MAGNETIC FIELDS OF ROTATING SUPERCONDUCTORS

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Quantum interference effects in rotating hollow thin-wall superconducting cylinders are investigated. It is shown that by a suitable choice of the cylinder radius, wall thickness, and superconductor parameters one can study the oscillating dependence of the London field on angular velocity ω for reasonable values of the latter when both the external field and angular velocity are effective. Stability of current states of the cylinder is analyzed and it is shown that for small radii a large part of the descending branch of the pair-breaking curve is stable and may be observed.

1. INTRODUCTION

THIS article is devoted to an investigation of quantum coherent effects in superconductors in a rotating coordinate system. The existence of such effects is based on the phenomenon of the "London moment," namely the appearance of a magnetic field in the volume of a superconductor when the latter is rotated[1] 1). The Meissner effect in the presence of rotation calls for the vanishing not of the magnetic induction **B**, but of the quantity $\mathbf{B} + 2mc\boldsymbol{\omega}/e$, where $\boldsymbol{\omega}$ is the angular velocity of the rotation. Consequently, the field in the volume of the superconductor differs from zero and equals $-2mc\boldsymbol{\omega}/e$, amounting to 7×10^{-7} G per revolution per second.

When a hollow thin-wall cylinder is rotated, conditions are produced for the quantization of the magnetic flux produced by the field of the London moment. The situation here, however, is more complicated than in the case of flux quantization in a thin-wall cylinder in an external field[3]. The macroscopic states of a ring with frozen-in field, corresponding to a specified number n of flux (more accurately, fluxoid) quanta, are stable to a high degree against switching into states with other n. Therefore the energy-minimum principle, which is usually used in the interpretation of the Parks-Little effect in hollow thin-wall cylinders[3,4], is not directly applicable. A calculation of quantum coherent effects calls for the determination of the lability boundaries (i.e., the "superheat" field) for the indicated states. It is shown in the present paper that observation of quantum coherent effects connected with rotation is perfectly feasible if certain requirements with respect to organization of the experiment are satisfied.

The London-moment effect was first observed by Hildebrandt[5] and subsequently by a number of workers[6-9]. All the indicated measurements confirm London's formula $H = -2mc\omega/e$, where m is the free-electron mass, and reveal no oscillations or jumps of the London field as a function of ω even when the cylinder wall thickness amounts to 27 Å[9]. In the case of superconducting circuits containing Josephson tunnel junctions, the effect of rotation leads to the appearance of quantum interference effects, perfectly analogous to the action of a magnetic field[10].

London's treatment of the effects in a rotating superconductor is based on an extension of his phenomenological equations to a rotating coordinate system. London's equation for the superfluid velocity

$$\mathbf{H} + \frac{mc}{\rho}\operatorname{rot}\mathbf{v}_s = 0 \tag{1.1}$$

can be written as the condition $\mathbf{p}_s = 2m\mathbf{v}_s + (2e/c)\mathbf{A}$ that the field of the superfluid momentum be potential:2)

$$\operatorname{rot}\mathbf{p}_s = 0. \tag{1.2}$$

In a rotating superconductor, the superfluid velocity in the volume of a bulky body should coincide with the lattice velocity $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$. From this we get $\operatorname{curl}\mathbf{v}_s = \operatorname{curl}\mathbf{u} = 2\boldsymbol{\omega}$, which yields, when substituted in (1.1), a field $\mathbf{H} = -2mc\boldsymbol{\omega}/e$ in the volume of the superconductor. From the point of view of modern concepts, $\mathbf{p}_s$ is the gradient of the phase of the Cooper-pair wave function. The expression for the current in the superconductor takes the form

$$\mathbf{j} = N_s e(\mathbf{v}_s - \mathbf{u}), \quad \mathbf{v}_s = \frac{1}{2m}\left(\hbar\nabla\varphi - \frac{2e}{c}\mathbf{A}\right). \tag{1.3}$$

The current inside the superconductor vanishes, all that remains is a certain surface current, and this produces the field of the London moment.

Determining $\nabla\varphi$ from (1.3) and integrating it over a certain closed circle Γ passing inside the superconducting body, we obtain, from the condition that the wave function ψ be unique, the relation

$$\int_\Gamma \nabla\varphi dl = 2\pi n$$

(n is an integer), whence

$$\Phi_L = \int_S \left(\mathbf{H} + \frac{2mc}{e}\boldsymbol{\omega}\right) d\mathbf{S} + \frac{mc}{N_s e^2}\oint_\Gamma \mathbf{j} dl = n\Phi_0. \tag{1.4}$$

The quantity Φ_L is called a fluxoid. The first integral in (1.4) represents the flux of the field $\mathbf{H} + 2mc\boldsymbol{\omega}/e$ through the surface S subtending the contour Γ. Relation (1.4) expresses the fluxoid quantization condition for a rotat-

1) An analogous effect was predicted earlier for an ideal conductor by Becker, Sauter, and Heller [2], but, as is well known, the properties of a superconductor are not identical with the properties of an ideal conductor.

2) Naturally, we substitute in London's formulas the doubled charge and mass of the electron, which pertain to the Cooper pair.

ing superconductor; $\Phi_0 = hc/2e$ is the magnetic-flux quantum, the numerical value of which is $\Phi_0 = 2 \times 10^{-7}$ G-cm^2.

2. SUPERCONDUCTING CURRENT IN A MOVING SUPERCONDUCTOR

The purpose of the present section is to derive an expression for the current in a moving superconductor on the basis of the microscopic theory of superconductivity[11]. We consider a homogeneous current state, in which the velocity of the superfluid motion and the current density do not depend on the spatial coordinates[12] (see also[13]). When formulating the equations of superconductivity theory, it is natural to change over to a coordinate system moving with the lattice.

