in the presence of a constant energy difference between the two helium volumes. According to the phenomenological theory[9,10] the superfluid density is in general not conserved, owing to the nonequilibrium processes, and the energy $E_0$ per unit volume is a function of the total density $\rho$, $|\Psi|^2$, T, and the entropy S in the coordinate frame in which the superfluid liquid is at rest (the normal component is assumed immobile). The equation for $\Psi$ is sought in the form

$$i\hbar\partial\Psi/\partial t = (A - iB)\Psi, \qquad (3.1)$$

where A and B are certain Hermitian operators. The operator A can be written in the form[9]

$$A = -\frac{\hbar^2}{2m}\Delta + U + m\left\{\left(\frac{\partial E_0}{\partial \rho}\right)_{\rho_s, S} + \left(\frac{\partial E_0}{\partial \rho_s}\right)_{\rho, S}\right\},$$

where the term U takes into account the energy produced by the difference between the level heights. This term plays in the present case the same role as the scalar potential in the case of superconductors. The operator B describes the relaxation of $\rho_S$, and is given by[9,10]

$$B = \Lambda\left\{-\frac{\hbar^2}{2m}\Delta + \left(\frac{\partial E_0}{\partial \rho_s}\right)_{\rho, S} m\right\},$$

where $\Lambda$ is a coefficient proportional to the reciprocal time of relaxation of $\rho_S$. We finally obtain the following equation for $\Psi$:

$$i\hbar\frac{\partial\Psi}{dt} = -\frac{\hbar^2}{2m}\Delta\Psi + U\Psi + \left\{\left(\frac{\partial E_0}{\partial \rho}\right)_{\rho_s, S} + \left(\frac{\partial E_0}{\partial \rho_s}\right)_{\rho, S}\right\} m\Psi \qquad (3.2)$$
$$- i\Lambda\left\{-\frac{\hbar^2}{2m}\Delta\Psi + \left(\frac{\partial E_0}{\partial \rho_s}\right)_{\rho, S} m\Psi\right\}.$$

As already noted, we can neglect the dissipative processes, and then, taking into account the fact that the normal component is immobile, the energy per unit volume $E_0$ can be regarded as a function of $|\Psi|^2$ and T. Such an approach presupposes considering only small deviations from the equilibrium state under the influence of the external perturbation causing the nonstationarity, and, as emphasized in[1-4,6], such a situation obtains in the case of the AJE. Taking all the foregoing into account, the equation for the ordering parameter is reduced to the form

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi - \alpha\Psi + U\Psi + \beta|\Psi|^2\Psi. \qquad (3.3)$$

The appearance of an oscillating flow component in the channel can be understood on the basis of Landau's theory of nonstationary liquid flow[13]. The Landau theory gives a physically very lucid picture, which accounts in the best manner for the inhomogeneous flow, and in which the first stage corresponds to a small deviation from the stationary case. During this stage the flow changes and acquires as a result a stable periodic character. We see that this is precisely what occurs in the AJE case. Coherence of the superfluid state is the main condition under which observation of the oscillating flow is possible. Indeed, for an ordinary liquid, each particle taking part in the periodic motion is characterized by its own initial phase, as a result of which only disordered pulsations can be observed.

For the coherent superfluid state, on the other hand, all the parameters characterizing the motion of the system coincide fully, and the oscillations of the flow become manifest in a macroscopic scale. We shall trace the Landau picture in the exposition that follows with allowance for the features of our model. In the hydrodynamic representation, Eq. (3.3) can be rewritten, in the units of (2.7), in the form

$$\frac{\partial f}{\partial t}=-f\frac{\partial^2\varphi}{\partial x^2}-2\frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x}, \tag{3.4}$$

$$f\frac{\partial\varphi}{\partial t}=\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial\varphi}{\partial x}\right)^2 f+(1-U)f-|f|^2 f; \tag{3.5}$$

here $f \equiv f(x, t)$ and $\varphi \equiv \varphi(x, t)$.

