Stability of Vacuum and Limiting Fields

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A study is made of polarization of vacuum in strong static fields when bound states with energies close to that of particle production in vacuum arise. It is shown that allowance for interaction between the particles ensures stability of vacuum since vacuum polarization leads to strong screening fields which do not permit the levels to approach dangerous values. Possible physical and astrophysical consequences of such alteration of the vacuum are considered.

I. POLARIZATION OF VACUUM IN STRONG FIELDS

IN the study of the polarization of vacuum one usually excludes from consideration fields in which there are deep bound states of particles. We consider in this article phenomena that arise when bound states with energy close to the energy of particle production from vacuum appear in a strong external field in the single-particle problem. In Sec. II we consider examples of such critical fields and critical levels.

The best known example of the appearance of critical levels is the point-like nucleus with charge $Z_c = 137$ or the nucleus with finite radius $R = r_0A^{1/3}$ and charge $Z_c = 170$[1]. It can be shown[2] that at $Z > Z_c$ the state of lowest energy corresponds to vacuum with an altered polarization charge. This charge lies in the region $\lesssim \hbar/mc$. Owing to the Pauli principle, which does not allow the particles to accumulate in a "dangerous" state, a relatively weak screening field is produced. The stability of the "new" vacuum is ensured by the Pauli principle.

A much more important realignment of the vacuum occurs in fields in which Bose particles can be produced. Allowance for the interaction between the particles ensures the stability of the vacuum, viz., when the wall deepens beyond a critical value, a strong polarization of the vacuum sets in and a screening field appears and prevents the level from approaching the limiting value. Thus, owing to the existence of Bose particles, the effective field, i.e., the external field plus the vacuum-polarization field, cannot exceed a limiting value at which the critical value of the particle energy is reached (Sec. III).

Particularly interesting phenomena occur in the scalar field that is realized in nuclear matter. The scalar field acting on the mesons in nucleon matter is determined by the formula

$$v = -4\pi nf,$$

where n is the density of the nucleons and f is the amplitude of scattering of a π^0 meson by a nucleon (formula (40)). At a sufficient nucleon density the mesonic vacuum becomes realigned and a phase transition occurs, in which the equation of state of the nuclear matter is altered (Sec. IV). Such a phase transition can apparently be realized in neutron stars, in regions of high neutron density.

In atomic nuclei, the dense phase, if it does exist at all, is separated from the usual one by a tremendous potential barrier. One can attempt to seek such superdense nuclei in cosmic rays. The charge-to-mass ratio in such nuclei would differ considerably from the usual one.

II. BOUND STATES OF RELATIVISTIC PARTICLES

1. Scalar Particle in Scalar Field

A. We start with a consideration of a scalar particle in a scalar external field. The particle is described by the Klein-Gordon equation

$$\Delta\Psi_\lambda + (\varepsilon_\lambda^2 - 1 - v(r))\Psi_\lambda = 0 \quad (\mu = c = \hbar = 1). \tag{1}$$

We assume that v is added in the Lagrangian to μ^2, and therefore v is an unbounded quantity. If a scalar field v_1 were to be added to μ (for which there are no grounds), then the quantity in (1) would be

$$v = v_1^2 + 2v_1\mu; \quad -v < \mu.$$

Let v have the form of a well:

$$v(r) < 0, \quad r < R; \quad v(r) \to 0, \quad r \to \infty.$$

The bound-state condition in our notation is $\epsilon_\lambda < 1$. As the well deepens, the lower level drops and at $v = v_c$ it approaches the value $\epsilon_0 = 0$. Let us trace the course of the level when the potential is varied near v_c. From the equations

$$\Delta\Psi + (\varepsilon^2 - 1 - v)\Psi = 0,$$
$$\Delta\Psi_c = (1 + v_c)\Psi_c$$

we obtain

$$\varepsilon^2 = \frac{(\Psi|v - v_c|\Psi_c)}{(\Psi, \Psi_c)}$$

When v is sufficiently close to v_c we have $\Psi \cong \Psi_c$ and

$$\varepsilon^2 = (\Psi, v\Psi) - (\Psi, v_c\Psi) \equiv \bar{v} - \bar{v}_c. \tag{2}$$

Thus, the level energy as a function of the parameter ν proportional to the depth of the well takes the form shown in Fig. 1.

B. The lower branches of the curves shown in Fig. 1 should be discarded for the following reason. Equation (1) without a field or in weak fields has superfluous solutions (corresponding to negative energy). In strong fields, when bound states arise, the selection rule for the physical solutions should be formulated in the following manner: It is necessary to discard all the solu-

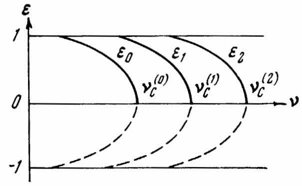


FIG. 1

tions that arise from states with negative energy when the field is turned on adiabatically, i.e., discard the solutions that come from the negative continuum.

The dangerous situation arises for the first time near the point $\nu = \nu_c^{(0)}$. The particle-production energy in this state is equal to zero. Particles should be produced from the vacuum. We shall show below that the solution of the field equation rather than the single-particle equation, with allowance for the interaction between the particles, changes strongly the curve of Fig. 1 and leads to stability of the vacuum, namely, the particle-production energy does not reach zero for any well.

