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Particle Production and Vacuum Polarization in an Anisotropic Gravitational Field

YA. B. ZEL'DOVICH AND A. A. STAROBINSKIĬ

Institute of Applied Mathematics, USSR Academy of Sciences

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Particle production and vacuum polarization of a scalar field with arbitrary mass are considered in a strong external gravitational field with a homogeneous spatially-flat nonstationary metric. The finite renormalized average values of the energy-momentum tensor are found in the general anisotropic case. It is established that vacuum polarization and production of particles of a zero mass field are absent only in the isotropic case. The behavior near the singularity is investigated; in the case of isotropic collapse the exact asymptotic forms are determined. It is shown that the average values found for anisotropic collapse increase like t^{-4}, and the contraction law may be substantially altered when the reciprocal effect of the scalar field on the metric is taken into account. The polarization operator is calculated to first order in h for the case of a weak gravitational field.

STATEMENTS made prior to 1960[1-3] about the possibility of particle production in empty space are of interest as an indication of an important area of investigation, but they do not contain precise physical-mathematical treatments of the problem.

The structure of the field corrections to the theory of gravitation[4-7] was investigated in a number of articles during the following ten-year period, right up to attempts to derive a complete theory of gravitation from a consideration of vacuum polarization.[8] The question of the cosmological constant has also been studied from this point of view.[9-10]

In recent years particle production in strong gravitational fields, in particular near the cosmological singularity, has been considered in a number of articles.[11-16] Interest in this problem has been "warmed-up" by the theory of a "hot" Universe containing, as is well known, a large preponderance of neutral particles over charged particles. It is also important to consider particle production during the process of relativistic collapse.

As is well known, the most general solution of the problem of collapse turns out to be locally anisotropic[17] near the singularity. Cosmological solutions are also known in which the expansion is anisotropic at first, near the singularity, and only later does the transition to isotropic expansion, which is observed at the present time, occur (the simplest of these solutions is the Heckmann-Schüking solution). Interest in such models has recently increased.[18-20]

The object of this article is primarily the investigation of the production of scalar particles in strong anisotropic gravitational fields, specifically near a singularity.

In the presence of a strong gravitational field, it is natural to treat it in the classical (not the quantum) approximation. On the contrary, the particles which are produced (photons, e^+e^- pairs, $\nu\bar{\nu}$ pairs, $N\bar{N}$ pairs, etc.) necessarily must be described within the spirit of the theory of quantized fields. The same also pertains to gravitons; this means that it is advantageous to represent the metric as a superposition of slowly and rapidly varying quantities (compare with the investigation of gravitational waves in the article by Isaacson[21]) and it is necessary to quantize only these quantities.

Vacuum polarization and particle production must be investigated concurrently; and what is more, so long as the metric does not become four-dimensionally flat, it is impossible to separate in the formulas the particles which are actually produced from the virtual particles responsible for the polarization. Paradoxes arise[22] if this fact is not taken into consideration; however, these paradoxes can be resolved in analogy to the situation in electrodynamics.[23]

It is found that the energy density and the other components of the energy-momentum tensor of the produced particles increase rapidly upon approach to an anisotropic singularity. At a characteristic time $t_p = \sqrt{G\hbar/c^5}$, allowance for particle production may substantially alter the asymptotic form of the collapse; however, no such complete calculation with allowance for the reaction of the produced particles on the metric has yet been made.

It is necessary to emphasize that the isotropic case (uniform contraction or expansion in all directions), which was treated in[11-14], is degenerate. Upon approach to an isotropic singularity, particle production ceases (see Sec. 3) and its reaction on the metric is small; however, in the anisotropic case the situation is just the opposite. The deep-seated reasons for this difference are related to conformal invariance of the fields.

The important concept of conformal invariance,[24] that is, invariance with reference to a conformal transformation of the metric $ds^2 \rightarrow v^2(x^i)ds^2$, exists in the theory of wave fields (classical or quantum fields—it doesn't make any difference). When v depends on the coordinates, a change occurs in the properties of space, such as the transformation of flat space into curved space (but not into every curved space).

It is obvious that the particle's rest mass violates the conformal invariance of its wave equation, since the length $\hbar/mc$ associated with the rest mass m does not transform in proportion to the scaling factor v. Only the fields of particles with m = 0 can be conformally invariant. For such particles a remarkable result is obtained: they are not produced (which is shown in the articles by Bronnikov and Tagirov[12] and by Parker[13]) and they do not give any contribution to the vacuum polarization in conformally-flat space-time. For particles with $m \neq 0$ the contribution at large momenta is proportional to the fourth power of the mass, which leads to the removal of the divergences from the polarization and to the smallness of the pair production.

The most important class of Friedmann's cosmological solutions pertain to conformally-flat metrics. However, the more general singular solutions and, in particular, the simplest of these, whose three scaling factors a, b, c along the three spatial axes depend on the time in different ways (see the metric (1) given below), are not conformally-flat. In such a metric one can expect strong particle production, independent of the mass (that is, particle production also occurs even for m = 0), increasing like t^{-4} in the case of power-law dependent a, b, and c.[16] The form of the answer, $\epsilon \sim \hbar c^{-3} t^{-4}$, follows from dimensional considerations: the action of the gravitational field on the particles and on the other fields do not contain the gravitational constant G; the energy density is constructed in a unique way out of $\hbar$, c, and t. One can formulate the result in the same way as the well-known properties of the viscosity of a gas, namely, the absence of pair production at m = 0 in the isotropic case means that the so-called second viscosity, associated with the divergence of the velocity and the change of volume, is equal to zero for a vacuum of relativistic particles. However, the first viscosity of the vacuum, associated with shear, does not vanish—the relations are the same as for a gas. We have carried out specific calculations for the metric (1), in which three-dimensional space (the section t = const) is homogeneous and three-dimensionally flat. The quantum theory can be treated in the Hamiltonian formalism owing to the distinguished role of the time. The problem reduces to solving the classical wave equation.[1)]

[1)]This connection between the production of particles in the quantum theory and the consideration of the classical equation is quite natural, since for Bose particles the classical theory is the exact asymptotic limit of the quantum theory for a large number of particles. Amplification of the classical wave due to nonadiabaticity and parametric resonance corresponds to the creation of particles—the quanta of the field.