An essential feature of this model is that the flow velocity is a function of the amplitude f and of the flux J. The behavior of the amplitude determines the character of the motion, and the state of superfluid flow corresponds to real positive solutions. The time variation of the phase determines the stability of this flow: an increase (decrease) of the phase with time is evidence of instability (stability) of the motion.

To investigate the stability, we make use of perturbation theory and seek the solution of the problem after separating the variables

$$f(x, t)=f(x)v(t),\quad \varphi(x, t)=\varphi(x)\Theta(t). \tag{3.6}$$

Perturbation theory is valid if the condition $U < 1$ is satisfied. We assume that the perturbation is turned on at the instant of time $t = 0$ and acts during the succeeding instants of time. Up to the instant $t = 0$ we had a stable stationary flow with velocity close to critical, described by the functions $f_0(x)$ and $\varphi_0(x)$. Since we are interested in small deviations from the stationary state, we assume henceforth $U \ll 1$. We can then confine ourselves to the first approximation for the function $\Theta(t)$ and to the stationary solution for $f(x)$. Substituting (3.6) in (3.4) we obtain the following equations with allowance for the fact that (3.4) is the imaginary part of (3.3):

$$\frac{\partial^2\varphi}{\partial x^2}+2\frac{1}{f}\frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x}=i\lambda, \tag{3.7}$$

$$i\frac{\partial v}{\partial t}=-\lambda\Theta v, \tag{3.8}$$

where $\lambda$ is the separation constant.

We seek $\varphi$ in the form $\varphi = \varphi_1 + i\varphi_2$. It follows from (3.7) that $\varphi_1$ is determined by the stationary equation (2.5), where $J = J_0$ is now the initial flux, and $\varphi_2$ describes the dissipative flux caused by a source with energy $\lambda$. The appearance of an imaginary part of the flux is not unexpected, and an analogous situation occurs in the Josephson nonstationary tunnel effect in superconductors. In accordance with the assumption made, we disregard the dissipative component of the flux. Equating the source energy to the external perturbation U at the entrance to the channel ($x = -L/2$), where the flow is uniform, we obtain the following condition for the determination of $\lambda$;

$$\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{x=-L/2}=\lambda=U. \tag{3.9}$$

Integration of (3.8) yields

$$v(t)=\exp\left\{i\lambda\int_0^t\Theta(t')\,dt'\right\}. \tag{3.10}$$

The constant coefficient is set equal to unity in accord with the initial conditions.

From (2.8) and (3.10) we get the overall result that the nonstationary flow can be described by periodic motion with increasing frequency, constituting a superposition of individual harmonics[13]. If the motion is stable, then $\Theta(t)$ can be replaced by the initial phase $\Theta_0$[13], and we obtain a stable oscillating motion with one degree of freedom. We obtain the form of the function $\Theta(t)$ from Eq. (3.5), which can be rewritten, with allowance for (3.7) and (3.10), as

$$\frac{\partial^2 f}{\partial x^2}+(1-U)f-f^3-\frac{J_0^2}{f^3}\Theta^2(t)=f\varphi\frac{\partial\Theta(t)}{\partial t}. \tag{3.11}$$

We differentiate (3.11) with respect to time, and obtain as a result

$$\varphi\frac{\partial^2\Theta}{\partial t^2}+2\frac{J_0^2}{f^4}\Theta\frac{\partial\Theta}{\partial t}=0. \tag{3.12}$$

We put

$$f=f_0+f_1,\quad \Theta=\Theta_0+\Theta_1, \tag{3.13}$$

and have in the first approximation (we assume for simplicity that $\Theta_0 = \varphi_0 = 1$)

$$\frac{\partial^2\Theta_1}{\partial t^2}+2\frac{J_0^2}{f_0^4}\frac{\partial\Theta_1}{\partial t}=0. \tag{3.14}$$

