THE CONTRIBUTION OF TWO-PHONON PROCESSES TO THE ANOMALOUS HALL EFFECT

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The anomalous Hall effect (AHE), due to the asymmetry of the electron scattering by phonons, is considered. In contrast to previous calculations not only one-phonon processes but also two-phonon processes are taken into consideration in the scattering part of the Hamiltonian. It is shown that this leads in the first place to a larger value for the AHE since here it arises in third-order perturbation theory with respect to the electron-phonon interaction, and secondly it permits us to describe the AHE over the entire temperature range by a certain interpolation formula, which is similar to the Grüneisen formula for the electrical resistance.

At the present time the most prevalent point of view is that the anomalous Hall effect (AHE) arises as a consequence of the asymmetry, due to the spin-orbit interaction, of the scattering of the conduction electrons.[1–6] [1)] In ferromagnetic d-metals, over a broad range of temperatures, the conduction electrons are primarily scattered by the phonons, which also determines the major part of the electrical resistance of these metals. Therefore some of the first articles on the AHE were devoted to an investigation of the asymmetry of the electron-phonon interaction.[2–4] Since a one-phonon Hamiltonian was used in all of these articles, in order to obtain the AHE it was necessary to expand the scattering probability up to the fourth order in the electron-phonon interaction. This led to extremely cumbersome calculations which could be carried out to the very end only for the case of high temperatures. As a result it was found that the AHE coefficient R_s is proportional to the square of the temperature for $T \gg \Theta$ (Θ denotes the Debye temperature).

Such a dependence of R_s satisfactorily describes the experimental data at high temperatures, breaking down only above the Curie point where in all the substances which have been investigated R_s turns out to be independent of the temperature.[8–10] It would appear as if the latter property testifies against the participation of phonons in the formation of the AHE; however recently special investigations have been carried out in the region of the para-process, and these have led to the conclusion that the Hall effect has a different nature below and above the Curie point and is described by different coefficients.[11–12] Therefore, the fact that the Hall coefficient does not depend on the temperature above the Curie point is not a convincing argument against a "phonon" mechanism for the AHE.

In the present article the contribution made to the AHE by the asymmetry of the electron-phonon interaction is investigated once again. It is shown that the earlier treatments[2–4] are inconsistent to the extent that they do not take into account the two-phonon terms in the scattering part of the Hamiltonian. Taking these terms into account leads, in the first place, to a larger value of the AHE and, in the second place, it substantially simplifies the investigation, since it is necessary to expand the scattering probability only to third-order in the electron-phonon interaction. This permits us to derive a certain formula, analogous to the Grüneisen interpolation formula for the electrical resistance, describing the temperature dependence of R_s over the entire range of temperatures.

THE HAMILTONIAN OF THE ELECTRON-PHONON INTERACTION

The displacement of the ions from their equilibrium positions gives rise to a corresponding change $V(\mathbf{r})$ in the interaction energy of the conduction electron with the ions of the metal. Assuming the usual approximations, we can write

$$V(\mathbf{r}) = \sum_m [W(\mathbf{r} - \mathbf{r}_m - \boldsymbol{\xi}_m) - W(\mathbf{r} - \mathbf{r}_m)] = \frac{1}{N} \sum_{m\mathbf{q}} W_\mathbf{q} e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}_m)} (e^{-i\mathbf{q}\boldsymbol{\xi}_m} - 1), \tag{1}$$

where $\mathbf{r}$ is the radius vector of the conduction electron; $\mathbf{r}_m$ and ξ_m are, respectively, the radius vector of the m-th ion and its displacement; N is the total number of elementary cells in the fundamental region of the crystal. Changing to the representation of second quantization, we obtain an expression for the Hamiltonian $\mathscr{H}$ of the electron-phonon interaction in the form

$$\mathscr{H} = \int \hat{\Psi}^+(\mathbf{r}) V(\mathbf{r}) \hat{\Psi}(\mathbf{r}) d\mathbf{r} = \frac{1}{N} \sum_{ll_1\mathbf{q}m} W_\mathbf{q} e^{-i(\mathbf{k}-\mathbf{k}_1, \mathbf{r}_m)} (e^{-i\mathbf{q}\boldsymbol{\xi}_m} - 1) \delta_{\mathbf{k};\,\mathbf{k}_1+\mathbf{q}} I_{ll_1} a_l^+ a_{l_1}. \tag{2}$$