The proposed method is generally covariant. In fact, the wave equation and the energy-momentum tensor of the scalar field are covariant. However, the choice of the quantum state, with respect to which the averaging is carried out, is not unique. As such a state we take the vacuum of the scalar field, which can be properly defined only in flat space-time (ST). However, if an arbitrary ST adjoins such an ST, then by solving the wave equation (and also the "n-equation" necessary for renormalization) one can uniquely determine the energy-momentum tensor T_{ij} of the considered wave field resulting from the vacuum. In curved ST it is not clear how to separate the contribution made to T_{ij} by previously created particles from the contribution due to vacuum polarization; however, perhaps this question pertains more to philology than to physics.

The considered form of the metric (1) contains both the conformally-flat case as well as the opposite case, which is not conformally reducible to the flat case. Further, in the mentioned case divergences of the polarization part of the energy-momentum tensor take place in full measure, and therefore a program of renormalization and elimination of the infinities is required. For this purpose a method is developed which is a modification of the Pauli-Villars method. Each individual "wave' with a given m and a wave vector **k** is associated with an n-wave with similarly increased nm and n**k** and an amplitude decreased by $\sqrt{n}$ times. The energy-momentum tensor $T_{ij}^{(n)}(\mathbf{k})$ of the n-wave and the necessary number of its derivaties with respect to n are subtracted from the $T_{ij}(\mathbf{k})$ of the physical particles under consideration in such a way that $\int d^3k T_{ij}^{\mathrm{Reg}}(\mathbf{k})$ converges, and then the limiting transition $n \rightarrow \infty$ is made. The increase of not only m, but also of **k**, substantially facilitates the procedure. Just like the T_{ij}, the quantities $T_{ij}^{(n)}$ are constructed from the solutions of the wave equation; therefore each of the quantities satisfies the conservation law $T^{ij}{}_{;j} = 0$. Therefore the renormalized quantities T_{ij}^{Reg} identically satisfy the conservation law. In comparison with the Pauli-Villars method, the principal advantage consists in the fact that in the new method one can indicate the renormalized contribution of each individual wave with a given **k**. The subtracted quantities correspond to an n-particle, whose trajectory is the classical limit (ray) of the considered wave with given values of m and **k**, which is not the case in the Pauli-Villars method. Real production of n-particles obviously does not occur in the limit $n \rightarrow \infty$.

In order to study the general properties of particle production, the case of small changes of the metric is instructive:

$$a^2(t) = 1 + h_1(t), \quad b^2(t) = 1 + h_2(t), \quad c^2(t) = 1 + h_3(t);$$
$$h_\alpha(\pm\infty) = 0, \quad |h_\alpha| \ll 1.$$

In this case, by having specified the vacuum at $t = -\infty$, one can uniquely determine the number of particles and all of the properties at $t = +\infty$, that is, after completion of the process. In this connection the answer for $t = +\infty$ turns out to be convergent for continuous h_α. The number of produced particles is connected with the imaginary part of the vacuum polarization. This imaginary part is finite and is not needed in the renormalization procedure. However, the real part for $t \neq \pm\infty$ turns out to be divergent, and a finite answer is obtained only after renormalization.

The consideration of small perturbations of the metric also enables us to classify quantities according to powers of the small parameter h. It is found that the spatial components $T_{\mu\nu}$ of the energy-momentum tensor are proportional to the first power of h, whereas the energy density $\epsilon = T_{00}$ is proportional to h^2. The energy conservation law, which has schematically the form $\dot{T}_{00} \sim \dot{h}T_{\mu\nu}$, is therefore satisfied. Only for $t \to +\infty$ do the spatial components $T_{\mu\nu}$ of first order in h vanish, leaving $T_{\mu\nu}(+\infty) \sim h^2$. Thus, the condition of "energy dominance" (the relation $|T_{\mu\nu}| < T_{00}$) is violated in the process of particle production; only after completion of the process do we deal with a definite number of produced real particles and the indicated condition is satisfied.[22,23]

Above, the production of particles was treated in a metric which was specified beforehand and without taking account of the reaction of the produced particles on the metric. By virtue of the equations of the general theory of relativity (GTR), the given metric corresponds to a quite definite energy-momentum tensor $T_{ik(ext)}$ of the "external" matter (in particular, it is possible that $T_{ik(ext)} = 0$). In this sense it is not completely correct to call the process under study particle production in vacuum; we are actually investigating particle production in the presence of "external" matter. However, only the gravitational interaction of the "external" matter with the produced particles is taken into consideration here.

The particular case of the Kasner metric with power-law dependences of a, b, and c on t, and with the exponents p_1, p_2, p_3 satisfying the well-known relations, corresponds to the absence of "external" matter. Incidentally, if we gradually effect the transition from the Minkowski metric to the Kasner metric, then external matter is certainly necessary. The natural mode of transition to the Kasner metric gives the Heckmann-Schücking solution with $a(t) \sim t^{p_1}(t + t_1)^{2/3 - p_1}$ and so forth. In this case as $t \to 0$ the "external" matter does not disappear, its density increases like t^{-1}, and its role is small in comparison with the major terms ($\sim t^{-2}$) in the equations of GTR. From the methodological point of view the formulation of the problem of particle production in a given metric is not altered thereby. The method of renormalization, verified with a weak field as an example, turns out to be applicable also to the problem of production in an asymptotically Kasner metric. We mention the case $p_1 = 1$, $p_2 = p_3 = 0$, whose importance has been insisted on by V. A. Belinskiĭ. Here space-time is four-dimensionally flat and is easily transformed to the Minkowski form. However, the metric with the transition region $a(t) \to \text{const}$ for $t < t_0$ and $a(t) \sim |t|$ for $t > t_0$, $b = c = \text{const}$, is essentially non-flat for $t \sim t_0$; "external" matter exists in it. As a consequence of the nonlocal nature of the theory, the influence of the transition at $t = t_0$ also appears for $t > t_0$. Therefore, the nonvanishing result for $p_1 = 1$ and $p_2 = p_3 = 0$ in such a formulation of the problem does not violate any general principles.