The Gor'kov equation for the Green's functions G and F of a superconductor are of the form[11]

$$(i\omega - \hat{H}_0 + \mu)G_\omega(\mathbf{r}, \mathbf{r}') + \Delta(\mathbf{r})F_\omega^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$
$$(-i\omega - \hat{H}_0^* + \mu)F_\omega^+(\mathbf{r}, \mathbf{r}') - \Delta^*(\mathbf{r})G_\omega(\mathbf{r}, \mathbf{r}') = 0, \tag{2.1}$$

where

$$\Delta^*(\mathbf{r}) = |\lambda| T \sum_\omega F_\omega^+(\mathbf{r}, \mathbf{r}), \tag{2.2}$$

$\Delta(\mathbf{r})$ is the superconducting ordering parameter, ω are odd frequencies ($\omega_n = (2n+1)\pi T$), λ is the Cooper interaction constant, and $\hat{H}_0$ is the one-electron Hamiltonian. In a nonmoving reference frame, in the quasiclassical approximation, $\hat{H}_0 = \epsilon(\hat{\mathbf{p}} - (e/c)\mathbf{A}$, where $\epsilon(\mathbf{p})$ is the dispersion law and $\hat{\mathbf{p}} = -i\nabla$. In the moving coordinate system, the quasiparticle energy is

$$E = \varepsilon\left(\hat{\mathbf{p}} - m\mathbf{u} - \frac{e}{c}\mathbf{A}\right) - \frac{mu^2}{2},$$

and we should put accordingly[3)]

$$\hat{H}_0 = \varepsilon\left(\hat{\mathbf{p}} - m\mathbf{u} - \frac{e}{c}\mathbf{A}\right) - \frac{mu^2}{2}. \tag{2.3}$$

In formula (2.3), m represents the mass of the free electron. This expression can be obtained on the basis of the laws for the Galilean transformation of the energy and momentum on going over to the moving coordinate system[14][4)]. We present here, however, a more direct derivation based on the microscopic model of a moving one-dimensional periodic lattice with δ-function potential (the Kronig-Penney model), which admits of an exact solution.

The Schrödinger equation for the time-dependent wave function is (a is the lattice period, $\hbar = 1$)

$$i\frac{\partial\Psi}{\partial t} = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + U_0\sum_{n=-\infty}^{\infty}\delta(x - na - ut)\right]\Psi. \tag{2.4}$$

We seek its solution in the form

$$\Psi = e^{-iEt}e^{ik\xi}u(\xi), \quad \xi = x - ut, \tag{2.5}$$

where $u(\xi)$ is the Bloch function

$$u(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{ik_n\xi}, \quad k_n = \frac{2\pi n}{a}.$$

Substitution of (2.5) in (2.4) leads to an equation for the coefficients a_n:

$$\left[\frac{1}{2m}(k + k_n)^2 - u(k + k_n)\right]a_n + \frac{U_0}{a}\sum_{n=-\infty}^{\infty} a_n = Ea_n. \tag{2.6}$$

For the energy we obtain the relation

$$\frac{U_0}{a}\sum_{n=-\infty}^{\infty}\left[E - \frac{1}{2m}(k + k_n)^2 + u(k + k_n)\right]^{-1} = 1. \tag{2.7}$$

At the same time, the equation determining the dispersion law $\epsilon(k)$ in the immobile lattice ($u = 0$) is

$$\frac{U_0}{a}\sum_{n=-\infty}^{\infty}\left[\varepsilon - \frac{1}{2m}(k + k_n)^2\right]^{-1} = 1. \tag{2.8}$$

Comparing (2.8) with (2.7) we see that the energy in the moving reference frame, as a function of k, is

$$E(k) = \varepsilon(k - mu) - {}^1\!/_2 mu^2, \tag{2.9}$$

thus proving formula (2.3).

Returning to the Gor'kov equations (2.1) and (2.2), we seek their solution in the form

$$\Delta = \Delta_0 e^{i\varphi}, \quad \varphi = 2\mathbf{kr},$$
$$G_\omega(\mathbf{r}, \mathbf{r}') = \mathcal{G}_\omega(\mathbf{r} - \mathbf{r}')\exp[i\mathbf{k}(\mathbf{r} - \mathbf{r}')],$$
$$F_\omega^+(\mathbf{r}, \mathbf{r}') = \mathcal{F}_\omega^+(\mathbf{r} - \mathbf{r}')\exp[-i\mathbf{k}(\mathbf{r} + \mathbf{r}')]. \tag{2.10}$$

Neglecting the small term $mu^2/2$, we obtain for $\mathcal{G}_\omega$ and $\mathcal{F}_\omega$ the equations

$$[i\omega - \varepsilon(-i\nabla + {}^1\!/_2\mathbf{p}_s) + \mu]\mathcal{G}_\omega(\mathbf{r} - \mathbf{r}') + \Delta_0(p_s)\mathcal{F}_\omega^+(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$
$$[-i\omega - \varepsilon(-i\nabla - {}^1\!/_2\mathbf{p}_s) + \mu]\mathcal{F}_\omega^+(\mathbf{r} - \mathbf{r}') - \Delta_0(p_s)\mathcal{G}_\omega(\mathbf{r} - \mathbf{r}') = 0. \tag{2.11}$$

Here $\mathbf{p}_s$ is the superfluid momentum vector, defined by

$$\mathbf{p}_s = 2\left(\mathbf{k} - \frac{e}{c}\mathbf{A} - m\mathbf{u}\right) = \nabla\varphi - \frac{2e}{c}\mathbf{A} - 2m\mathbf{u}. \tag{2.12}$$

Subtracting $\mathcal{F}_\omega$ and $\mathcal{G}_\omega$ from (2.10) and substituting in the self-consistency equation (2.2), we arrive at a relation defining the function $\Delta_0(p_s)$:

$$1 = |\lambda| T \sum_\omega \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{(\omega + i\xi_+)(\omega - i\xi_-) + \Delta_0^2(p_s)}. \tag{2.13}$$

Here $\xi_\pm = \epsilon(\mathbf{p} \pm {}^1\!/_2\mathbf{p}_s) - \mu = \xi \pm {}^1\!/_2\mathbf{v}_0(\mathbf{p})\mathbf{p}_s$, and ξ is the energy in the normal metal, reckoned from the Fermi level μ.