Here

$$\hat{\Psi}^+(\mathbf{r}) = \sum_l \psi_l^* a_l^+, \qquad \Psi(\mathbf{r}) = \sum_l \psi_l a_l$$

are the wave-function operators of the conduction electrons; $\psi_l = \Omega^{-1/2} e^{i\mathbf{k}\cdot\mathbf{r}} u_l(\mathbf{r})$ are the Bloch functions, which are normalized to the volume Ω of the fundamental region of the crystal; the quantum number l denotes the band in-

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[1)] Recently another mechanism for the AHE has been proposed which is not related to the asymmetry of the scattering of conduction electrons. [7]

dex n and the wave number k of the conduction electron; a_l^+ and a_l are the operators for the creation and annihilation of these electrons in the state l;

$$I_{ll_1} = \frac{1}{\Omega_0}\int u_l^*(\mathbf{r})\, u_{l_1}(\mathbf{r})\, d\mathbf{r}, \tag{3}$$

where Ω_0 denotes the volume of the elementary cell. The vector ξ_m characterizing the displacement can be expressed in terms of the phonon creation and annihilation operators $b_\mathbf{q}^+$ and $b_\mathbf{q}$:

$$\xi_m = \frac{1}{\sqrt{N}}\sum_\mathbf{q} \mathbf{e}_\mathbf{q} A_\mathbf{q}\left(e^{i\mathbf{q}\mathbf{r}_m} b_\mathbf{q} + e^{-i\mathbf{q}\mathbf{r}_m} b_\mathbf{q}^+\right), \tag{4}$$

$$A_\mathbf{q} = \sqrt{\hbar / 2m\omega_\mathbf{q}}, \tag{5}$$

where $\mathbf{e}_\mathbf{q}$ is the polarization vector, which for acoustic phonons can be chosen along the phonon wave vector $\mathbf{q}$; m is the mass of a metal ion; $\hbar\omega_\mathbf{q}$ is the phonon energy.

Usually in the expansion of expression (2) in powers of ξ_m, only the first term is considered, which corresponds to taking into account only one-phonon scattering processes. To take also the two-phonon scattering processes into account, it is necessary to retain also the terms of second order in ξ_m. As a result the Hamiltonian of the electron-phonon interaction is obtained in the following form:

$$\mathscr{H} = \frac{1}{\sqrt{N}}\sum_{nn_1\mathbf{q}}\ \sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{q}} Q_{ll_1\mathbf{q}}(b_\mathbf{q} - b_{-\mathbf{q}}^+)\, a_l^+ a_{l_1}$$
$$+ \frac{1}{N}\sum_{nn_1\mathbf{q}\mathbf{q}_1}\ \sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{q}+\mathbf{q}_1} B_{ll_1\mathbf{q}\mathbf{q}_1}(b_\mathbf{q} b_{\mathbf{q}_1} + b_\mathbf{q}^+ b_{\mathbf{q}_1}^+ - b_\mathbf{q}^+ b_{-\mathbf{q}_1} - b_{-\mathbf{q}} b_{\mathbf{q}_1}^+)\, a_l^+ a_{l_1}, \tag{6}$$

where

$$Q_{ll_1\mathbf{q}} = iqA_\mathbf{q} W_\mathbf{q} I_{ll_1}, \tag{7}$$

$$B_{ll_1\mathbf{q}\mathbf{q}_1} = -\frac{1}{2}[q^2 + (\mathbf{q}\mathbf{q}_1)]\,[q_1^2 + (\mathbf{q}\mathbf{q}_1)]\frac{A_\mathbf{q} A_{\mathbf{q}_1}}{qq_1} W_{\mathbf{q}+\mathbf{q}_1} I_{ll_1}. \tag{8}$$

Since at higher temperatures the specific form of $W_\mathbf{q}$ as a function of $\mathbf{q}$ does not have any influence on the temperature dependence of R_S, and at low temperatures ($\mathbf{q} \to 0$) $W_\mathbf{q}$ tends to a certain constant W, then in what follows the function $W_\mathbf{q}$ will be replaced everywhere by W. As is well known, for a nondegenerate structure of the spectrum in the linear approximation with respect to the spin-orbit interaction, the energy of the conduction electrons is not changed, but only their wave functions are changed (in our case the latter only appear in I_{ll_1}). It is precisely this fact which leads to the result that the terms responsible for the AHE are not contained in the usual Boltzmann kinetic equation. To investigate this effect it is therefore necessary to derive the kinetic equation in higher-order approximations than the first Born approximation.