The present article does not claim to be an exhaustive investigation of the problem. The immediate fundamental problem, which is in principle solvable (although it is also difficult) within the framework of the concepts developed here, is the systematic investigation of the collapse to a singularity with a self-consistent calculation of the reaction of the produced particles and of the vacuum polarization. As indicated above, such an effect is small in the isotropic case; however, it may become large in the case of anisotropic collapse. It is very probable that the reaction of particle production on the metric leads to isotropization of the contraction, the last stage of the approach to the singularity is switched over to the tracks of the quasi-isotropic solution described by Lifshitz and Khalatnikov.[17] On the other hand, it is improbable that the reaction might lead to the replacement of contraction by expansion. For this to happen it would be necessary for the energy density to be negative at some stage.

Finally, let us mention problems which are natural components of the general problem, but which require new ideas: 1) a general-covariant formulation of the theory, in particular a formalism without a distinguished role for the time; 2) allowance for the direct (electromagnetic, strong, etc.) nongravitational interaction between the particles; 3) the most difficult and important problem—the cosmological problem of emergence from the singularity, and the formulation of the initial conditions in the singular state. It is possible that this problem is inseparable from the general problem of the quantization of the metric, where the separation into high- and low-frequency dependences on the coordinates is still not satisfactory because of the contraction of the horizon. Here one can also express the enticing hypothesis that, just as in the case of collapse, the reaction of vacuum polarization on the gravitational field leads to the transition of an anisotropic expansion of the Kasner type into quasi-isotropic expansion at $t \sim t_p$ (see[16] for the appropriate estimates).

1. QUANTIZATION OF THE FIELD AND THE AVERAGE VALUE OF THE ENERGY-MOMENTUM TENSOR

Let us consider a real scalar field $\varphi(x^i)$ in a homogeneous cosmological model with a spatially-flat nonstationary metric:

$$ds^2 = dt^2 - a^2(t)\,dx_1^2 - b^2(t)\,dx_2^2 - c^2(t)\,dx_3^2, \tag{1}$$

where a, b, and c are certain given non-negative functions of the time. We take the Lagrangian density of this field in the form

$$\mathscr{L} = {}^1/_2[g^{ik}\varphi_{,i}\varphi_{,k} - (m^2 - {}^1/_6 R)\varphi^2], \tag{2}$$

where $R = g^{ik}R_{ik}$ is the scalar curvature we use a system of units in which $\hbar = c = 1$; the notation for products and the choice of signs in the definitions of g_{ik}, R_{iklm} and R_{ik} coincide with those adopted in[25]). The corresponding field equation is

$$\varphi_{;i}^{;i} + (m^2 - {}^1/_6 R)\varphi = 0. \tag{3}$$

It is conformally invariant for $m = 0$,[24] that is, under the conformal transformation of the metric

$$(ds = v(x^i)\widetilde{ds}, \qquad g_{jk} \to \widetilde{g}_{jk} = v^{-2}(x^i)g_{jk})$$

and under the corresponding transformation of the field function

$$(\varphi \to \tilde{\varphi} = v(x^i)\varphi)$$

Eq. (3) preserves its form. Tagirov and Chernikov[11] advance arguments in favor of the necessity of choosing the scalar field Lagrangian in the form (2).

The energy-momentum tensor of this field[11] is of the form

$$T_{ik} = \varphi_{,i}\varphi_{,k} - g_{ik}\mathscr{L} + {}^1/_6(R_{ik} + g_{ik}\Box - \nabla_i\nabla_k)\varphi^2, \qquad (4)$$

where $\Box = g^{ik}\nabla_i\nabla_k$; ∇ denotes the operator of covariant differentiation. This tensor possesses the following properties:

$$T_{ik} = T_{ki}, \quad T \equiv T_i^i = m^2\varphi^2, \quad T^{ik}{}_{;k} = 0. \qquad (5)$$

We note that in the case of flat space-time T_{ik} differs from the usually employed energy-momentum tensor of a scalar field by the quantity $(1/6)(g_{ik}\Box - \nabla_i\nabla_k)\varphi^2$, which has the form of a divergence.

The metric is assumed to be given and is not quantized. We carry out quantization of the scalar field by the standard method, using the Hamiltonian formalism (for the case of the isotropic Friedmann model, this was done by Bronnikov and Tagirov[12] and also by Parker[13]), namely: we introduce the canonical equal-time commutation relations

$$[\varphi(\mathbf{x}, t), \varphi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0,$$
$$[\varphi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \qquad (6)$$

where the generalized momentum of the field is given by

$$\pi = \partial(\sqrt{-g}\mathscr{L}) / \partial\dot{\varphi} = V\dot{\varphi}, \qquad V \equiv \sqrt{-g} = abc.$$

Relations (6) are satisfied if we represent the function $\varphi(x^i)$ in the form

$$\varphi(x^i) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,[\hat{A}_k\varphi_k(t)e^{-i\mathbf{kx}} + \hat{A}_k^+\varphi_k^*(t)e^{i\mathbf{kx}}]\,. \qquad (7)$$

Here $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{k} = (k_1, k_2, k_3)$ is the constant wave vector; the function $\varphi_k(t)$ satisfies the equation

$$\ddot{\varphi}_k + \frac{\dot{V}}{V}\dot{\varphi}_k + \left(\omega_k^2(t) - \frac{R}{6}\right)\varphi_k = 0, \qquad (8)$$

where

$$\omega_k^2(t) = m^2 + \frac{k_1^2}{a^2} + \frac{k_2^2}{b^2} + \frac{k_3^2}{c^2},$$

and the condition

$$\varphi\dot{\varphi}^* - \dot{\varphi}\varphi^* = i / V \qquad (9)$$

(a dot indicates differentiation with respect to t), and the A_k are certain constant operators with the commutation relations:

$$[\hat{A}_k\hat{A}_{k'}] = [\hat{A}_k^+\hat{A}_{k'}^+] = 0, \quad [\hat{A}_k\hat{A}_{k'}^+] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \qquad (10)$$

Further, we shall everywhere assume that, as $t \to -\infty$, a, b, and c $\to$ const and space-time becomes flat, and also the initial condition for $\varphi_k(t)$ is

$$\varphi_k(t)\,|_{t\to-\infty} = e^{-i\omega_{k0}t} / \sqrt{2\omega_{k0}V_0},$$

where $\omega_{k_0} = \omega_k(-\infty)$ and $V_0 = V(-\infty)$. Only in this case do the operators $\hat{A}_k$ and $\hat{A}_k^+$ coincide with the operators for the annihilation and creation of quanta of the free field at $t = -\infty$ and one can introduce the constant Heisenberg state vector $|0\rangle$ (satisfying the condition: $A_k|0\rangle = 0$ for all $\mathbf{k}$), which is the properly defined vacuum of the scalar field for $t = -\infty$. A complete system of in-states for $t = -\infty$ is introduced in similar fashion.