The time behavior of the phase, as seen from (3.14), depends on the distribution of the velocity in the stationary case $V_{st}$. From the general analysis of the hydrodynamic stability of an inviscid fluid[14] it is known that if the velocity distribution has an inflection point in the stationary case, then the flow is stable, and if there is no such point, then the flow is unstable. In this problem the character of the velocity distribution in the stationary case, at values not exceeding critical, is given by

$$V_{st}=V_0+V_0\frac{a_1\,\mathrm{cn}^2(2^{-1/2}a^{1/2}x;\,k)}{[z_2'-a_1\,\mathrm{cn}^2(2^{-1/2}a^{1/2}x;\,k)]}, \tag{3.15}$$

where the first term $V_0$ corresponds to the velocity of the flow as a whole, and the second describes the velocity oscillation. We see that in the present case the velocity in the stationary state has an inflection point.

We seek the solution for $\Theta_1(t)$ in the form

$$\Theta_1(t)=c_1e^{i\omega t}+c_2, \tag{3.16}$$

where $c_1$ and $c_2$ are certain constants. For the frequency $\omega$ we obtain the expression

$$\omega=2iJ_0^2/f_0^4.$$

In the first approximation, if the velocity of the stationary state does not exceed the critical value, the change of phase with time is described by the formula (see[13])

$$\Theta_1(t)=\mathrm{const}\cdot e^{-2V_0^2t}+U/2V_0^2. \tag{3.17}$$

From an analysis of (3.17) we conclude that $\Theta_1(t)$ tends to a constant value as $t \to \infty$, and consequently the motion is stable and has an oscillating component with specified amplitude. In other words, the action of the perturbation can be described as self-excitation of an oscillator, leading ultimately to the appearance of a steady periodic motion with definite amplitude. It is important here that the phase of this oscillation is not determined uniquely by the external conditions, but depends on the random initial values of the phase of the perturbation, i.e., in fact it can be arbitrary. In the case of the AJE we have in essence excitation of an oscillator whose frequency varies with time, and this variation is due to fluctuations in the system, both internal and external, introduced upon synchronization. This distinguishes the present effect from nonstationary Josephson tunnel effect in superconductors. The flow velocity changes with time like (see (2.8), (3.10), and (3.17))

$$V = V_0 \cos^2(\bar{\omega} t) = {}^1/_2 V_0 + {}^1/_2 V_0 \cos(2\bar{\omega} t). \quad (3.18)$$

The frequency of the periodic motion must be regarded as the frequency of a randomly modulated oscillator

$$\cos(\bar{\omega} t) = \cos\left(\omega_0 t + \int_0^t \nu(t')\,dt'\right), \quad (3.19)$$

where $\omega_0 = mgZ/\hbar$ is the natural or Josephson frequency and $\nu(t) = \bar{\omega} - \omega_0$ is a certain random quantity.

We can confine ourselves to the stationary solution for $f(x)$ only for a channel of short length, since we see from (3.11) that the amplitude attenuates with increasing channel length. It is physically clear that in a long channel an important role is assumed by inhomogeneity-accumulation processes which stop the superfluid flow. The calculation in terms of high approximations is quite difficult, and we shall therefore use a simplified procedure. We average (3.11) over the time period $2\pi/\omega_0$ and assume that the total change of phase during this time is $2\pi$, and in addition we put $\varphi(x) = V_0 x$, which is equivalent to regarding the flux as a single entity. By investigating the conditions for the superfluid flow on the basis of the obtained equation, as in the stationary case, assuming the coefficients of the equation constant, we obtain an approximate expression for the critical channel length

$$L_{cr} \sim 1/UV_0. \quad (3.20)$$

For $U \approx 0.01$ and $V_0 \sim 0.02$ we have $L_{cr} \sim 5 \times 10^3$ or $L_{cr} \sim 10^{-3}$ cm in ordinary units.