THE KINETIC EQUATION

Let us consider a system of mutually interacting electrons and phonons in the presence of an external electric field. The Hamiltonian of such a system is written down in the form

$$\mathscr{H}_T = \mathscr{H}_0 + \mathscr{H} + \mathscr{H}_F, \tag{9}$$

where $\mathscr{H}_0$ is the energy operator of the non-interacting electrons and phonons, $\mathscr{H}$ takes their interaction into account, and $\mathscr{H}_F$ describes the interaction of the conduction electrons with the external electric field. We shall assume that the spin-orbit interaction is included in $\mathscr{H}_0$. Starting from the density-matrix operator equations linearized with respect to the external electric field, we obtain the kinetic equation, which takes the terms of the second Born approximation into account, in the form[6]

$$\operatorname{Sp} \rho[\mathscr{H}_F;\ a_l^+ a_l] = \operatorname{Sp} D[\mathscr{H};\ a_l^+ a_l]\tilde{\mathscr{H}} + {}^1\!/_2 \operatorname{Sp} D\tilde{\mathscr{H}}[a_l^+ a_l;\ \mathscr{H}]\tilde{\mathscr{H}}. \tag{10}$$

Here ρ denotes the equilibrium density matrix operator of the electron-phonon system; D is the part of the nonequilibrium correction to the density matrix which is diagonal in the representation of the Hamiltonian $\mathscr{H}_0$:

$$\tilde{\mathscr{H}}_{\chi\chi_1} = 2\pi i \mathscr{H}_{\chi\chi_1}\delta(E_\chi - E_{\chi_1}), \tag{11}$$

where E_χ denotes the eigenvalue of the Hamiltonian $\mathscr{H}_0$, which is determined by a complete set of the appropriate quantum numbers χ.

Not being interested in the corrections to the electrical resistance, which arise from the last term in Eq. (10), we shall henceforth understand that only the part proportional to the spin-orbit interaction has been separated from this term and retained. It should be noted that if only the one-phonon scattering processes are taken into account, then the third term in Eq. (10) is equal to zero, and therefore terms of higher order of smallness in $\mathscr{H}$ are responsible for the AHE. It is not difficult to verify that the basic contribution made to the AHE by the anisotropy of the electron-phonon interaction can be obtained only by simultaneously taking account of both one- and two-phonon scattering processes. In fact, the order of magnitude of the relative displacement of the ions from their equilibrium positions is given by the dimensionless parameter $qA_\mathbf{q}$, and the ratio W/E_F (E_F denotes the electron energy on the Fermi surface) determines the magnitude of the electron-ion interaction. Upon considering the contribution to the AHE from only the one-phonon scattering processes, the Hall term is proportional to

$$(Q / E_\Phi)^4 \sim (qA_\mathbf{q})^4 (W / E_F)^4.$$

However, upon taking the two-phonon processes into account, the order of magnitude of the AHE is determined by the product

$$(Q / E_\Phi)^2 (B / E_\Phi) \sim (qA_\mathbf{q})^4 (W / E_F)^3,$$

that is, it turns out to be (E_F / W) times larger.