Because of the nonstationary nature of the metric (1), in the general case it is impossible to identify the operators $\hat{A}_k$ and $\hat{A}_k^+$ with the creation and annihilation operators for $t > -\infty$, and the state $|0\rangle$ is not the vacuum state for $t > -\infty$. The question as to how to define the creation and annihilation operators and the "vacuum" state for all values of t was investigated by Bronnikov and Tagirov[12] in the case of the isotropic Friedmann model, and they showed that the answer is not unique (in this connection, see also the articles by Grib and Mamaev[14]).

For the problems considered in the present article, this ambiguity is unimportant, since we are only interested in the average value $\langle 0|T_i^k(t)|0\rangle$. In a space with the metric (1), the average values of the diagonal components of the energy-momentum tensor are different from zero:

$$\mathscr{E} \equiv \langle 0|T_0^0|0\rangle = \frac{1}{(2\pi)^3}\int d^3k\left\{\frac{1}{2}(|\dot{\varphi}_k|^2 + \omega_k^2|\varphi_k|^2) + \frac{1}{6}\left(R_0^0 - \frac{1}{2}R\right)|\varphi_k|^2 + \frac{1}{6}\frac{\dot{V}}{V}\frac{d}{dt}|\varphi_k|^2\right\},$$

$$\mathscr{P}_1 \equiv -\langle 0|T_1^1|0\rangle = \frac{1}{(2\pi)^3}\int d^3k\left\{\frac{k_1^2}{a^2}|\varphi_k|^2 + \frac{1}{2}(|\dot{\varphi}_k|^2 - \omega_k^2|\varphi_k|^2) - \frac{1}{6}\left(R_1^1 - \frac{1}{2}R\right)|\varphi_k|^2 - \frac{1}{6V}\frac{d}{dt}\left(V\frac{d}{dt}|\varphi_k|^2\right) + \frac{1}{6}\frac{\dot{a}}{a}\frac{d}{dt}|\varphi_k|^2\right\} \qquad (11)$$

and so forth.

It is clear that in such a formulation the quantum problem reduces to a consideration of the classical wave equation (3) for $\varphi(x^i)$ or Eq. (8) for $\varphi_k(t)$.

We immediately note that the vacuum expectation values (11) include the energy density and the pressure of the zero-point oscillations of the vacuum and necessarily diverge. Therefore, in what follows we investigate the renormalized quantities

$$\varepsilon = \mathscr{E} - \mathscr{E}_0, \qquad p_\alpha = \mathscr{P}_\alpha - \mathscr{P}_{\alpha 0} \qquad (\alpha = 1, 2, 3),$$

where

$$\mathscr{E}_0 = \frac{1}{(2\pi)^3V}\int d^3k\frac{\omega_k}{2}, \quad \mathscr{P}_{10} = \frac{1}{(2\pi)^3V}\int d^3k\frac{k_1^2}{2a^2\omega_k}$$

and so forth; $\mathscr{E}_0$ and $\mathscr{P}_{\alpha 0}$ do not depend on the rate of change of the metric, and they diverge like k^4.

The renormalized energy density ϵ and pressure p_α satisfy the following conservation law:

$$\frac{1}{V}\frac{d}{dt}(V\varepsilon) = -\left(p_1\frac{\dot{a}}{a} + p_2\frac{\dot{b}}{b} + p_3\frac{\dot{c}}{c}\right). \qquad (12)$$

The question of their convergence will be investigated in the following section.

Because of the spatial homogeneity of the metric (1), the total momentum of the scalar field, $G_\alpha = \int d^3x\sqrt{-g}\,T_\alpha^0$, is conserved. In the case of a complex scalar field $\varphi(x^i)$ (whose quantization is carried out in analogous fashion) the total charge $Q = \int d^3x\sqrt{-g}\,J^0$ is also conserved, that is, the particles are created in pairs with opposite charges and momenta.

2. GENERAL ANISOTROPIC CASE

Let us make the following change of the variable t and of the field function φ:

$$\eta = \int \frac{dt}{V^{1/3}(t)} \tag{13}$$

$$\varphi(\eta, x) = \frac{\chi(\eta, x)}{V^{1/3}(\eta)}, \quad \varphi_k(\eta) = \frac{\chi_k(\eta)}{V^{1/3}(\eta)}.$$

We introduce the notation $v(\eta) = V^{1/3}$, $g_1(\eta) = a/v$, and $g_2(\eta) = b/v$; then Eq. (8) reduces to the form

$$\chi_k'' + [\Omega_k^2 + Q(\eta)]\chi_k = 0, \tag{14}$$

where

$$\Omega_k^2 = v^2\omega_k^2 = m^2v^2 + \frac{k_1^2}{g_1^2} + \frac{k_2^2}{g_2^2} + g_1^2 g_2^2 k_3^2,$$

$$Q(\eta) = \frac{1}{18}\left[\left(\frac{a'}{a} - \frac{b'}{b}\right)^2 + \left(\frac{a'}{a} - \frac{c'}{c}\right)^2 + \left(\frac{b'}{b} - \frac{c'}{c}\right)^2\right]$$
$$= \frac{1}{3}\left[\left(\frac{g_1'}{g_1}\right)^2 + \left(\frac{g_2'}{g_2}\right)^2 + \frac{g_1'g_2'}{g_1 g_2}\right],$$

and a prime denotes differentiation with respect to η. It is clear that $Q(\eta) = 0$ only in the isotropic case.

We have obtained the equation of a classical oscillator with a variable frequency.