Calculating the current in the system K′ moving together with the lattice, we obtain

$$\mathbf{j} = e\,\mathrm{Re}\sum_\alpha\left\langle \psi_\alpha^+\mathbf{v}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A} - m\mathbf{u}\right)\psi_\alpha\right\rangle$$
$$= 2e\,\mathrm{Re}\,T\sum_\omega \mathbf{v}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A} - m\mathbf{u}\right)G_\omega(\mathbf{r}, \mathbf{r}')\big|_{\mathbf{r}'=\mathbf{r}}. \tag{2.14}$$

Here ψ_α^+ is the electron wave-function operator in the second-quantization representation, and $\mathbf{v}(\hat{\mathbf{p}}) = \partial\epsilon/\partial\hat{\mathbf{p}}$ is the quasiparticle velocity operator. In particular, in the case of quadratic dispersion with an effective mass m^*, the current will be

$$\mathbf{j} = \frac{e}{m^*}\,\mathrm{Im}\sum_\alpha\left\langle \psi_\alpha^+\left(\nabla - \frac{ie}{c}\mathbf{A} - im\mathbf{u}\right)\psi_\alpha\right\rangle$$

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3)The last term is immaterial. It can be included in the renormalization of the chemical potential, and will henceforth be omitted.

4)After this article was written, we learned of a paper [15] in which an analogous derivation was obtained using the Larmor theorem. The free electron mass $m = m_0$ enters as a result in the electron-inertial experiments.

$$= \frac{2e}{m^*} \operatorname{Im} T \sum_\omega \left(\nabla - \frac{ie}{c} \mathbf{A} - imu \right) G_\omega(\mathbf{r}, \mathbf{r}') |_{r'=r}. \quad (2.15)$$

In (2.14) and (2.15), m is again the free-electron mass. The remainder of the calculation duplicates exactly the derivation of the expression for the current in the homogeneous state[12,13]. As $p_S \to 0$ we obtain

$$\mathbf{j} = \frac{N_s e}{2m^*} \left(\nabla \varphi - \frac{2e}{c} \mathbf{A} - 2mu \right), \quad (2.16)$$

where $N_S(T)$ is the concentration of the "superconducting electrons"[11]

$$\frac{N_s}{N} = \pi T \sum_\omega \frac{\Delta_0^2}{(\omega^2 + \Delta_0^2)^{3/2}}. \quad (2.17)$$

In conclusion, we make the following remark. In the derivation of (2.16) we have assumed that $u \neq$ const. Actually, we shall be interested below in a situation wherein the velocity varies in space. When the gradient terms $\partial u_i / \partial x_k$ are taken into account, the current contains corrections proportional to these gradients:

$$J_i' = N_s e \lambda_{ikl}^{(1)} \partial u_k / \partial x_l + \ldots, \quad (2.18)$$

where in order of magnitude we have $\lambda^{(1)} \sim \xi_0$ (ξ_0 is the coherence length), etc. Consequently, the correction terms are smaller than those taken into account in (2.16) by the factor ξ_0/a, where a is the characteristic distance over which the velocity changes. The role of a will be played below by the rotating-cylinder radius $R \gg \xi_0$.

Thus, all the formulas of superconductivity theory retain their form, with the required degree of accuracy, provided the vector potential A is replaced by $\tilde{\mathbf{A}} = \mathbf{A} + (mc/e)\mathbf{u}$, where e and m are the charge and mass of the free electron.

3. QUANTUM STATES OF ROTATING THIN-WALL CYLINDER

An analysis of the behavior of a hollow cylinder in a magnetic field,[16–18] and elsewhere, shows that the distribution of the induction and the magnitude of the field in the hollow are determined by the dimensionless ratio

$$\alpha = rd / 2\delta^2, \quad (3.1)$$

where d is the cylinder-wall thickness, r the cylinder radius, and δ the depth of penetration. In the case of a rotating cylinder (Fig. 1), the induction distribution is determined by the total field

$$H = H_0 + 2mc\omega / e, \quad (3.2)$$

which consists of the applied field H_0 and the field of the London moment $2mc\omega/e$. Substituting the expression for the superconducting current (2.16) in Maxwell's equation curl $\mathbf{H} = 4\pi\mathbf{j}/c$, we obtain the field in the hollow

$$H_1 = -\frac{2mc}{e}\omega + \left[\left(H_0 + \frac{2mc}{e}\omega \right) (I_0(\rho_1) K_1(\rho_1) + K_0(\rho_1) I_1(\rho_1)) - \frac{n\Phi_0}{2\pi r \delta} (I_0(\rho_1) K_0(\rho_2) - K_0(\rho_1) I_0(\rho_2)) \right] [I_0(\rho_2)(K_1(\rho_1) + {}^1\!/_2 \rho_1 K_0(\rho_1)) + K_0(\rho_2)(I_1(\rho_1) - {}^1\!/_2 \rho_1 I_0(\rho_1))]^{-1}; \quad (3.3)$$

here $\rho_1 = r/\delta$, $\rho_2 = R/\delta$, r and R are respectively the inside and outside radii of the cylinder, K and I are Bessel functions of imaginary argument, and $n\Phi_0$ is the fluxoid. At a fixed value of n, the derivative $dH_1/d\omega$ is given by the relation