C. We present the dependence of the energy of the lower level on the depth of the well for a spherical square well. The lowest level corresponds to a zero orbital angular momentum. Separating in (1) the angle variables and putting $\Psi_0 = (1/\sqrt{4\pi})u/r$, we obtain

$$u'' + (\varepsilon_0^2 - 1 + v_0)u = 0, \quad r < R,$$
$$u'' + (\varepsilon_0^2 - 1)u = 0, \quad r > R,$$

where $v_0 = -v$ and $r < R$. The condition for continuity at $r = R$ yields

$$K \operatorname{ctg} KR = -\lambda_0, \quad K^2 = \varepsilon_0^2 - 1 + v_0, \quad \lambda_0 = \sqrt{1 - \varepsilon_0^2}.$$

In the case of a narrow well ($R \ll 1$) we have $K \sim 1/R \gg 1$, and consequently

$$KR \cong \frac{\pi}{2} + \frac{\lambda_0}{K} \cong \frac{\pi}{2}(1 + \lambda_0 R).$$

The energy ϵ_0 is determined by the expression

$$\varepsilon_0^2 - 1 + v_0 \simeq \frac{\pi^2}{4R^2}(1 + 2\lambda_0 R), \quad -v_c \simeq 1 + \frac{\pi^2}{4R^2}. \tag{3}$$

In the case of a broad well ($R \ll 1$) we have $K \sim 1/R \gg 1$ and for ϵ_0 we easily obtain

$$\varepsilon_0^2 - 1 + v_0 = \frac{\pi^2}{R^2}\left(1 - \frac{2}{\lambda_0 R}\right), \quad -v_c \simeq 1 + \frac{\pi^2}{R^2}. \tag{4}$$

D. In the case of a broad well, when $R \gg 1$, the distance between the first levels is of order $1/R^2$ and at $v = v_c$, when $\epsilon_0 = 0$, many levels have near-zero energy. All these levels contribute to the polarization of the vacuum.

It is convenient to consider the case of the broad well in the quasiclassical approximation. We confine ourselves to presenting an expression for the energy levels in a cubic well of dimension L. From (1) it follows that

$$\varepsilon_k^2 = 1 - v_0 + k^2, \tag{5}$$

where **k** is equal to

$$\mathbf{k} = \pi \mathbf{n} / L, \qquad \mathbf{n} = (n_x, n_y, n_z).$$

Here n_x, n_y, and n_z are the numbers of nodes of Ψ along the axes x, y, and z; $v_0 = -v$ at x, y, z < L. The bound states correspond to $\epsilon_k < 1$ and $v_c \cong -1 - \pi^2/L^2$. Thus, the energy spectrum inside a broad well differs from the spectrum of the free particles in that the mass is replaced by the quantity $1 - v_0$, which vanishes when $v = v_0$.

2. Scalar Particle in Electric Fields. Bound States in the Case of a Repulsion Potential

A. The Klein-Gordon equation in a static electric field is of the form

$$\Delta\Psi_\lambda + [(\varepsilon_\lambda - V)^2 - 1]\Psi_\lambda = 0. \tag{6}$$

It is somewhat difficult to operate with this equation, since the eigenfunctions Ψ_λ are not orthogonal. Proceeding in the usual manner, i.e., multiplying the equations for Ψ_1 and Ψ_2 by Ψ_2 and Ψ_1 respectively, and subtracting, we obtain (for $\Psi_1 \neq \Psi_2$)

$$(\varepsilon_1 - \varepsilon_2)(\Psi_1, \Psi_2) = 2(\Psi_1, V\Psi_2). \tag{7}$$

Relation (7) replaces the orthogonality condition.

One more feature of (6) is that at sufficiently large $|V|$ ($|V| > 1$), regardless of the sign of V, there is a region of r where $\Psi_\lambda(r)$ has an oscillating solution, i.e., there is effective attraction[5]. This can give rise to a bound state even in the case when $V(r) > 0$ for all r.

The existence of an attraction region becomes particularly clear if we write the equivalent Schrödinger equation, putting $\epsilon^2 - 1 \equiv 2E$, and then the equivalent potential energy is

$$v = -\tfrac{1}{2}V^2 + \varepsilon V.$$

In the region where $V^2 > 2\epsilon V$ we have $v < 0$, i.e., attraction (one must not forget that the analogy with the Schrödinger equation is incomplete, since v depends on ϵ, and this leads to non-orthogonality of the functions).

B. Let us find the character of the dependence of the level energy on the depth of the potential well. We show first that $\partial\epsilon_\lambda/\partial\nu = \infty$ at $\epsilon_\lambda = (\Psi_\lambda, V\Psi_\lambda)$; here, as above, ν is a parameter proportional to the depth of the well.

Indeed, writing down the two equations of (6) for two close values of ν, multiplying by the corresponding functions, and subtracting, we obtain

$$[\varepsilon - (\Psi, V\Psi)]\frac{\partial \varepsilon}{\partial \mathrm{v}}\mathrm{v} = \varepsilon(\Psi, V\Psi) - (\Psi, V^2\Psi), \tag{8}$$

from which it follows that

$$\partial\varepsilon_\lambda / \partial \mathrm{v} = \infty \text{ when } \varepsilon_\lambda = (\Psi_\lambda, V\Psi_\lambda). \tag{9}$$

Therefore the plot of ϵ_λ vs. ν has the form shown in Fig. 2[1].