Let us consider the special case of a metric whose evolution is such that $a, b, c|_{\eta=-\infty} = a, b, c|_{\eta=+\infty} = 1$. Let us take a single mode $\mathbf{k}$. As $\eta \to -\infty$ let the function $\chi(\eta)$ corresponding to this mode have the form $\chi(\eta) = e^{-i\Omega_0\eta}$, where $\Omega_0 = \sqrt{m^2 + \mathbf{k}^2}$. Then, as $\eta \to +\infty$ this same function χ has the asymptotic form

$$\chi = \alpha e^{-i\Omega_0\eta} + \beta e^{i\Omega_0\eta},$$

where $|\alpha|^2 - |\beta|^2 = 1$, $\beta \neq 0$ in the general case. Thus, amplification of the wave occurs—its energy increases by $1 + 2|\beta|^2$ times. The same thing also pertains to the second elementary wave: if $\chi = e^{i\Omega_0\eta}$ as $\eta \to -\infty$, then for $\eta \to +\infty$ one has

$$\chi = \alpha^* e^{i\Omega_0\eta} + \beta^* e^{-i\Omega_0\eta}.$$

The energy of this wave also increases by $1 + 2|\beta|^2$ times. An arbitrary linear combination of both waves with different signs of the frequency for $\eta \to -\infty$ obviously can be both intensified and weakened.

The wave equation is invariant with regard to the replacement of η by $-\eta$; one can easily construct the initial combination

$$\chi = C_1 e^{-i\Omega_0\eta} + C_2 e^{i\Omega_0\eta}$$

for $\eta \to -\infty$, which gives a decrease of the energy on emergence, that is, for $\eta \to +\infty$. However, if for $\eta \to -\infty$ the ratio of the moduli of C_1 and C_2 is fixed and the answer is averaged over the relative phase $(\mathrm{Arg}(C_1/C_2))$, then one again obtains an increase of the total energy by that same factor of $1 + 2|\beta|^2$ times.

From the quantum point of view, the energy increase associated with this process implies the creation of new quanta of the field. In the classical theory the increase of energy is proportional to its initial magnitude by virtue of the linearity of the field equations. The quantum theory of Bose particles is equivalent to a classical theory with a nonvanishing energy $(\hbar\Omega_0/2)$ of the state without any particles, and therefore gives a non-zero result for the production of particles from this state. Since the vacuum is a state with an undetermined phase, the energy always only increases.

It is obvious that β is a measure of the nonadiabaticity of the process. The departure from adiabaticity decreases with increasing wave vector $\mathbf{k}$ and frequency Ω_k; therefore $|\beta|^2$ turns out to be a rapidly decreasing function of $|\mathbf{k}|$. For the calculations it is convenient to write the corrections to the adiabatic approximation in explicit form; therefore, in order to solve Eq. (14) we shall use the method of Lagrange.

Thus, we shall seek the solution of Eq. (14) in the form

$$\chi_k(\eta) = \frac{\alpha_k(\eta)}{\sqrt{2\Omega_k}}e_- + \frac{\beta_k(\eta)}{\sqrt{2\Omega_k}}e_+, \qquad e_\pm = \exp\left\{\pm i\int \Omega_k d\eta\right\}, \tag{15}$$

with the additional condition

$$\chi_k'(\eta) = -i\Omega_k\left(\frac{\alpha_k(\eta)}{\sqrt{2\Omega_k}}e_- - \frac{\beta_k(\eta)}{\sqrt{2\Omega_k}}e_+\right) \tag{16}$$

(that is, the derivative of χ_k is taken as if α_k, β_k, and Ω_k did not depend on the time). From Eqs. (9), (15), and (16) it follows that

$$|\alpha_k(\eta)|^2 - |\beta_k(\eta)|^2 \equiv 1. \tag{17}$$

As a result, instead of the single second-order differential equation (14) for χ_k, a system of two linear first-order equations is obtained for α_k and β_k:

$$\alpha_k' = \frac{1}{2}\left(\frac{\Omega_k'}{\Omega_k} - i\frac{Q}{\Omega_k}\right)e_+^2\beta_k - i\frac{Q}{2\Omega_k}\alpha_k,$$
$$\beta_k' = \frac{1}{2}\left(\frac{\Omega_k'}{\Omega_k} + i\frac{Q}{\Omega_k}\right)e_-^2\alpha_k + i\frac{Q}{2\Omega_k}\beta_k \tag{18}$$

with the initial conditions: for $\eta = -\infty$ $(t = -\infty)$ the quantities $\alpha_k = 1$, $\beta_k = 0$.

Sometimes it is convenient to change from the two complex variables α_k and β_k, which are related by the condition (17), to three real variables:

$$s_k = |\beta_k|^2, \quad u_k = \alpha_k\beta_k^* e_-^2 + \alpha_k^*\beta_k e_+^2, \qquad \tau_k = i(\alpha_k\beta_k^* e_-^2 - \alpha_k^*\beta_k e_+^2). \tag{19}$$

For these variables one can obtain a system of three linear equations:

$$\frac{ds_k}{d\eta} = \frac{1}{2}\frac{\Omega_k'}{\Omega_k}u_k + \frac{1}{2}\frac{Q}{\Omega_k}\tau_k,$$
$$\frac{du_k}{d\eta} = \frac{\Omega_k'}{\Omega_k}(1 + 2s_k) - \left(\frac{Q}{\Omega_k} + 2\Omega_k\right)\tau_k, \tag{20}$$
$$\frac{d\tau_k}{d\eta} = \frac{Q}{\Omega_k}(1 + 2s_k) + \left(\frac{Q}{\Omega_k} + 2\Omega_k\right)u_k$$

with the initial conditions: $s_k = u_k = \tau_k = 0$ for $\eta = -\infty$. From the asymptotic form of the solution of the system (20) (or the system (18) which is equivalent to it), which is given in Appendix II, it is seen that as $\Omega_k \to \infty$

$$s_k \sim \Omega_k^{-2}, \ u_k \sim \Omega_k^{-2}, \ \tau_k \sim \Omega_k^{-1},$$

so that ϵ and p_α diverge like k^2 at the upper limit (also see formula (22)).