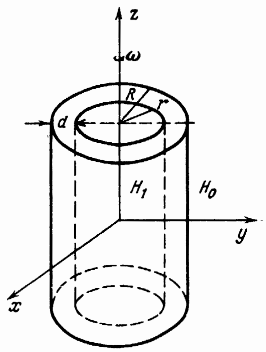

FIG. 1

$$\frac{dH_1}{d\omega} = -\frac{2mc}{e} \left\{ 1 - \frac{I_0(\rho_1) K_1(\rho_1) + K_0(\rho_1) I_1(\rho_1)}{[K_1(\rho_1) + {}^1\!/_2 \rho_1 K_0(\rho_1)] I_0(\rho_2) + [I_1(\rho_1) - {}^1\!/_2 \rho_1 I_0(\rho_1)] K_0(\rho_2)} \right\} \quad (3.4)$$

If the cylinder wall thickness $d = R - r$ is small compared with the penetration depth δ, formula (3.4) assumes the simple form

$$\frac{dH_1}{d\omega} = -\frac{2mc}{e} \frac{\alpha}{1 + \alpha} \quad (3.5)$$

where α is given by (3.1). An analogous expression was obtained earlier by Griffin[19].

We see from (3.5) that at small thicknesses ($\alpha \to 0$) we have $dH_1/d\omega \to 0$, as should be the case. When $\alpha \gg 1$, to the contrary, $dH_1/d\omega$ is equal to the London value $-2mc/e$. The transition from the case of "large" thicknesses to the case of small ones is realized, in accordance with (3.5), not at $d \sim \delta$, but at much smaller values of d:

$$d \lesssim d_c = 2\delta^2 / r \ll \delta. \quad (3.6)$$

The critical thickness d_c is usually very small. Under normal conditions at $r \sim 1$ cm and $\delta \sim 10^{-5}$ cm, it amounts to $\sim 10^{-10}$ cm. Nonetheless, an appreciable increase of the parameter d_c is possible if one uses extremely "dirty" systems, in which the penetration depth is $\delta \sim \delta_0(T) (\xi_0/l)^{1/2} \gg \delta_0$ (l is the mean free path, δ_0 the penetration depth of a pure superconductor, and ξ_0 the coherence length in the absence of scattering at T = 0). In granular or amorphous films with $l \sim 1$ Å (see[20,21]) we obtain $\delta \sim 10^{-3}$ cm, which gives a perfectly reasonable value $d_c \sim 2 \times 10^2$ Å. Thus, the situation of the "thin" cylinder is, in principle, experimentally attainable. At the same time, the most interesting phenomena come into play precisely in the thickness region $d \lesssim d_c$, where effects of "switching" of the values of the quantum number n (the fluxoid in units of Φ_0) come into play when the angular velocity ω is varied.

In order to consider effects of this kind, we use the Ginzburg-Landau equations[22], which, allowing for the expression obtained in Sec. 2 for the current in a rotating superconductor, take the form[5)]

[5)] As is well known, these equations are valid near T_c [11]. For "dirty" systems, the region of their applicability is broader [23]. In one way or another, the Ginzburg-Landau theory describes the character of all the effects qualitatively correctly also at temperatures far from T_c.

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla-\frac{2e}{c}\mathbf{A}-2m\mathbf{u}\right)^2\psi-\alpha\psi+\beta|\psi|^2\psi=0,$$

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j},\quad \mathbf{H}=\operatorname{rot}\mathbf{A},\quad \operatorname{rot}\mathbf{u}=2\boldsymbol{\omega}, \tag{3.7}$$

$$\mathbf{j}=\frac{e}{2m^*}\operatorname{Im}\left[\psi^*\left(\hbar\nabla-\frac{2ie}{c}\mathbf{A}-2im\mathbf{u}\right)\psi\right].$$

The wave function ψ is normalized in such a way that $|\psi|^2$ is the concentration N_S of the superconducting electrons (the depth of penetration of a weak field is $\delta = (m^*c^2/4\pi N_S e^2)^{1/2}$). Introducing the customary dimensionless variables of the Ginzburg-Landau theory[22], we obtain ($\kappa = \delta/\xi$)

$$\left(\frac{\nabla}{i\kappa}-\mathbf{A}-[\omega\rho]\right)^2\psi-\psi(1-|\psi|^2)=0, \tag{3.8}$$

and

$$\operatorname{rot}\operatorname{rot}\mathbf{A}=\operatorname{Im}\left[\psi^*\left(\frac{\nabla}{\kappa}-i(A+[\omega\rho])\right)\psi\right]; \tag{3.9}$$

ω is measured in units of $(e/mc)H_c\sqrt{2}$, where H_c is the thermodynamic critical field (ξ is the coherence length):

$$H_c=\Phi_0/2\pi\sqrt{2}\delta\xi. \tag{3.10}$$

The role of the boundary conditions for (3.9) are played by the requirements that the normal component of the current vanish on the surface of the superconductor:

$$\mathbf{n}\left(\frac{\nabla}{\kappa}-i(\mathbf{A}+[\omega\rho])\right)\psi|_s=0 \tag{3.11}$$

and that the tangential component of the field H be equal to the field in vacuum H_S:

$$[\mathbf{n}\operatorname{rot}\mathbf{A}]|_s=[\mathbf{n}\mathbf{H}_s]. \tag{3.12}$$

We introduce further, as usual, the modulus and phase ψ and the superfluid velocity vector **Q**:

$$\mathbf{Q}=-\frac{1}{\kappa}\nabla\varphi+\mathbf{A}+[\omega\rho],\quad \psi=Fe^{i\varphi}. \tag{3.13}$$