Curve 1 corresponds to particles for which $V < 0$, and curve 2 to particles of opposite charge ($V > 0$). As follows from (6), curve 2 differs from curve 1 in that ϵ is replaced by $-\epsilon$. It follows from (9) that the point

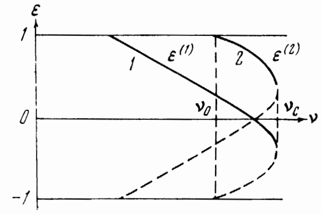


FIG. 2

$\partial\epsilon/\partial\nu = \infty$ on curve 1 corresponds to the value $\epsilon_\lambda^{(1)} < 0$, and on curve 2 to $\epsilon_\lambda^{(2)} > 0$.

C. How are these curves to be interpreted? Repeating the selection rule given above for the physical solutions, it is necessary to discard all those solutions that result from negative energies when the field is adiabatically turned on. Thus, it is necessary to leave only those branches of curves 1 and 2 which go from the positive continuum.

At $\nu = \nu_0$, a bound state appears in spite of the repulsion potential!

At $\nu = \nu_c$ we arrive at the ctitical situation referred to above, namely, the pair-production energy $W = \epsilon_\lambda^{(1)} + \epsilon_\lambda^{(2)}$ vanishes at $\nu = \nu_c$. Indeed, at the critical point $\epsilon_\lambda'^{(1)} = (\Psi_\lambda V \Psi_\lambda)$ and $\epsilon_\lambda'^{(2)} = -(\Psi_\lambda V \Psi_\lambda)$. As will be shown in the next section, a strong polarization of the vacuum sets in when $\nu = \nu_c$ is approached, and leads to the appearance of a compensating field such that the pair energy cannot reach zero at any value of V.

D. The energy $\epsilon^{(1)}$ at the critical point lies in the interval $(-1, 0)$. For potentials with large radius $(R \gg 1)$, $\epsilon_c^{(1)}$ and $\epsilon_c^{(2)}$ tend to -1 and $+1$, respectively.

To explain the character of these tendencies, we present without proof a formula for the critical energy of the first level for a spherical square well with a penetrability barrier $e^{-\gamma}$:

$$\varepsilon_c^{(1)} = -1 + R^{-1}e^{-2\gamma} = -\varepsilon_c^{(2)}.$$

For a narrow well $(1 + \epsilon_c^{(1)})/|\epsilon_c^{(1)}| \sim 1$; thus, for example, for a square spherical well with $R \ll 1$ we have $\epsilon_c^{(1)} \cong -0.8$. For a broad well with a flat bottom we obtain from (6) for the lower state, in analogy with (5),

$$(\varepsilon_0^{(1)} - V)^2 = 1 + C_1\frac{\pi^2}{L^2},\quad \varepsilon_0^{(1)} = -V_0 + 1 + \frac{C_1\pi^2}{2L^2},\quad V_c \cong -2, \qquad (5')$$

where L are the characteristic dimensions of the well and C_1 is a number of the order of unity ($C_1 = 1$ for a cubic well).

III. POLARIZATION OF VACUUM. SCREENING FIELDS

1. Polarization of Vacuum in the Case of a Scalar Field

A. Let us explain the role of polarization, using first as an example a scalar particle in a scalar field. This will facilitate the transition to the case of an electric field.

Assume we have a field $\hat\varphi$ of scalar particles in an external scalar field v with an interaction

$$H' = \frac{\lambda}{4}\int \hat\varphi^4 dV,\quad 0 < \lambda \ll 1.$$

We expand $\hat\varphi$ in terms of the eigenfunctions of (1):

$$\hat\varphi = \sum \frac{1}{\sqrt{2\varepsilon_\lambda}}(a_\lambda \Psi_\lambda + a_\lambda^+ \Psi_\lambda^*). \qquad (10)$$

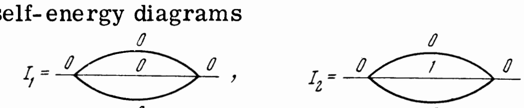
We confine ourselves for the time being to the case of a narrow well with $|v - v_c|/|v_c| \ll 1$. Then there is only one level near the critical value ($\epsilon = 0$) and it is precisely this level that makes the decisive contribution to the polarization of the vacuum, i.e., to the change of the energy spectrum of one or several particles in the field v[1)]. It is therefore necessary to retain in the expansion (1) one term:

$$\hat\varphi = \frac{1}{\sqrt{2\omega_0}}(a + a^+)\Psi. \qquad (10')$$

We used the fact that Ψ is real for a bound state, and denoted by ω_0 the unperturbed energy of the level under consideration.

It is easy to verify, for example, graphically that allowance for the remaining terms in the expansion (10) leads to corrections that contain powers of λ. Moreover, it can be assumed that allowance for the discarded terms leads only to a renormalization of the interaction H′ and does not alter the results qualitatively even when $\lambda \sim 1$.

Let us consider by way of illustration two very simple self-energy diagrams



The symbols 0 and 1 denote respectively the dangerous state and the state with energy ϵ_1. We obtain directly the estimate

$$I_2/I_1 \sim \varepsilon_0^2/\varepsilon_1^2;$$

Near V_c we obtain $I_2/I_1 \sim \lambda_1^{2/3}$.

B. The Hamiltonian of the system can be written in the form

$$H = (a^+a + aa^+)\omega_0 + \frac{\lambda_1}{4}\left(\frac{a^+ + a}{\sqrt{2\omega_0}}\right)^4 \qquad (11)$$

$$\lambda_1 = \lambda\int \Psi^4 dV.$$

We put

$$q = \frac{a + a^+}{\sqrt{2\omega_0}}\quad p = -i\sqrt{\frac{\omega_0}{2}}(a - a^+).$$

Then

$$H = \frac{p^2 + \omega_0^2 q^2}{2} + \frac{\lambda_1}{4}q^4. \qquad (11')$$

Thus, the problem of determining the levels of the system has reduced to the problem of the energy spectrum of an anharmonic oscillator.