Using the explicit form of $\mathscr{H}$ and performing the indicated operations in the Hall term of Eq. (10), we represent it in the form

$$\operatorname{Sp} D\tilde{\mathscr{H}}[a_l^+ a_l : \mathscr{H}]\tilde{\mathscr{H}} = \left(\frac{4\pi}{N}\right)^2 \operatorname{Im}\sum_{l_1 l_2} (\mathbf{k}_2 \mathrm{C}(\varepsilon_2))\{[ll_1l_2] + [l_1l_2l] + [l_2ll_1]\},$$

where

$$[ll_1l_2] = \sum_{\mathbf{q}\mathbf{q}_1} Q_{ll_1\mathbf{q}_1} B_{l_1l_2\mathbf{q}_1\mathbf{q}} Q_{l_2l\mathbf{q}} \delta_{\mathbf{k},\mathbf{k}_1+\mathbf{q}_1} \delta_{\mathbf{k}_2,\mathbf{k}+\mathbf{q}} N_q N_{q_1} \rho_{l_1} \rho_{l_2}$$
$$\times [e^{x+x_1}\delta(E_{l_1l\mathbf{q}_1})\,\delta(E_{ll_2\mathbf{q}}) + e^{x}\delta(E_{ll_1\mathbf{q}_1})\,\delta(E_{ll_2\mathbf{q}})$$
$$+ \delta(E_{ll_1\mathbf{q}_1})\,\delta(E_{l_2l\mathbf{q}}) + e^{x_1}\delta(E_{l_1l\mathbf{q}_1})\,\delta(E_{l_2l\mathbf{q}})]. \tag{13}$$

Here the following notation has been introduced: ρ_l and $N_\mathbf{q}$ are respectively the Fermi and Bose distribution functions of the electrons and phonons; $E_{ll_1\mathbf{q}} = E_l - E_{l_1} - \hbar\omega_{\mathbf{q}}$; $x = \hbar\omega_{\mathbf{q}}/\kappa T$; $\epsilon = (E_l - E_F)/\kappa T$; κ is the Boltzmann constant, and T is the absolute temperature.

In order to obtain Eq. (12) the nonequilibrium part of the electron distribution function was represented in the usual form:

$$f_l = -(\mathbf{k}\mathbf{C}(\varepsilon))\partial\rho(\varepsilon)/\partial\varepsilon. \tag{14}$$

Let us assume that the distances between the energy bands are so large that the electrons interacting with the phonons cannot undergo transitions from one band to another.[2] It follows therefore that $\delta(E_{ll_1\mathbf{q}}) = \delta_{nn_1} \times \delta(E_{\mathbf{kk}_1\mathbf{q}})$ and the subscript indicating the number of the band can be omitted. In this case, for the expansion of the corresponding matrix elements with respect to the spin-orbit interaction, one can utilize the results of [1], from which it follows that

$$J_{\mathbf{k},\mathbf{k}+\mathbf{q}_1}J_{\mathbf{k}+\mathbf{q}_1,\,\mathbf{k}+\mathbf{q}}J_{\mathbf{k}+\mathbf{q},\mathbf{k}} = -\frac{i}{2}\sum_{\alpha\beta} q_\alpha q_{1\beta}\left[\frac{\partial J_\alpha(\mathbf{k})}{\partial k_\beta} - \frac{\partial J_\beta(\mathbf{k})}{\partial k_\alpha}\right], \tag{15}$$

where $J_\alpha(\mathbf{k})$ is the regular part of the coordinate matrix element. Since after averaging over the spin variables[2]

$$J_\alpha(\mathbf{k}) = \frac{Aa^2}{M_s}[\mathbf{kM}]_\alpha, \tag{16}^*$$

then

$$J_{\mathbf{k},\mathbf{k}+\mathbf{q}_1}J_{\mathbf{k}+\mathbf{q}_1,\,\mathbf{k}+\mathbf{q}}J_{\mathbf{k}+\mathbf{q},\mathbf{k}} = i\frac{Aa^2}{M_s}(\mathbf{M}[\mathbf{q}_1\mathbf{q}]), \tag{17}$$

where M/M_S is the relative magnetization, a is the lattice constant, and the order of magnitude of A is determined by the strength of the spin-orbit interaction.