In a cylindrical coordinate system (ρ, θ, z), for the geometry of interest (Fig. 1), only the θ component of the vector **Q** differs from zero, the phase φ is equal to $n\theta$, where n is an integer, and the quantities F and Q depend on ρ[6)] and satisfy the equations

$$\frac{1}{\kappa^2}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dF}{d\rho}\right)+F(1-F^2-Q^2)=0, \tag{3.14}$$

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho Q)\right]=F^2Q,$$

with the boundary conditions

$$\frac{dF}{d\rho}=0|_{\rho=r},\quad \frac{dF}{d\rho}=0|_{\rho=R},$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho Q)=H_0+2\omega|_{\rho=R},\quad \frac{1}{\rho}\frac{d}{d\rho}(\rho Q)=H_1+2\omega|_{\rho=r}. \tag{3.15}$$

Equations (3.14) coincide with the corresponding equations describing vortices in a superconductor[24]. The only difference from the latter case lies in the form of the boundary conditions; to Eqs. (3.15) (H_0 is the field applied from the outside and H_1 is the field in the hollow), it is necessary to add a relation expressing the continuity of the fields at ρ = r. The latter can be written in the form (compare with[18])

$$2\pi\rho\left(Q+\frac{n}{\kappa\rho}-\omega\rho\right)_{\rho=r}=\pi r^2H_1, \tag{3.16}$$

which can be regarded as the continuity condition for the flux $\Phi = \Phi(\rho)$ at ρ = r, expressed in dimensionless variables.

We shall obtain the solution of (3.14)–(3.16) in the following manner. Assuming $r \gg 1$, we can replace the radial parts of the Laplace operators in (3.14) by $d^2/d\rho^2$, and $\rho^{-1}d(\rho Q)/d\rho$ reduces to $dQ/d\rho$. We shall assume furthermore that the cylinder wall d is thin in the sense of the criterion (3.6), which, in dimensionless variables, takes the form $rd \ll 1$, and expand formally all the quantities in powers of d (actually, rd). Obviously, at d = 0 the field H_1 should coincide with H_0. The indicated expansion therefore takes the form

$$H_1=H_0+dh_1+d^2h_2+\ldots$$

Making the substitution $\rho = r + xd$, $0 < x < 1$, we represent the functions $F(\rho)$ and $Q(\rho)$ in the form

$$F=\sum_{n=0}^{\infty}d^nF_n(x),\quad Q=\sum_{n=0}^{\infty}d^nQ_n(x), \tag{3.17}$$

where F_n and Q_n no longer depend on d. Substitution of (3.17) in the initial equations (3.14) and (3.15) leads to a system of equations and boundary conditions for the functions $F_n(x)$ and $Q_n(x)$. The first three approximations for F_n are of the form

$$F_0''(x)=0,\quad F_1''(x)=0,\quad F_n'=0|_{x=0,1};$$
$$\kappa^{-2}F_2''+F_0(1-F_0^2-Q_0^2)=0, \tag{3.18}$$

from which it follows that F_0, F_1, and F_2 are certain constants that do not depend on x, and from the last equation follows the relation $F_0^2 + Q_0^2 = 1$. The equations for $Q_n(x)$ are written and solved analogously.

Omitting the trivial derivations, we present the final result. We have, accurate to terms of zeroth order in d for the quantities Q and F and first order for the field H_1,

$$Q={}^1\!/_2r(H_0+2\omega-2n/\kappa r^2),\quad F=\sqrt{1-Q^2}, \tag{3.19}$$

$$H_1=H_0-dQ(1-Q^2),\quad |Q|<1. \tag{3.20}$$

The field difference $H_0 - H_1$ is proportional to the total current in ring j. As seen from (3.19) and (3.20), the current distribution is determined by the summary field $H_0 \to H_0 + 2\omega$ or (in dimensional variables) by the field $H_0 + 2mc\omega/e$. Thus, the problem of destruction of superconductivity by rotation is equivalent to the problem of

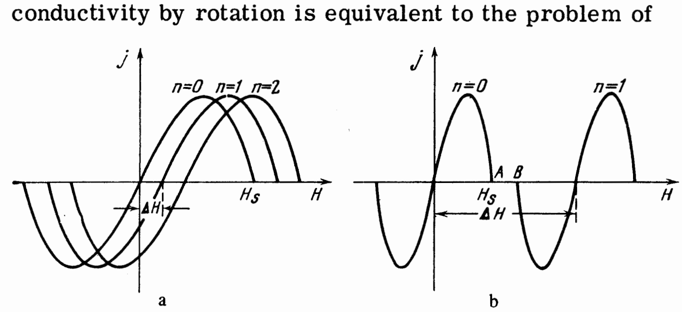

FIG. 2. Dependence of current circulating over the surface on the magnetic field: a—case $\kappa r > 1/2$, b—case $\kappa r < 1/2$.

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6)In principle, inhomogeneous solutions (that depend on the angle θ) are possible, but analysis shows (see footnote 11 below) that they are unstable.

its destruction by a magnetic field of magnitude $2mc\omega/e$, in accord with the Larmor theorem[15].

Figure 2 shows the dependence of a current j circulating over the surface of a cylinder on H at different values of the parameter κr. When $\kappa r > 1/2$, the state of the cylinder in a specified field H is ambiguous, and can be realized by a number of values of n; when $\kappa r < 1/2$ there is no such ambiguity and in certain intervals of variation of H (for example between the points A and B in Fig. 2b), the superconductivity is destroyed[7)]. At $n = 0$, the critical value of the field H, at which the ordering parameter vanishes, is determined from the condition $Q = 1$, and equals $H = H_S = 2/r$ or, after changing over to dimensional units,

$$H_s = \Phi_0 / \pi r \xi. \qquad (3.21)$$

A comparison of this expression with the thermodynamical critical field (3.10) shows that $H_S/H_C \sim \delta/r \ll 1$. Thus, for the example considered above we have $H_S/H_C \sim 10^{-3}$. The "critical" angular velocity of rotation ω_S is obtained by multiplying H_S by the coefficient $e/2mc$. At $H_C \sim 10^2$ G, it amounts to $\sim 10^5$ rps, i.e., it is quite large (experimentally unattainable).