According to formula (2) we have

$$\omega_0^2 = \bar v - \bar v_c. \qquad (2')$$

Expression (2′), when substituted in (11′), makes it possible to continue analytically the solution of the problem from the subcritical region $(\bar v - \bar v_c > 0)$ to the transcritical region $(\bar v - \bar v_c < 0)$, since the operator (11′) has no singularity at $\omega_0 = 0$.

C. We can obtain a Hamiltonian suitable in the transcritical region also without an analytic continuation. To this end, we choose the basis functions Ψ_λ for a fixed potential $v = v_1$ which is sufficiently close to v_c. The Hamiltonian takes the form

$$H = \sum (a_\lambda a_\lambda^+ + a_\lambda^+ a_\lambda)\varepsilon_\lambda^{(1)} + \int (v - v_1)\hat\varphi^2 dr + \lambda\int \hat\varphi^4 dr.$$

Retaining again only one term, we obtain

$$H = \frac{p^2 + \omega_1^2 q^2 + (\bar v - \bar v_1)q^2}{2} + \frac{\lambda_1 q^4}{4}. \qquad (11'')$$

In particular, we can take $v_1 = v_c$. Then $\omega_1 = 0$ and (11″) goes over into (11′) with the condition (2′). By the same token we have proved the validity of the analytic continuation and refined the constant λ_1, namely, in such a

[1)]The case of a broad well is considered in Sec. IV.

basis the Ψ function which enters in λ_1 is the eigenfunction of the considered level at $v = v_c$.

D. The energy spectrum of the Hamiltonian (11′) is obtained with good accuracy quasiclassically:

$$\int_{q_1}^{q_2} \{2[E(n)-U(q)]\}^{1/2} dq = (n+1/2)\pi, \quad U(q) = 1/2(\bar{v}-\bar{v}_c)q^2 + 1/4\lambda_1 q^4. \tag{12}$$

Comparison with computer calculations (see below) shows that the error in the determination of the single-particle energy ($\omega = E(1) - E(0)$ is of the order of 5%.

For clarification, let us consider separately regions far from the critical point and the region close to it. The parameter separating the different limiting cases is the quantity

$$\eta = \lambda_1 / 4|\bar{v}-\bar{v}_c|^{3/2}.$$

At $\eta \ll 1$ and $\bar{v} - \bar{v}_c > 0$ (weak anharmonicity) we have

$$E(n) = (n+1/2)\omega_0 + 1/4\lambda_1\langle n|q^4|n\rangle.$$

The averaging is over the eigenfunction of the n-th state of the oscillator. It is easy to obtain

$$E(n) = (n+1/2)\omega_0[1+\eta(n+1/2)]. \tag{13}$$

Consequently the corrected energy of the single-particle state is

$$\omega = E(1) - E(0) = \omega_0(1+2\eta), \quad \eta \ll 1. \tag{13'}$$

When $\eta \gg 1$ or $|\bar{v} - \bar{v}_c| \ll \lambda_1^{2/3}$, i.e., in the immediate vicinity of the critical point, we can obtain from (12)

$$E(n) = A\lambda^{1/3}(n+1/2)^{4/3}, \quad A = \left(\frac{\pi}{B(1/4, 3/2)}\right)^{4/3}, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{14}$$

We present for comparison the values of $\theta(n) = E(n)/\lambda^{1/3}$, obtained quasiclassically (I) and with a computer (II):

	I	II
$\theta(0)$:	0.35	0.4208
$\theta(1)$:	1.50	1.5079
$\theta(2)$:	2.96	2.9587
$\theta(3)$:	4.62	4.6210

The energy of the single-particle state is[3]

$$\omega = E(1) - E(0) = 1.09\lambda^{1/3}, \tag{14'}$$

and the quasiclassical value is

$$\omega = 1.15\lambda^{1/3}.$$

The energy ω as a function of $\bar{v} - \bar{v}_c$, obtained quasiclassically, is given in Fig. 3. At $|\bar{v} - \bar{v}_c| \gg \lambda_1^{2/3}$ the energy $\omega = E(1) - E(0)$ approaches zero asymptotically.

E. Let us find the asymptotic form of ω when $|v - v_c| \gg \lambda_1^{2/3}$. In this region, the potential-energy curve of the oscillator (11′) has the form shown in Fig. 4. U(q) has two symmetrical minima separated by a barrier. The eigenfunctions break up into two classes, symmetrical and antisymmetrical with respect to the reversal of the sign of q.

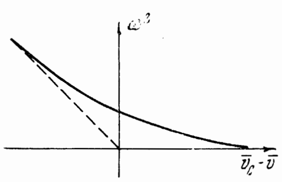


FIG. 3

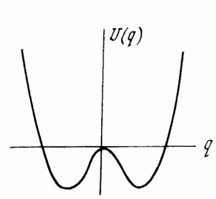


FIG. 4

The character of the spectrum can be easily understood: if the barrier separating the two wells has low penetrability, then the energy of the symmetrical state will be lower than the energy of the corresponding antisymmetrical state by a small amount proportional to the barrier penetrability. On the other hand, the distance between levels that differ in the number of nodes in each of the wells is not only not small, but, as we shall show, increases with increasing distance from the critical point.