Therefore, an additional renormalization is required in the anisotropic case. We shall use the method discussed in the introduction to this article. Let $T_{ij}(\mathbf{k}, m)$ be the energy-momentum tensor for the single mode $\mathbf{k}$, and let $T_{ij}^{(n)} = n^{-1}T_{ij}(n\mathbf{k}, nm)$ be the energy-momentum tensor for the n-wave. Then the renormalized energy-momentum tensor for the given mode will be given by

$$T_{ij}^{Reg}(\mathbf{k}, m) = \lim_{n\to\infty}\Big[T_{ij}(\mathbf{k}) - T_{ij}^{(n)}(\mathbf{k}) - \frac{\partial}{\partial(n^{-2})}T_{ij}^{(n)}(\mathbf{k}) - \frac{1}{2}\frac{\partial^2}{\partial(n^{-2})^2}T_{ij}^{(n)}(\mathbf{k})\Big]. \quad (21)$$

By expanding $T_{ij}(\mathbf{k})$ in inverse powers of Ω_k as $\Omega_k \to \infty$, one can easily show that for large values of Ω_k we have $T_{ij}^{Reg}(\mathbf{k}) \sim \Omega_k^{-5}$; so the total energy-momentum tensor of the field, $T_{ij}^{Reg} = \int d^3k T_{ij}^{Reg}(\mathbf{k})$, converges. In addition, for $|\mathbf{k}| \to 0$ and $m \neq 0$, $|T_{ij}^{Reg}(\mathbf{k})| < \infty$. We note that the first subtraction corresponds to discarding the energy-momentum tensor of the zero-point oscillations of the vacuum, which was done at the end of Sec. 1. It is clear from the method of regularization that the tensor T_{ij}^{Reg} satisfies the conservation law: $T^{ij}_{Reg;\,j} = 0$.

The present method of renormalization is not applicable when m = 0, owing to the appearance of an infrared logarithmic divergence (the analogous phenomenon is observed in quantum electrodynamics). In actual fact this divergence is fictitious, and evidently it can be eliminated by the same method as in electrodynamics.

With the aid of this procedure, one can determine the finite renormalized average values of the field's energy-momentum tensor (for $m \neq 0$):

$$\varepsilon_{Reg} = \frac{1}{(2\pi)^3 v^4}\int d^3k\Big\{\Omega_k(s_k - s_k^{(2)} - s_k^{(4)}) - \frac{1}{2}\frac{Q}{\Omega_k}\Big[s_k - s_k^{(2)} + \frac{1}{2}(u_k - u_k^{(2)})\Big]\Big\};$$

$$p_{1Reg} = \frac{1}{(2\pi)^3 v^4}\int d^3k\Big\{\frac{k_1^2}{g_1^2\Omega_k}(s_k - s_k^{(2)} - s_k^{(4)}) + \frac{1}{6\Omega_k}\Big(3\frac{k_1^2}{g_1^2} - \Omega_k^2\Big)(u_k - u_k^{(2)} - u_k^{(4)}) + \frac{1}{6\Omega_k}\Big[\Big(\frac{g_1'}{g_1}\Big)' - Q\Big] \times\Big[s_k - s_k^{(2)} + \frac{1}{2}(u_k - u_k^{(2)})\Big] - \frac{g_1'}{6g_1}(\tau_k - \tau_k^{(1)} - \tau_k^{(3)})\Big\}\ \text{etc.}, \quad (22)$$

where $s_k^{(2)}$, $s_k^{(4)}$, $u_k^{(2)}$, $u_k^{(4)}$, $\tau_k^{(1)}$, and $\tau_k^{(3)}$ represent the corresponding terms of the asymptotic expansion of s_k, u_k, and τ_k in powers of Ω_k^{-1} for $\Omega \to \infty$ (see Appendix II); the superscripts indicate the order of decrease with respect to Ω_k^{-1}.

The case of a weak gravitational field (a small difference between the metric (1) and the Minkowski metric) is considered in Appendix I.

The quasi-Euclidean model of Friedmann is an interesting and important special case; in this model one can assume a = b = c = v, and then the metric (1) is isotropic and conformally-flat:

$$ds^2 = v^2(\eta)(d\eta^2 - dx_1^2 - dx_2^2 - dx_3^2).$$

As a consequence of this, $Q(\eta) \equiv 0$ and all the formulas in Sec. 2 simplify considerably.

In the first place, in the isotropic case for $k \equiv |\mathbf{k}| \to \infty$ we have $s_k \sim k^{-6}$, $u_k \sim k^{-4}$, $\tau_k \sim k^{-3}$ so that the quantities ϵ and p, introduced at the end of Sec. 1, turn out to be finite (that is, the first subtraction is already sufficient for regularization). They have the following form:

$$\varepsilon = \frac{1}{(2\pi)^3 v^4}\int d^3k\Omega_k s_k, \quad p = \frac{1}{(2\pi)^3 v^4}\int\frac{d^3k}{3\Omega_k}\Big(k^2 s_k - \frac{m^2v^2}{2}u_k\Big), \quad (23)$$

where $\Omega_k = \sqrt{k^2 + m^2v^2}$. From the expression for ϵ it follows that one can interpret s_k as the average number of pairs (real and virtual) created in the mode **k**.

Furthermore, for particles with m = 0, one will have $\Omega_k' = 0$ and $\beta_k \equiv 0$, so that $\epsilon = p = 0$. Thus, in the isotropic case with an arbitrary dependence of $v(\eta)$ not only the production of real massless particles does not occur, which is shown in articles[12,13], but also vacuum polarization is completely absent. We emphasize that in the general anisotropic case $\beta_k \neq 0$ for m = 0, which is clear from the system (18).

3. BEHAVIOR OF THE ENERGY DENSITY AND PRESSURE OF A SCALAR FIELD AS THE SINGULARITY IS APPROACHED

The investigation of the behavior of the field near a singularity is of the greatest physical interest. Let us consider the case of collapse, when we have flat space-time at $t = -\infty$ and a singularity at t = 0. Let the quantities a, b, and c be power-law functions of t as $t \to 0$.