A transition between neighboring curves in Fig. 2a corresponds to a field change amounting to $\Delta H = 2/\kappa r^2$ or, in dimensional variables,

$$\Delta H = \Phi_0 / \pi r^2 \qquad (3.22)$$

and corresponds to a change of the flux in the ring by one quantum. At $r = 1$ cm, the change of the angular velocity corresponding to the increase of the field by an amount ΔH is $\sim 10^{-1}$ rps.

The changeover from the situation shown in Fig. 2a to the case shown in Fig. 2b is realized at $\Delta H = 2H_S$, which yields $\kappa r = 1/2$. It is easy to show that the character of the destruction of the superconductivity by the field (or by rotation) shown in Fig. 2b agrees with the diagram of state of the superconductor in the field. Indeed, in the case of a large-radius ring, according to the linearized Ginzburg-Landau equation (3.7), the quantity T_C is obtained as a solution of the equation

$$\left[\frac{n}{r} - \frac{2e}{\hbar c}\left(H_0 + \frac{2mc}{e}\omega\right) r\right]^2 = \xi^{-2}(T). \qquad (3.23)$$

Since $\xi^{-2}(T)$ is proportional to $T_{C0} - T$, this leads to the $T_C(H)$ curve shown in Fig. 3[8)]. From this we obtain for the smallest critical temperature $T_C = T_{C1}$ (Fig. 3), by minimizing the left-hand side with respect to n, the condition

$$\xi(T_{c1}) = 2r, \qquad (3.24)$$

corresponding to oscillations of T_C with an amplitude[4]

$$\Delta T_c / T_{c0} \sim \xi^2(0) / r^2. \qquad (3.25)$$

It is easily seen that the condition $\xi(T) = 2r$, when expressed in dimensionless variables, corresponds ex-

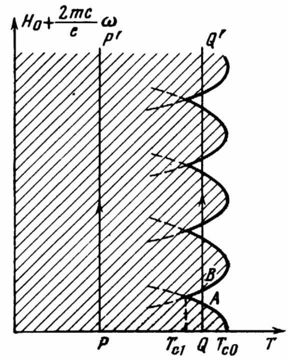

FIG. 3. Diagram of states of cylinder in field. The superconducting region is shaded.

actly to the relation $\kappa r = 1/2$. Depending on how close T is to T_{C0}, we obtain a dependence of j on H of the type shown either in Fig. 2a or in Fig. 2b. Curves of the type of Fig. 2a correspond to motion along the line PP′ on Fig. 3 ($T < T_{C1}$), and the curves of Fig. 2b ($T_{C1} < T < T_{C0}$) to motion along the line QQ′ in the same figure. The points A and B in Fig. 2b correspond exactly to the points A and B in Fig. 3. Between these points, the cylinder is in the normal state, and the circulating current is $j = 0$.

4. STABILITY OF MACROSCOPIC STATES OF A RING AND QUANTUM COHERENT EFFECTS

The analysis presented in the preceding section cannot, of course, answer in the general case the question of how the field inside the cylinder actually depends on the external field H_0 or on the angular velocity ω. If at the initial instant the quantum number is $n = 0$, then in principle, when H_0 (or ω) is increased, we can move up to the point H_S in Fig. 2a, although it is already clear intuitively that the "switching over" into a state with another value of n will occur earlier, apparently near the vertex of the j(H) curve. In the case shown in Fig. 2b, however, realization of the entire "pair breaking curve" is possible, up to its termination at the point A.

To determine the limit of thermodynamic stability it is necessary to analyze the second variation of the free-energy functional G, which has a minimum at the equilibrium state for the specified external conditions[25]. In the case of type-II superconductors, such a problem was considered in[26,27]. In our case the functional G is of the form

$$G = \int \Big[F^2 Q^2 + \frac{1}{\varkappa^2}(\nabla F)^2 + \frac{1}{2}(1-F^2)^2 + (\mathbf{H}_0 + 2\boldsymbol{\omega} - \operatorname{rot}\mathbf{Q})^2 \Big] d\rho + (H_0 - H_i)^2 \pi r^2 L_z, \qquad (4.1)$$

where L_Z is the cylinder length in the direction of the z axis. Considering small perturbations, we assume that the quantities F and Q acquire small increments[9)]

$$F_1 = f(\rho,\theta), \quad Q_{1\theta} = q(\rho,\theta), \quad Q_{1\rho} = s(\rho,\theta), \quad Q_{1z} = 0. \qquad (4.2)$$

In addition, we should take into account the change of the z-th component in the magnetic field inside the cylin-

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7) We note that since we have put $r \gg 1$, the relation $\kappa r < 1/2$ can be realized only at sufficiently small κ.

8) Actually, at large values of the field H, the envelope of the oscillations in Fig. 3 should shift to the left, corresponding to a monotonic decrease of T_C with increasing field [3]. The corresponding characteristic values of the fields $H_{C\parallel} \sim \Phi_0/d\xi$ in the case of films are very large, so that in our case we can assume this envelope to be practically vertical.