This can be easily verified by expanding U(q) near the minimum:

$$q_{min}^2 = |\omega_0^2|/\lambda_1, \quad q_{min} = \pm\sqrt{|\omega_0^2|/\lambda_1}, \quad U(q) = \omega_0^4/4\lambda_1 + |\omega_0^2|(q-q_+)^2. \tag{15}$$

Thus, the question of the gross structure of the spectrum has been reduced to the problem of an oscillator with frequency $\sqrt{2}\,|\omega_0|$. Let us verify that the oscillations of the oscillator do not take the expansion (15) beyond the limits of applicability. Indeed,

$$\langle (q-q_+)^2\rangle / q_+^2 \cong \lambda_1 / |\omega_0|^3 \ll 1.$$

Therefore, the gross structure of the levels will be

$$E'(n') = -\omega_0^4/4\lambda_1 + (n'+1/2)\sqrt{2}|\omega_0|,$$

where n′ is the number of nodes in each well. The splitting of each of these levels is exponentially small because of the influence of the second "well." The magnitude of the splitting is proportional to the penetrability of the barrier separating the wells, i.e., it decreases exponentially with increasing separation between the minima and with increasing depth of each well. In particular, there are two exponentially close states with n′ = 0: a symmetrical one, corresponding to E(0) and an antisymmetrical one corresponding to E(1). The quasiclassical calculation yields

$$E(1) - E(0) = \frac{\omega_0\sqrt{2}}{\pi}\exp\left\{\frac{-\pi|\omega_0|^3}{4\lambda_1}\right\}.$$

The energy of the two particles

$$\omega_2 = E(2) - E(0) \cong 2\sqrt{2(\bar{v}_c - \bar{v})}$$

increases with increasing distance from the critical point.

Attention should be called to the fact that the energy of the ground state decreases with deepening of the well:

$$E(0) \cong -\omega_0^4/4\lambda_1. \tag{16}$$

2. Polarization of Vacuum in Electric Field

A. Let us consider the polarization of the vacuum of a field of charged mesons placed in a static electric field. The Lagrangian of the system takes the form

$$\mathscr{L} = \left(\frac{\partial}{\partial t} - iV\right)\hat{\varphi}^+\left(\frac{\partial}{\partial t} + iV\right)\hat{\varphi} - \nabla\hat{\varphi}^+\nabla\hat{\varphi} - \mu^2\hat{\varphi}^+\hat{\varphi} - \frac{\lambda}{2}(\hat{\varphi}^+\hat{\varphi})^2, \quad \lambda > 0. \tag{17}$$

We have assumed the simplest interaction between the mesons. It can be shown that stability of the vacuum is ensured also by a purely electric interaction, but it is natural to assume that the hadron interaction is more significant, i.e., $\lambda > e^2$. From the character of the conclusion it will become clear that a change in the form of the hadron interaction does not influence the qualitative results.

It is first necessary to carry out the program of second quantization with respect to the basis functions of Eq. (6), which, as we have seen, are not orthogonal. We write $\hat{\varphi}$ in the form

$$\hat{\varphi}=\sum_{\lambda}(c_\lambda^{(1)}a_\lambda\Psi_\lambda^{(1)}+c_\lambda^{(2)}b_\lambda^+\Psi_\lambda^{(2)*}). \qquad (18)$$

Here $\Psi^{(1)}$ and $\Psi^{(2)}$ are the solutions of Eq. (6) for V and $-V$, respectively ($\Psi_\epsilon^{(2)}=\Psi'^{(1)}$). The coefficients $c_\lambda^{(1)}$ and $c_\lambda^{(2)}$ are obtained from the requirement that the free Hamiltonian ($\lambda = 0$) take the form

$$H_0=\sum(a_\lambda^+a_\lambda\varepsilon_\lambda^{(1)}+b_\lambda^+b_\lambda\varepsilon_\lambda^{(2)}). \qquad (19)$$

Let us explain the scheme of action and present the results. From the Lagrangian (17) without the interaction term we obtain the Hamiltonian

$$H_0=\int T_{00}d\mathbf{r},\quad T_{00}=\frac{\partial\mathscr{L}_0}{\partial\dot{\varphi}}\dot{\varphi}+\frac{\partial\mathscr{L}_0}{\partial\dot{\varphi}^+}\dot{\varphi}^+-\mathscr{L},$$

in which, after substituting (18), the terms with $\lambda\neq\lambda'$ vanish by virtue of relation (7), which replaces the orthogonality condition. Comparison with (19) yields

$$|c_{\lambda^{(1)}}|^2=\frac{1}{2(\varepsilon_{\lambda^{(1)}}-V_{\lambda\lambda^{(1)}})},\quad |c_{\lambda^{(2)}}|^2=\frac{1}{2(\varepsilon_{\lambda^{(2)}}+V_{\lambda\lambda^{(2)}})}; \qquad (20)$$

where

$$V_{\lambda\lambda}^{(1)}=(\Psi_\lambda^{(1)}V\Psi_\lambda^{(1)}),\quad V_{\lambda\lambda}^{(2)}=(\Psi_\lambda^{(2)}V\Psi_\lambda^{(2)}).$$

We take the eigenfunctions of Eq. (6) with potential $V = V_1$ close to critical to be the system of basis functions, and consider the "dangerous" state. Then $\Psi_1^{(1)}\cong\Psi_1^{(2)}$, since $\epsilon_1^{(1)}\cong-\epsilon_1^{(2)}$ near the critical point. From Eq. (6) for $\Psi_1^{(1)}$ and $\Psi_1^{(2)}$ we obtain, after multiplying respectively by $\Psi_1^{(2)}$ and $\Psi_1^{(1)}$,

$$(\varepsilon_1^{(1)}-\varepsilon_1^{(2)})(\Psi_1^{(1)},\Psi_1^{(2)})=2(\Psi_1^{(1)}V\Psi_1^{(2)}). \qquad (21)$$