Substituting (17) into (12), we obtain

$$\operatorname{Sp} D\tilde{\mathscr{H}}[a_{\mathbf{k}}^+a_{\mathbf{k}},\ \mathscr{H}]\tilde{\mathscr{H}} = [1]_{\mathbf{k}} + [2]_{\mathbf{k}}, \tag{18}$$

where

$$\begin{aligned}[1]_{\mathbf{k}} = \int d\mathbf{q}\, d\mathbf{q}_1 F(\mathbf{qq}_1)(\mathbf{C};\mathbf{q})e^{\varepsilon}\rho_{\mathbf{k}-\mathbf{q}_1}\rho_{\mathbf{k}+\mathbf{q}}[e^{x_1}\delta(E_{\mathbf{k},\mathbf{k}-\mathbf{q}_1,\mathbf{q}_1})\delta(E_{\mathbf{k},\mathbf{k}+\mathbf{q},\mathbf{q}})\\ +\delta(E_{\mathbf{k},\mathbf{k}-\mathbf{q}_1,\mathbf{q}_1})\delta(E_{\mathbf{k},\mathbf{k}+\mathbf{q},\mathbf{q}}) + e^{x}\delta(E_{\mathbf{k},\mathbf{k}-\mathbf{q}_1,\mathbf{q}_1})\delta(E_{\mathbf{k}+\mathbf{q},\mathbf{kq}})\\ + e^{x+x_1}\delta(E_{\mathbf{k}-\mathbf{q}_1,\mathbf{kq}_1})\delta(E_{\mathbf{k}+\mathbf{q},\mathbf{kq}})],\end{aligned} \tag{19}$$

$$\begin{aligned}[2]_k = \int d\mathbf{q}\, d\mathbf{q}_1 F(\mathbf{qq}_1)(\mathbf{C};\mathbf{q})e^{\varepsilon}\rho_k\rho_{\mathbf{k}-\mathbf{q}-\mathbf{q}_1}[e^{x+x_1}\delta(E_{\mathbf{k}-\mathbf{q}_1,\mathbf{kq}_1})\delta(E_{\mathbf{k}-\mathbf{q}_1-\mathbf{q},\mathbf{k}-\mathbf{q}_1,\mathbf{q}})\\ + e^{x}\delta(E_{\mathbf{k},\mathbf{k}-\mathbf{q}_1,\mathbf{q}_1})\delta(E_{\mathbf{k}-\mathbf{q}_1-\mathbf{q},\mathbf{k}-\mathbf{q}_1,\mathbf{q}}) + \delta(E_{\mathbf{k},\mathbf{k}-\mathbf{q}_1,\mathbf{q}_1})\delta(E_{\mathbf{k}-\mathbf{q}_1,\mathbf{k}-\mathbf{q}_1-\mathbf{q},\mathbf{q}})\\ + e^{x_1}\delta(E_{\mathbf{k}-\mathbf{q}_1,\mathbf{k},\mathbf{q}_1})\delta(E_{\mathbf{k}-\mathbf{q}_1,\mathbf{k}-\mathbf{q}_1-\mathbf{q},\mathbf{q}})]\end{aligned} \tag{20}$$

$$F(\mathbf{qq}_1) = -\frac{2Aa^2W^3\Omega_0^2}{(2\pi)^6M_s}N_{\mathbf{q}}N_{\mathbf{q}_1}A_{\mathbf{q}}^2A_{\mathbf{q}_1}^2(\mathbf{M}[\mathbf{qq}_1])[q^2q_1^2 + (q^2+q_1^2) + (qq_1)^2] \tag{21}$$

Then assuming, in accordance with the method of Kroll, that C is a constant vector, after integration over ϵ we obtain

$$\begin{aligned}\operatorname{Sp} D\tilde{\mathscr{H}}[a_{\mathbf{k}}^+a_{\mathbf{k}};\mathscr{H}]\tilde{\mathscr{H}} = 5i\left(\frac{3\pi}{4}\right)^2\left(\frac{\hbar^2k}{m}\frac{\partial k}{\partial E_k}\right)_F^2\\ \times\frac{Aa^2q_0^2W^3}{(\varkappa\Theta)^2}\frac{(\mathbf{C}[\mathbf{kM}])}{M_s}I(T),\end{aligned} \tag{22}$$

where

$$\begin{aligned}I(T) = \left(\frac{T}{\Theta}\right)^{10}\int_0^{\Theta/T}\int_0^{\Theta/T} dx\,dx_1\frac{x^4x_1^4e^{x+x_1}}{(e^x-1)(e^{x_1}-1)}\left(\frac{x-x_1}{e^x-e^{x_1}}\right.\\ \left.+\frac{x+x_1}{e^{x+x_1}-1}\right)P\left(\frac{T}{2\Theta}x;\frac{T}{2\Theta}x_1\right),\end{aligned} \tag{23}$$