9) We consider perturbations that the homogeneous along the z axis. Perturbations inhomogeneous with respect to z correspond to an additional positive contribution to G, which in turn corresponds to greater stability (compare with [27]).

der, which we denote by h. By virtue of the flux quantization condition (3.16), we have

$$h=\frac{1}{\pi r}\left[\int_0^{2\pi} q(\rho,\theta)\,d\theta\right]_{\rho=r}. \qquad (4.3)$$

We expand G in terms of the small perturbations f, q, and s, accurate to second-order terms

$$G=G_0+G_1+G_2; \qquad (4.4)$$

G_0 does not contain small terms, and G_1 and G_2 are proportional to the first and second degrees of the perturbation, respectively. We are interested in an expression for G_2. Referring this quantity to unit length in the z direction, we obtain

$$G_2=\int_0^{2\pi} d\theta\int_r^R \rho\,d\rho\Big\{F^2(q^2+s^2)-(1-3F^2-Q^2)f^2+4FQfq+\frac{1}{\varkappa^2}\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\varkappa^2\rho^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho q)-\frac{1}{\rho}\frac{\partial s}{\partial\theta}\right]^2\Big\}+\pi r^2h^2. \qquad (4.5)$$

The functional G_2 ceases to be positive definite at the point at which $\delta G_2 = 0$. By virtue of (4.3), this leads to the following system of equations and boundary conditions for the functions f, q, and s:

$$\frac{1}{\varkappa^2}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right]+(1-3F^2-Q^2)f-2FQq=0,$$
$$-\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho q)-\frac{1}{\rho}\frac{\partial s}{\partial\theta}\right]+F^2q+2FQf=0,$$
$$\frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho q)-\frac{1}{\rho}\frac{\partial s}{\partial\theta}\right]+F^2s=0,$$
$$\frac{\partial f}{\partial\rho}=0\Big|_{\rho=r,R},\qquad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho q)-\frac{1}{\rho}\frac{\partial s}{\partial\theta}=0\Big|_{\rho=R},$$
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho q)-\frac{1}{\rho}\frac{\partial s}{\partial\theta}=\frac{1}{\pi r}\int_0^{2\pi} q\,d\theta\Big|_{\rho=r}. \qquad (4.6)$$

These equations are solved by the method described in Sec. 3, namely we put $r \gg 1$ and $d \to 0$, and expand all the quantities f, q, and s in powers of the cylinder wall thickness d (see (3.17)). The result is a system of equations for the functions $f_n(x,\theta)$, $q_n(x,\theta)$, and $s_n(x,\theta)$, where $x = (\rho - r)/(R - r)$ and $0 < x < 1$. An analysis of these equations shows that f_0, f_1, q_0, q_1, s_0, s_1 and f_2 do not depend on x. For f_0 we obtain the relation

$$-\frac{1}{\varkappa^2r^2}\frac{d^2f_0}{d\theta^2}-(1-3F^2-Q^2)f_0+2FQq_0=0; \qquad (4.7)$$

F and Q are determined by formulas (3.19). Inasmuch as the quantities f_0, q_0, etc. are periodic functions of the angle θ, they can be expanded in Fourier series:

$$f_0=\sum_{m=-\infty}^{\infty} f_{0m}e^{im\theta} \qquad (4.8)$$

etc. The boundary conditions for Eqs. (4.6) lead to the relations

$$\int_0^{2\pi} q_0\,d\theta=0,\qquad \frac{1}{\pi r}\int_0^{2\pi} q_1\,d\theta+F^2q_0+2FQf_0=0. \qquad (4.9)$$

At m = 0 Eq. (4.7) yields, by virtue of the first relation in (4.9)

$$1-3F^2-Q^2=0. \qquad (4.10)$$

This quantity is equal to $-2(1-Q^2)$. It vanishes only at the point Q = 1, at which the current $j = Q(1 - Q^2)$ becomes equal to zero. Consequently, homogeneous perturbations do not lead to loss of stability at any value of Q.

At nonzero m, by virtue of the second relation in (4.9) we obtain

$$q_{0m}=-\frac{2Q}{F}f_{0m}, \qquad (4.11)$$

which leads after substitution in (4.7) to an equation determining the point where stability is lost.

$$2(3Q^2-1)-m^2/\varkappa^2r^2=0,\quad m\neq 0. \qquad (4.12)$$

It is clear therefore that the first occurrence of instability (the minimum value of $Q = Q_c$) will be determined by the smallest value of $|m|$. Inasmuch as $m \neq 0$, we should put m = 1, which leads to the expression

$$Q_c=\frac{1}{\sqrt{3}}\left(1+\frac{1}{2\varkappa^2r^2}\right)^{1/2}. \qquad (4.13)$$

The foregoing analysis could be realized also in the following manner. Instead of determining the second variation of the functional G_2, one can use the time-dependent Ginzburg-Landau equation in the form[28-30]

$$\frac{\partial\psi}{\partial t}+\left(\frac{\nabla}{i\varkappa}-\mathbf{A}-[\omega\rho]\right)^2\psi-\psi(1-|\psi|^2)=0 \qquad (4.14)$$

and define the point at which stability is lost as the point of appearance of perturbations that increase with time, $(f, q, s) \sim e^{\gamma t}$ with $\gamma > 0$. Such an approach is perfectly equivalent to the one given above. We note that in this case we are not interested in a formal derivation of (4.14), which can be rigorously justified only in certain particular cases[30]. The diffusion equation (4.14) can be represented in the form

$$\dot\psi=-\delta G/\delta\psi^*, \qquad (4.15)$$

which gives the relation $\delta G_2 = 0$ at the point where the instability appears first ($\gamma = 0$, i.e., $\psi = 0$), by virtue of the fact that ψ satisfies the stationary Ginzburg-Landau equation ($\delta G_1 = 0$).

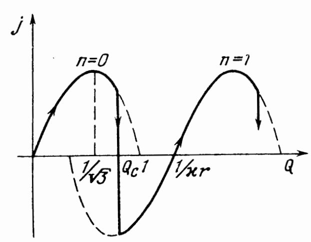

FIG. 4. Dependence of the current on the superfluid velocity. $\varkappa r = 0.75$.