In the anisotropic Kasner case the following region gives the basic contribution to the integrals in formula (22): $\omega_k \sim 1/t$, where $s_k \sim 1$, and for $t \to 0$ and $|mt| \ll 1$ we have $\epsilon \sim p_\alpha \sim t^{-4}$ (correct to, possibly, logarithmic terms). The exact coefficients associated with ϵ and p_α have still not been determined. Since the components of the Riemann tensor increase like t^{-2} as $t \to 0$, but the energy density of the "external" matter increases like $t^{-4/3}$ (for an ultrarelativistic gas), it follows that as the characteristic time $t \sim t_p = \sqrt{G}$ is approached the reciprocal effect of the created particles on the metric becomes large and it may substantially alter (of course, in connection with the consideration of the self-consistent problem) the subsequent course of the contraction.

In the degenerate case of the isotropic contraction of space, the following case is physically of interest: $a(t) \sim t^q$ for $t \to 0$; $0 < q < 1$. Let us choose η such that $\eta = 0$ when t = 0; then $a(\eta) \sim \eta^{q_1}$, where $q_1 = q/(1 - q)$. From Eq. (II.2) (see Appendix II) it is seen that if $ma(\eta) \ll k \equiv |\mathbf{k}| \ll 1/\eta$ (and for a sufficiently small value of η, any arbitrary k falls in this region), then s_k, u_k, and $\tau_k \to$ const; therefore pair production stops as $\eta \to 0$.

The reason for this is very simple: $\omega_k \approx k/a$ as $\eta \to 0$, that is, one cannot neglect the mass, but particles with m = 0 are generally not created during isotropic collapse. Upon further contraction the created particles behave as a relativistic gas with the equation of state $p = (1/3)\epsilon$, so that for $t \to 0$ and $|mt| \ll 1$ we find

$$\varepsilon = 3p = \frac{1}{2\pi^2 a^4}\int_0^\infty dk\ k^3 n(k) \sim m^4|mt|^{-4q},$$

where $n(k) = \lim_{\eta\to\infty} s_k$. One can show that $n(k) \sim k^{-1}$ as $k \to 0$, and $n(k) \sim k^{-4/(1-q)}$ as $k \to \infty$; therefore the written integral converges. The corresponding expression for the asymptotic density of particles (also including vacuum polarization) is given by

$$n(t) = \frac{1}{2\pi^2 a^3}\int_0^\infty dk\ k^2 n(k) \sim m^3|mt|^{-3q}.$$

matter is an ultrarelativistic gas), then the ratio of the energy of the created particles to the energy of the external matter will be: $\epsilon/\mathscr{E}_{ext} \sim Gm^2 \ll 1$ (we are considering "ordinary" elementary particles with masses from 10^{-24} to 10^{-27} g).

We note that in the isotropic case one can set up the opposite "cosmological" problem, specifying the "vacuum" state at t = 0 with the aid of the condition: $s, u, \tau|_{t=0} = 0$ (for $C_1 = 1/2\sqrt{k}$, $C_2 = \sqrt{k}/2$; see Eq. (II.2)). In the anisotropic case s, u, and $\tau \to \infty$ as $t \to 0$ and it is impossible to impose such an initial condition.

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APPENDIX I

WEAK GRAVITATIONAL FIELD

Let us consider the case of a weak external field (that is, there is a small difference between the metric (1) and the Minkowski metric) and let us calculate ϵ and p_α to first order in h(t) using perturbation theory. The following anisotropic case is of the greatest interest:

$$a^2(t) = 1 + h_1(t),\quad b^2(t) = 1 + h_2(t),\quad c^2(t) = 1 + h_3(t),$$

where

$$|h_\alpha(t)| \ll 1,\quad h_\alpha(t)|_{t=\pm\infty} = 0,\quad \sum_{\alpha=1}^{3} h_\alpha = 0.$$

To first order in h_α, in the Fourier representation we have

$$\varepsilon_{Reg} = 0,\quad \sum_{\alpha=1}^{3} p_{\alpha\,Reg} = 0,\quad p_{\alpha\,Reg}(q) = A(q)h_\alpha(q), \tag{I.1}$$

where

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(q)e^{iqt}dq,\quad p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(q)e^{iqt}dq.$$

The imaginary part of A(q) is finite and does not require renormalization:

$$\operatorname{Im} A(q) = -\frac{1}{1920\pi}(q^2-4m^2)^2\sqrt{1-\frac{4m^2}{q^2}}\,\Theta(q^2-4m^2)E(q), \tag{I.2}$$

where

$$\Theta(x) = \begin{cases}1, & x>0\\ 0, & x<0\end{cases},\quad E(x) = \begin{cases}1, & x>0\\ -1, & x<0\end{cases}$$

The quantity Im A(q) determines the energy at $t = +\infty$ in the second-order approximation (that is, the energy of the particles which are actually created):

$$\varepsilon(+\infty) = -\frac{1}{2}\int_{-\infty}^{\infty} dt \sum_{\alpha=1}^{3} p_\alpha \dot{h}_\alpha = -\frac{1}{4\pi}\int_{-\infty}^{\infty} dq\, q \operatorname{Im} A(q) \sum_{\alpha=1}^{3} |h_\alpha(q)|^2 \geq 0. \tag{I.3}$$

The real part of the polarization operator, which has been regularized with the aid of three subtractions, has the form

$$A(q) = \frac{1}{960(2\pi)^2}\frac{q^6}{m^2}\left[\frac{1}{\gamma^3} - \frac{7}{3\gamma^2} + \frac{23}{15\gamma} - \frac{(1-\gamma)^{5/2}}{\gamma^{7/2}}\operatorname{arctg}\sqrt{\frac{\gamma}{1-\gamma}}\right] \tag{I.4}$$

for $0 \leq \gamma \leq 1$; $\gamma = q^2/4m^2$;

$$\operatorname{Re} A(q) = \frac{1}{960(2\pi)^2}\frac{q^6}{m^2}\left[\frac{1}{\gamma^3} - \frac{7}{3\gamma^2} + \frac{23}{15\gamma} - \frac{(\gamma-1)^{5/2}}{\gamma^{7/2}}\ln(\sqrt{\gamma}+\sqrt{\gamma-1})\right]$$

for $\gamma > 1$.