Assuming that $\Psi_1^{(1)}\cong\Psi_2^{(2)}$, we obtain

$$\tfrac{1}{2}(\omega_1-\omega_2)=\bar{V}, \qquad (21')$$

where $\epsilon_1=\epsilon_1^{(1)}$ and $\epsilon_2=\epsilon_1^{(2)}$. Substitution in (20) yields

$$|c_1^{(1)}|^2=|c_1^{(2)}|^2=1/(\omega_1+\omega_2). \qquad (20')$$

Thus, for the "dangerous" state the coefficients $c_1^{(1)}$ and $c_1^{(2)}$ tend to infinity near the critical point, whereas the remaining coefficients remain finite. Therefore near the critical potential, just as in the case of a scalar well, we are justified in retaining only one term in the expansion (18):

$$\hat{\varphi}=\frac{a+b^+}{\sqrt{\omega_1+\omega_2}}\Psi. \qquad (22)$$

We have omitted the identifying symbols of a, b^+, and Ψ.

B. We can improve the expansion (18) by choosing an optimal system of eigenfunctions such that the corrections to the main term (22) become minimal. To this end we write $\hat{\varphi}$ in the form

$$\hat{\varphi}=\hat{q}\Psi/\sqrt{2} \qquad (22')$$

and obtain an equation for Ψ from the exact equation for the operators $\hat{\varphi}$:

$$\Delta\hat{\varphi}+\left[-\left(\frac{\partial}{\partial t}+iV\right)^2-\mu^2\right]\hat{\varphi}-\lambda\hat{\varphi}^+\hat{\varphi}^2=0. \qquad (23)$$

We obtain the equation for Ψ from the condition

$$q_{01}\Psi=\hat{\varphi}_{01},\quad i\left(\frac{\partial\hat{\varphi}}{\partial t}\right)_{01}=([H,\hat{\varphi}])_{01}=\tilde{\omega}_1\hat{\varphi}_{01}.$$

The matrix elements are taken for the exact states, namely vacuum and vacuum plus one particle. Then Ψ is the exact (normalized) functions, and $\tilde{\omega}_1$ is the exact energy of one particle with allowance for the interaction. The equation for Ψ follows from (23):

$$\Delta\Psi+[(\tilde{\omega}_1-V)^2-\mu^2]\Psi-\lambda\Psi^3\frac{(q^+q^2)_{01}}{q_{01}}=0. \qquad (24)$$

Near the critical point at $\lambda\ll 1$, this equation differs little from Eq. (6) for $V\cong V_c$.

Equation (24) makes it possible to consider also the case of a field $|V|\gtrsim|V_c|$.

In order for the problem to be closed, it remains to diagonalize the Hamiltonian, which depends on $\hat{q}$ and $\dot{\hat{q}}$ and which we now obtain. It then becomes easy to obtain the matrix elements $(q^+q^2)_{01}$ and q_{01}.

C. Substituting expression (22′) in the density of the Lagrangian function (17) and integrating over the volume, we obtain

$$L=\int\mathscr{L}\,d\mathbf{r}=-(\mu^2+\overline{p^2}-\overline{V^2})\frac{\hat{q}^+\hat{q}}{2}+\frac{\dot{\hat{q}}^+\dot{\hat{q}}}{2}+\frac{i\bar{V}(\dot{\hat{q}}^+\hat{q}-\hat{q}^+\dot{\hat{q}})}{2}-\frac{\lambda_1}{4}(\hat{q}^+\hat{q})^2,$$

where the bar denotes averaging over Ψ,

$$\lambda_1=\frac{\lambda}{2}\int\Psi^4d\mathbf{r},\quad \overline{p^2}=\int(\nabla\Psi)^2d\mathbf{r}.$$

We introduce in place of q and q^+ the Hermitian operators q_1 and q_2:

$$q=q_1+iq_2,\quad q^+=q_1-iq_2;$$

The generalized momenta corresponding to q_1 and q_2 are

$$p_1=\frac{\partial L}{\partial\dot{q}_1}=\dot{q}_1-\bar{V}q_2,\quad p_2=\frac{\partial L}{\partial\dot{q}_2}=\dot{q}_2+\bar{V}q_1. \qquad (25)$$

We obtain the Hamiltonian

$$H=p_1\frac{\partial L}{\partial\dot{q}_1}+p_2\frac{\partial L}{\partial\dot{q}_2}-L=\frac{p_1^2+p_2^2}{2}+\frac{\omega_0^2(q_1^2+q_2^2)}{2}+\frac{\lambda_1(q_1^2+q_2^2)^2}{4}+\bar{V}(p_1q_2-p_2q_1). \qquad (26)$$

Here

$$\omega_0^2=\bar{V}^2-\overline{V^2}+\mu^2+\overline{p^2}. \qquad (27)$$

Near $V = V_c$ at $(|V|-|V_c|)/|V_c|\ll 1$ the function Ψ differs little from Ψ_c, and ω_0^2 can be written in the form

$$\omega_0^2=\bar{V}^2-\overline{V^2}-\bar{V}_c^2+\overline{V_c^2}\equiv\gamma(\nu-\nu_c),$$

where ν is a parameter proportional to the depth of the well.