According to (4.13), the stable states will be those located somewhat to the right of the maximum of the pair-breaking curve j(Q) (Fig. 4), i.e., it becomes possible to observe at least a part of the decreasing branch of this curve[10)]. The physical reason for this fact is the "blocking" of the most unstable homogeneous oscillations with m = 0, corresponding to strong perturbation of the field in the entire internal region of the cylinder, which leads to an increase of the free energy G. It is important that the quantity Q_c changes in the range from

10)This remark has no bearing on the data of [31], where an attempt was made to investigate states located to the right of the maximum point of the j(Q) curve. The situation considered in [31] corresponds to the case $\varkappa r \gg 1$, and in this case the stability limit $Q_c \approx 1/\sqrt{3}$ corresponds to the point of maximum current.

$1/\sqrt{3}$ to 1 when κr changes from ∞ to 1/2. At $\kappa r = 1/2$, we have $Q_c = 1$, corresponding to stability of the entire j(Q) curve up to its termination at the point Q = 1. Naturally, this conclusion remains in force also when $\kappa r < 1/2$. Thus, in the case shown in Fig. 2b, it is possible to observe the entire pair-breaking curve, which is stable up to the point of transition to the normal state.

Taking all the foregoing into account, let us analyze the variation of the current j in the ring as a function of the angular velocity ω (Fig. 5). The quantity $H_0 - H_1$ is proportional to j. We consider for simplicity the case $\kappa r \gg 1$.

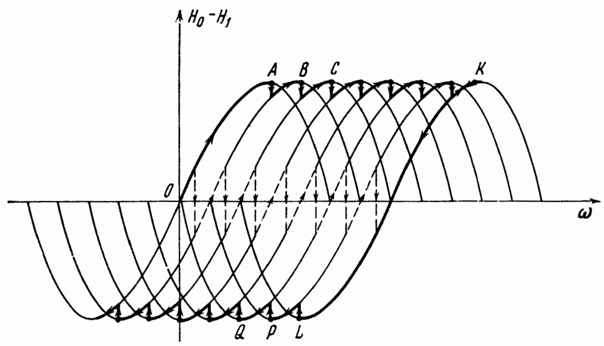

FIG. 5. Dependence of the field in the hollow of the cylinder (H_1) on the angular velocity ω.

Assume that the quantum number in the initial state is n = 0. Then, with increasing ω, we move to the vertex A of the curve corresponding to n = 0. At this point, stability is lost and a change takes place to the state with a value of n differing by unity from the initial state. This process repeats at the points B, C, ..., etc. If at a certain point K we begin to decrease ω, then we shall move along the line KL, i.e., appreciable hysteresis takes place. The analogous process of switching at the points P, Q, ..., etc. then repeats.[11)] We note that complete thermodynamic equilibrium (a minimum value of G for each specified value of ω) would correspond to motion along the dashed line of Fig. 5. Such a state, however, cannot be realized in samples with macroscopic dimensions when ω changes at any nonzero rate, since the corresponding relaxation times are very large.

Both in the case of thermodynamic equilibrium and when account is taken of effects of "superheating" (OAB...) and "supercooling" (KLPQ...), the dependence of the field inside the cylinder (H_1) on the angular velocity oscillates with a period $\Delta\omega$ determined by the flux quantization condition $\Delta\omega = e\Delta H/2mc$. We note that situations corresponding to superheating and supercooling are not fully equivalent, it being connected with boundary effects not taken into account in our calculation. It is known that in experiment it is usually much easier to observe supercooling than superheating. It may therefore turn out that it is more convenient to study these effects experimentally in a decreasing rather than increasing field $H = H_0 + 2mc\omega/e$.

As already noted, to reach the stability-loss point A it is necessary to have very large angular velocities, $\omega \sim \omega_S/\sqrt{3} \sim 10^5$ rps. Nonetheless, in view of the already noted additivity of the quantities H_0 and ω, it is easy to shift the origin by applying a constant magnetic field $H_0 \sim 0.1$ G. If we choose a value of H_0 close to $H_S/\sqrt{3}$, then the loss of stability with increasing angular velocity will occur already at rather small values of ω, which in principle can be made of the order of $\Delta\omega \sim 0.1-1$ rps. It is similarly possible to initiate the oscillations in a decreasing field. The order of the amplitude of the oscillations of H_1 is given by the relation

$$\frac{\Delta H_1}{\Delta\omega} \sim \frac{2mc}{|e|}\frac{rd}{\delta^2}\sqrt{3}, \quad d \leqslant d_c. \tag{4.16}$$

At values of ω corresponding to the points P, Q, etc., the derivative $dH_1/d\omega$ becomes very large (formally, infinite), i.e., a significant sensitivity to small changes of ω appears. By setting the working point in the region of the jump with the aid of an externa l field H_0, it is possible to measure the angular velocity with a very high degree of accuracy. Of course, this requires very accurate stabilization of the external field H_0.

Observation of all the foregoing effects is of considerable interest from the point of view of studying the so-called macroscopic coherent phenomena in superconductors[1,13].

In conclusion, we consider it our pleasant duty to express deep gratitude to I. M. Dmitrenko for constant interest in the work and for numerous discussions. We are also grateful to A. A. Abrikosov for a discussion of this article and significant remarks.

[11)] An investigation of the kinetics of such a switching is an independent problem. As shown by analysis, the transition occurs with involvement of inhomogeneous states of the ring, corresponding to values of F that depend on the angle θ [32]. All such states are unstable, and their development decreases the values of the modulus of the ordering parameter $|\psi|$, and it is vanishing of this parameter at a certain point that makes possible the switching into a state with another n. A unity change in n is the most natural assumption corresponding to a transition of the system into the state closest to the given one, which corresponds in turn to a local minimum of the potential curve.

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