The asymptotic form of A(q) is given by:

$$\begin{aligned} &1)\quad A(q) = \frac{1}{960(2\pi)^2}\frac{1}{7}\frac{q^6}{m^2} \quad \text{for } \gamma \ll 1;\\ &2)\quad A(q) = -\frac{1}{960(2\pi)^2}2q^4\ln\frac{q^2}{m^2} \quad \text{for } \gamma \gg 1. \end{aligned} \tag{I.5}$$

Also the following expressions in the coordinate representation are instructive:

$$\varepsilon(+\infty) = -\frac{1}{2}\int_{-\infty}^{\infty} dt \sum_{\alpha=1}^{3}\dot{h}_\alpha(t)\int_{-\infty}^{t} h_\alpha^{\rm VI}(\tau)\mathscr{K}(t-\tau)\,d\tau, \tag{I.6}$$

where $\mathscr{K}(t-\tau) \sim (t-\tau)\ln(t-\tau)$ for particles with m = 0. The quantity $\int_{-\infty}^{t}\ldots$ plays the role of the components of the stress tensor, denoted by $p'_\alpha = -T'^{\alpha}_{\alpha}$. However, this is only part of p_α; it is obvious that the addition of terms of the form $h_\alpha^{(2n)}(t)$ to p'_α does not change the value of $\epsilon(+\infty)$. It is precisely such terms, which do not give any contribution to $\epsilon(+\infty)$, which are subjected to renormalization.

We note that the found p_α are proportional to h_α, provided $\sum_\alpha h_\alpha = 0$, just as for the viscous stresses associated with the deformation of an incompressible liquid. However, viscosity would give $p_\alpha \sim \dot{h}_\alpha$; vacuum polarization (and even just its imaginary part) has a more complicated nonlocal dependence on the form of h as a function of t.

APPENDIX II

THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE SYSTEM OF EQUATIONS (20)

Let us consider the case of large momenta ($\Omega_k \to \infty$) and let us expand the solution of the system of Eqs. (20) in an asymptotic series in powers of Ω_k^{-1}. This expansion is valid in the quasiclassical region: $|\Omega'_k| \ll \Omega_k^2$. In this region s_k, $|u_k|$, $|\tau_k| \ll 1$. It is found that $\tau = \tau^{(1)} + \tau^{(3)} + \ldots$, $s = s^{(2)} + s^{(4)} + \ldots$, $u = u^{(2)} + u^{(4)} + \ldots$ (here and in what follows, the subscript k is omitted), where

$$\begin{aligned} \tau^{(1)} &= \frac{1}{2}\frac{\Omega'}{\Omega^2},\quad u^{(2)} = \frac{1}{2\Omega}\left[\frac{d\tau^{(1)}}{d\eta} - \frac{Q(\eta)}{\Omega}\right],\\ s^{(2)} &= \frac{1}{2}\int\left[\frac{\Omega'}{\Omega}u^{(2)} + \frac{\Omega(\eta)}{\Omega}\tau^{(1)}\right]d\eta = \frac{1}{16}\frac{\Omega'^2}{\Omega^4}\\ \tau^{(3)} &= -\frac{1}{2\Omega}\left[\frac{du^{(2)}}{d\eta} - 2\frac{\Omega'}{\Omega}s^{(2)} + \frac{Q(\eta)}{\Omega}\tau^{(1)}\right],\\ u^{(4)} &= \frac{1}{2\Omega}\left[\frac{d\tau^{(3)}}{d\eta} - \frac{Q(\eta)}{\Omega}(2s^{(2)} + u^{(2)})\right],\\ s^{(4)} &= \frac{1}{2}\int\left[\frac{\Omega'}{\Omega}u^{(4)} + \frac{Q(\eta)}{\Omega}\tau^{(3)}\right]d\eta \end{aligned} \tag{II.1}$$

and so forth. The superscript inside the brackets indicates the order of decrease in powers of Ω_k^{-1} for $\Omega_k \to \infty$.

Near the singularity ($\eta = 0$, t = 0) one can obtain another important asymptotic form of the system (18), (20). Let

$$\lim_{t\to 0} \frac{a, b, c}{t} = \infty,$$

then for sufficiently small η one has

$$\left| \int_0^\eta \Omega_k d\eta \right| \ll 1$$

for arbitrary **k**. In this essentially non-quasiclassical region we have

$$\begin{aligned} \alpha &= C_1\left(\sqrt{\Omega} - \frac{i}{\sqrt{\Omega}} \int Q(\eta)\, d\eta\right) + \frac{C_2}{\sqrt{\Omega}}, \\ \beta &= C_1\left(\sqrt{\Omega} + \frac{i}{\sqrt{\Omega}} \int Q(\eta)\, d\eta\right) - \frac{C_2}{\sqrt{\Omega}}, \\ s &= |C_1|^2 \left[\Omega + \frac{1}{\Omega}\left(\int Q\, d\eta\right)^2\right] + \frac{|C_2|^2}{\Omega} \\ &\quad + \frac{i}{\Omega}(C_1^* C_2 - C_1 C_2^*) \int Q\, d\eta - \frac{1}{2}, \\ u &= 2|C_1|^2 \left[\Omega - \frac{1}{\Omega}\left(\int Q\, d\eta\right)^2\right] - \frac{2|C_2|^2}{\Omega} - \frac{2i}{\Omega}(C_1^* C_2 - C_1 C_2^*) \int Q\, d\eta, \\ \tau &= 4|C_1|^2 \int Q\, d\eta + 2i(C_1^* C_2 - C_1 C_2^*), \end{aligned} \qquad \text{(II.2)}$$

where C_1 and C_2 are complex numbers (depending on **k**), satisfying the condition: $C_1^* C_2 + C_1 C_2^* = 1/2$. In the case of power-law dependences for the a, b, and c we have $Q(\eta) \sim \eta^{-2}$ so that, in the anisotropic case the quantity $s \approx -2u \sim 1/\Omega\eta^2$ and $\tau \sim \eta^{-1}$ as $\eta \to 0$; in the isotropic case s, u, and $\tau \to$ const as $\eta \to 0$.

By joining the two asymptotic expressions in the region

$$\left| \int_0^\eta \Omega_k\, d\eta \right| \sim 1,$$

one can derive the law $\epsilon \sim p_\alpha \sim t^{-4}$, which was cited in Sec. 3.

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