We now investigate the resonant case. Assume that an alternating external signal (perturbation) of the type $E_1 \exp(i\nu_0 t)$, where $E_1$ is the signal energy and $\nu_0$ the frequency, is added to the constant perturbation. The signal energy is assumed to be small enough to cause a change in the state of the flow, in other words, the role of the signal reduces to modulation of the flow (synchronization of the vortex formation). We can then confine ourselves, as before, to the stationary solution for $f(x)$ and replace $\Theta(t)$ by the initial phase. The function $v(t)$ satisfies the following equation (in ordinary units)

$$i\frac{\partial v}{\partial t} + (i\omega_1 \sin \nu_0 t + \bar{\omega})v = 0, \quad (3.21)$$

where $\omega_1 = E_1/\hbar$.

Solving (3.21), we obtain the following expression for the flow velocity in the case of modulation by an external signal:

$$V = V_0 \cos^2(\bar{\omega} t) \exp\left\{-2\frac{\omega_1}{\nu_0}(1 - \cos \nu_0 t)\right\}. \quad (3.22)$$

The flow velocity decreases to zero upon satisfaction of the relation $n_1\omega_0 \sim n_0\nu_0$, which gives the resonance condition (1.1).

A resonant state of the system corresponds to a step on the relaxation curve of the difference of the levels Z. The time during which the system is at resonance can be arbitrary, since synchronization is accompanied by blocking of the oscillator frequency by the external signal. Loss of resonance is determined only by the fluctuations that violate the synchronization conditions. The hydrodynamic description does not make it possible, using such a simple model, to present a detailed picture of the influence of the fluctuations. All that can be done here is to present a qualitative picture of the influence of the fluctuations. The quantity $\omega_1$ can be interpreted as the synchronization band. A large synchronization band favors a better locking of the oscillator frequency with the external signal. It can be concluded from this that an increase of the signal energy improves the synchronization conditions, But this is far from correct. Since we have a randomly modulated oscillator, the increase of the signal power increases the power of the noise acting on the given oscillator. The latter plays a decisive role in the determination of the synchronization conditions. Thus, in the case of a high-power oscillator the relaxation curves will have more steps corresponding to the harmonics (integer $n_1/n_2$) and subharmonics (fractional $n_1/n_2$), whereas the stability on these steps decreases. This is precisely the situation realized in the experimental conditions of[1-4]. As seen from (3.22), the flow velocity is smaller with modulation than without modulation, a fact noted in the experiment as a pumping effect[3,4]. The higher the signal energy, the larger the pumping, as follows from (3.22).

In conclusion, the authors thank L. P. Pitaevskiĭ for many valuable remarks and I. M. Khalatnikov for a discussion of the present work.

[1] P. L. Richards and P. W. Anderson, Phys. Rev. Lett., 14, 540 (1965).
[2] B. M. Khorana and B. S. Chandrasekhar, Phys. Rev. Lett., 18, 230 (1967).
[3] B. M. Khorana, Phys. Rev., 185, 299 (1969).
[4] P. L. Richards, Phys. Rev., A, 2, 1532 (1970).
[5] I. O. Kulik and I. K. Yanson, Éffekt Dzhozefsona v sverkhprovodyashchikh tunnel'nykh strukturakh (The Josephson Effect in Superconducting Tunnel Structures), Nauka, 1970.
[6] P. W. Anderson, Rev. Mod. Phys., 38, 298 (1966).
[7] R. P. Feynman, Progress in Low Temperature Physics, ed. Gorter, 1, 1955.
[8] V. L. Ginzburg and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 34,1240 (1958)[Sov. Phys.-JETP 7, 858(1958)].
[9] L. P. Pitaevskiĭ, ibid. 35, 408 (1958) [8, 282 (1959)].
[10] I. M. Khalatnikov, Vvedenie v teoriyu sverkhtekuchesti (Introduction to the Theory of Superfluidity), Nauka, 1965.
[11] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Fizmatgiz, 1964.
[12] A. M. Zhuravskiĭ, Spravochnik po éllipticheskim funktsiyam (Handbook of Elliptic Functions), AN SSSR, 1941.
[13] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1954 [Addison-Wesley, 1958].
[14] C. C. Lin, Hydrodynamic Stability, Cambridge U.P.

Translated by J. G. Adashko
208