$P(x; x_1) = (1 - x^2)(1 - x_1^2)(1 + 3x^2 + 3x_1^2 - 7x^2x_1^2)$, and q_0 denotes the limiting wave number of the phonon. Without taking account in the second term of (10) of the contribution from two-phonon processes, which give a contribution to the electrical resistance, we find that

$$\operatorname{Sp} D[a_{\mathbf{k}}^+a_{\mathbf{k}},\mathscr{H}]\tilde{\mathscr{H}} = 3\pi i\frac{\hbar^2k_\Phi}{m}\left|\frac{\partial k}{\partial E_h}\right|_F\frac{W^2}{\varkappa\Theta}(\mathbf{kC})I_0(T), \tag{24}$$

where

$$I_0(T) = \left(\frac{T}{\Theta}\right)^5\int_0^{\Theta/T}\frac{x^5dx}{(e^x-1)(1-e^{-x})}. \tag{25}$$

Substituting (22) and (24) into Eq. (10), we obtain the kinetic equation in the following form:

$$\begin{aligned}\frac{e(\mathbf{kF})}{\varkappa Tk_F^2} = 3\pi I_0(T)\frac{\hbar^2k_F^2}{m\varkappa\Theta}\left(\frac{W\partial k}{k\partial E_k}\right)_F^2(\mathbf{kC})\\ -\frac{45\pi^2A(aq_0)^2I(T)}{32}\left(\frac{\hbar^2k_\Phi^2}{m\varkappa\Theta}\right)^2\left(\frac{W\partial k}{k\partial E_k}\right)_F^3\frac{(\mathbf{k}[\mathbf{CM}])}{M_s}.\end{aligned} \tag{26}$$

Let us consider the case when the magnetization and the intensity of the electric field are mutually perpendicular. Then, by directing the z axis along the magnetization, from Eq. (26) we obtain the following system of equations for C_x and C_y:

$$\begin{aligned}\frac{eF_x}{\varkappa Tk_F^2} = 3\pi I_0(T)\frac{\hbar^2k_F^2}{m\varkappa\Theta}\left(\frac{W\partial k}{k\partial E_k}\right)_F^2C_x\\ -\frac{45\pi^2A(aq_0)^2I(T)}{32}\left(\frac{\hbar^2k_F^2}{m\varkappa\Theta}\right)^2\left(\frac{W\partial k}{k\partial E_k}\right)_F^3\frac{C_yM_z}{M_s},\\ \frac{eF_y}{\varkappa Tk_F^2} = 3\pi I_0(T)\frac{\hbar^2k_F^2}{m\varkappa\Theta}\left(\frac{W\partial k}{k\partial E_k}\right)_F^2C_y\\ +\frac{45\pi^2A(aq_0)^2I(T)}{32}\left(\frac{\hbar^2k_F^2}{m\varkappa\Theta}\right)^2\left(\frac{W\partial k}{k\partial E_k}\right)_F^3\frac{C_xM_z}{M_s}\end{aligned}$$

Determining C_x and C_y, with the aid of (14) we calculate the current density and then also calculate the components σ_{xx} and σ_{xy} of the electrical conductivity tensor, which determine the AHE coefficient. Since for $\sigma_{xy} \ll \sigma_{xx}$

$$R_s = \sigma_{xy}/4\pi M_z\sigma_{xx}^2, \tag{28}$$

then as a result we obtain

$$R_s = cI(T), \tag{29}$$

where

$$c = \frac{45\pi}{128}\frac{\hbar^5}{e^2n}\frac{Aa^2q_0^2k_\Phi^3W^3}{M_sm^2(\varkappa\Theta)^2}\left(\frac{\partial k}{\partial E_k}\right)_F^3 \tag{30}$$

and n denotes the number of electrons per unit volume.

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[2] As follows from the data on the electrical resistance, the role of interband transitions induced by phonons (the Mott-Wilson mechanism [13-14]) is extremely important, especially for transition elements at the beginning of the period. However, for elements at the end of the period (for example, for platinum or nickel), as a consequence of the filling-up of the d-band the role of such scattering processes becomes less significant. Therefore, one can anticipate that in cobalt, and especially in iron, s–d transitions may give a substantial contribution to the AHE.