Thus, the problem was reduced to that of a two-dimensional anharmonic oscillator with a potential energy that is independent of the angle. The stability of the problem is ensured by the fact that $\lambda > 0$. The energy of such an oscillator depends on two quantum numbers: the radial quantum number n and the angular momentum m relative to the axis perpendicular to the oscillator plane. It is easy to verify that

$$m=p_1q_2-p_2q_1=b^+b-a^+a=z,$$

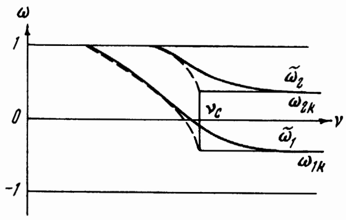


FIG. 5

i.e., the angular momentum m has the meaning of the total charge of the particle. The radial quantum number n has the meaning of the number of pairs.

D. The energy E(n, m) is determined in terms of the self-energy of the equation for the radial oscillator function

$$\chi'' + 2\left[E'(n,m) - U(\xi) - \frac{m^2 - {}^1/_4}{\xi^2}\right]\chi = 0,$$

namely,

$$E(n,m) = \bar{V}m + E'(n,m), \quad \xi = (q_1^2 + q_2^2)^{1/2},$$
$$U(\xi) = \omega_0^2\xi^2/2 + \lambda_1\xi^4/4.$$

In the quasiclassical approximation, which, as we have seen with the scalar well as an example, gives good accuracy[4], we have

$$\int_0^{\xi_1}\left\{2\left[E'(n,m) - U(\xi) - \frac{m^2}{\xi^2}\right]\right\}^{1/2} d\xi = (n + {}^1/_2)\pi. \tag{28}$$

Figure 5 shows the result of a quasiclassical calculation of the quantities $\tilde{\omega}_1$ and $\tilde{\omega}_2$, which play the role of single-particle energies for two signs of the charge:

$$\tilde{\omega}_1 = E(0,-1) - E(0,0), \quad \tilde{\omega}_2 = E(0,1) - E(0,0).$$

Let us find E(n, m) in explicit form for the transcritical region

$$-\omega_0^2 = \gamma(\nu_c - \nu) \gg \lambda^{2/3}.$$

In this case the potential energy $U(\xi)$ has a minimum at

$$\xi^2 = \xi_0^2 = -\omega_0^2/\lambda_1$$

and the problem reduces, as above, to the problem of the harmonic oscillator with frequency $\omega' = \sqrt{2|\omega_0^2|}$.

We obtain for E(n, m):

$$E(n,m) = U(\xi_0) + \frac{m^2}{\xi^2} + \left(n + \frac{1}{2}\right)\sqrt{2|\omega_0^2|} + \bar{V}m \tag{29}$$
$$= \bar{V}m - \frac{\omega_0^4}{4\lambda_1} + \frac{m^2}{|\omega_0^2|}\lambda_1 + \left(n + \frac{1}{2}\right)\sqrt{2|\omega_0^2|},$$

hence

$$\tilde{\omega}_1 = E(0,-1) - E(0,0) = \omega_{1c} + \frac{1}{\xi_0^2} = \bar{V} + \frac{\lambda_1}{|\omega_0^2|}, \tag{30}$$
$$\tilde{\omega}_2 = E(0,1) - E(0,0) = -\bar{V} + \frac{\lambda_1}{|\omega_0^2|}.$$

The pair energy does not depend on z = m, and is equal to

$$E(1,m) - E(0,m) = \sqrt{2|\omega_0^2|},$$

i.e., it increases with increasing distance from the critical point. Since we have put $\Psi = \Psi_c$, our results are bounded by the condition

$$(|V| - |V_c|)/|V_c| \ll 1. \tag{31}$$

Therefore, at our accuracy, $V = V_c$ in formulas (30). To get rid of the limitation (31), it is necessary to find Ψ from Eq. (24). It can be shown that at any V we have

$$\tilde{\omega}_1 + \tilde{\omega}_2 > 0, \quad \tilde{\omega}_1 > -1, \quad \tilde{\omega}_2 < 1.$$

3. Screening Fields

A. The main result obtained above reduces to the following:

In spite of the increase of the external field, the "dangerous" levels do not go through the critical value corresponding to the potential $V = V_c$. Consequently, the deepening of the well beyond the critical value is compensated for by the additional field resulting from the polarization of the vacuum.

Let us first find the screening field for the case of a scalar well. The equation for the field operators in the scalar well is analogous to (23) and takes the form

$$\Delta\hat{\varphi} + \left(-\frac{\partial^2}{\partial t^2} - \mu^2 - v\right)\hat{\varphi} - \lambda\hat{\varphi}^3 = 0. \tag{32}$$

Introducing, just as in the case of an electric field, $\hat{\varphi} = \hat{\varphi}\Psi$, we obtain for Ψ an equation similar to (24)

$$\Delta\Psi + (\tilde{\omega}^2 - \mu^2 - v)\Psi - \lambda\Psi^3(q^3)_{01}/q_{01} = 0. \tag{33}$$

The equation of the free particle is

$$\Delta\Psi_0 + (\omega_0^2 - \mu^2 - v)\Psi_0 = 0.$$

From the comparison we see that the role of the compensating field is played by the quantity

$$v_p(r) = -\lambda\Psi^2(r)(q^3)_{01}/q_{01}. \tag{34}$$

B. The matrix elements of the operators q^2 and q should be calculated from the eigenfunctions of the Hamiltonian of the anharmonic oscillator (11′). These functions can be approximately written in quasiclassical form. We confine ourselves to deriving an expression for v_p in the transcritical region and present an estimate for the case $v \cong v_c$.