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*[kM] $\equiv$ k × M.

Let us consider the cases of high and low temperatures. If T ≫ Θ, then

$$R_s \approx 0.157\, c(T / \Theta)^2. \tag{31}$$

If T ≪ Θ, then one can replace the upper limits on the integral (23) by infinity. Then

$$R_s \approx 0.34 \cdot 10^6 c(T / \Theta)^{10}. \tag{32}$$

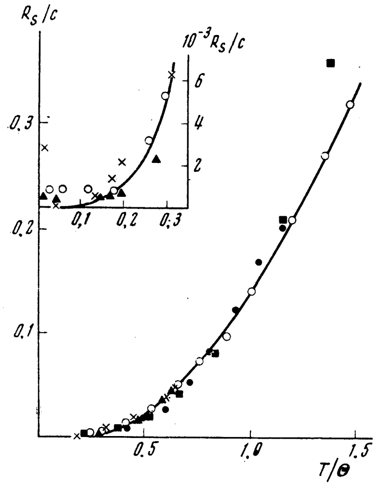
In the region of average temperatures, formula (29) can be regarded as a certain interpolation, analogous to the Grüneisen interpolation formula for the electrical resistance. The temperature dependence I(T), calculated for this case on an ÉVM M-20 computer, is shown in the figure.

DISCUSSION OF THE RESULTS

As the cited calculations indicate, taking the two-phonon terms in the scattering part of the Hamiltonian into account already leads to the AHE in third-order perturbation theory and, as a consequence, gives a value for R_s which is larger by a factor of E_F/W. Since the temperature dependence of R_s for T ≫ Θ turns out to be the same as upon taking account of only the one-phonon terms in the Hamiltonian, then it is impossible to regard the earlier investigations[2-4] of the AHE as consistent, since the one-phonon approximation was used in all of these articles. Apparently this applies to the entire range of temperatures although the AHE resulting from the one-phonon Hamiltonian was not investigated for T ≪ Θ.

As is evident from the figure, formula (29) agrees satisfactorily with the experimental data for Ni,[15-16] Fe,[8,15] and Gd[17] over a broad range of temperatures from 0.2 Θ to the Curie point. If such an agreement is to be regarded as an indication of the "electron-phonon" origin of the AHE, then in accordance with articles [11-12] it is necessary to assume that the AHE has a different nature below and above the Curie point.

As to the low-temperature region (T < 0.2 Θ), then in the first place, as a consequence of the rapid "freezing-out" of the phonons in the first scheme, other mechanisms can be introduced, as is done for the electrical resistance, such as electron-electron collisions, the scattering of electrons by spin waves, etc.



Temperature dependence of R_s/c: the solid line corresponds to the theoretical curve, calculated according to formula (23); ○–Gd, [17] ×–Ni, [15] ●–Ni, [16] ▲–Fe, [15] and ■–Fe. [8]

In the second place the theoretical curve and the experimental data (given in the upper left-hand corner of the Fig.) should still differ because the experimental points also include the permanent AHE coefficient and the various interference terms, which are certainly not small, when the temperature-dependent part of the AHE becomes smaller than the permanent part.

Therefore, in order to make a more complete comparison between theory and experiment at low temperatures, investigations of R_s in more pure samples would be extremely desirable. For a wide range of temperatures, it appears to us that it would be interesting to investigate the AHE in materials in which one could substantially change the Curie point while keeping the Debye temperature essentially unchanged. Alloys of palladium containing a few percent iron[18-19] can serve as an example of such materials. Depending on whether R_s is a universal function of Θ/T up to the Curie point T_c or whether it is determined by the ratio T_c/T, it may be possible to reach a conclusion about the "phonon" or "magnon" nature of the anomalous Hall e.m.f.

We note that since in our case the AHE arises in the third-order approximation with respect to the scattering potential, then the spin-orbit interaction can be described by the introduction of an effective internal magnetic field.[20] Therefore, in the infrared region of the spectrum the dispersion of the magneto-optical Kerr and Faraday effects will coincide with the results calculated on the basis of classical mechanics.[21]

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211