At $\omega_0^2 = 0$ ($v \cong v_c$) we have

$$\lambda\langle q^4\rangle \sim \lambda_1^{1/3}, \quad \langle q^2\rangle \sim \lambda_1^{-1/3}.$$

Consequently

$$v_p \sim -\frac{\lambda^{2/3}}{I^{1/3}}\Psi^2(r), \quad I = \int \Psi^4 d\mathbf{r}.$$

Multiplying by Ψ^2 and integrating we obtain

$$\tilde{\omega}^2 \sim -\bar{v}_p \sim \lambda_1^{2/3},$$

in accord with the earlier result.

Let us consider now the particularly interesting region, when

$$|\bar{v}| - |\bar{v}_c| \gg \lambda_1^{2/3}.$$

In this case the operator q has a classical term $q_0^2 = -\omega_0^2/\lambda_1$. We write the matrix element $(q^3)_{01}$ in the form

$$(q^3)_{01} = (q^2)_{00}q_{01} + (q^2)_{01}q_{11} + (q^2)_{02}q_{21}.$$

By virtue of the symmetry of the problem $q_{11} = 0$. The last term is small compared with the first in the ratio $\lambda_1/|\bar{v} - \bar{v}_c|$. As a result, (34) yields

$$v_p(r) = -\lambda\Psi^2(r)q_{00}^2 = \frac{\omega_0^2}{I}\Psi^2(r) = \frac{\bar{v}_c - \bar{v}}{I}\Psi^2(r). \tag{35}$$

Thus, the screening field in the case of a narrow well arises in a region with radius $\sim 1/\mu$, which is much larger than the radius of the well. The screening field is practically independent of the interaction constant.

In the case of an electric field we have from (24)

$$v_p(r) = -\lambda\Psi^2(r)\,(q^+q^2)_{01}/q_{01}.$$

The state 1, which differs from the ground state only in m, we obtain

$$\frac{(q^+q^2)_{01}}{q_{01}} = (q_1^2+q_2^2)_{00} = -\frac{\omega_0^2}{\lambda_1}.$$

Substituting in v_p, we obtain

$$v_p = -\frac{|\omega_0^2|}{I}\Psi^2(r), \tag{36}$$

where ω_0^2 is given by expression (27).

C. Besides the screening field v_p, an electrically polarized field is also produced in the case of charged particles. Let us find the charge-density operator:

$$\hat{\rho} = \frac{1}{i}(\varphi^+\dot{\varphi} - \varphi\dot{\varphi}^+) - 2V\varphi^+\varphi.$$

Using $\varphi = q\Psi$ and the expressions in (25), we obtain

$$\hat{\rho} = z\Psi^2 + 2e(\bar{V} - V)\,(q_1^2+q_2^2)\Psi^2. \tag{37}$$

The charge density in the ground state (z = 0) differs from zero:

$$\rho_{00}(r) = 2e(\bar{V} - V(r))\,(\xi^2)_{00}\Psi^2.$$

In the transcritical region we obtain

$$\rho_{00}(r) = 2e(\bar{V} - V(r))\,\frac{|\omega_0^2|}{\lambda_1}\Psi^2(r). \tag{38}$$

The additional electric field is determined from the Poisson formula

$$\Delta V_1 = -4\pi\rho.$$

The electromagnetic interaction between the particles can be written in analogy with the hadron interaction in the form

$$H_e' = {}^1\!/_4\lambda_e(q^+q)^2, \tag{39}$$

where

$$\lambda_e = 4e^2\int\frac{[(\bar{V}-V)\Psi^2]_r[(\bar{V}-V)\Psi^2]_{r'}}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{r}\,d\mathbf{r}'. \tag{39'}$$

At $\lambda_e \ll \lambda$, which we assume, the screening field is determined by the hadron interaction.

Let us explain how the formula (39′) comes about. We note first that the electromagnetic interaction between the mesons can be assumed to be non-retarded. Indeed, the frequencies for the production of one or several pairs near $V = V_c$ is of the order of $\lambda_1^{1/3}$ (or $\lambda_e^{1/3}$), whereas the spatial gradients are proportional to $1/R$ or to μ. Therefore the square of the wave vector is $k^2 = \omega^2 - \mathbf{k}^2 = -\mathbf{k}^2$, and the D-function in the coordinate representation takes the form

$$D(x-x') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t').$$

In addition, the vector part of the current is

$$j_\alpha = \frac{1}{i}\left(\varphi^+\frac{\partial\varphi}{\partial x_\alpha} - \varphi\frac{\partial\varphi^+}{\partial x_\alpha}\right) = 0,$$

According to (22) and (22′). Therefore the exact formula for the interaction

$$H' = \frac{1}{2}\int j_i(x)D_{ik}(x-x')j_k(x')\,dx\,dx'$$

goes over into

$$H' = \frac{1}{2}\int\frac{\hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{r}\,d\mathbf{r}'. \tag{39''}$$

The first term in $\hat{\rho}$, as can be readily seen, results in small changes (on the order of e^2) in H_0, and we are left only with expression (39) with λ_e given by (39′).

IV. POSSIBLE PHYSICAL CONSEQUENCES OF REALIGNMENT OF VACUUM

The discussion in this section is preliminary in nature—a more detailed analysis will be presented in our next paper.

1. Phase Transformation of Nuclear Matter at High Density

We consider first the question of the possible phase transition in nuclear matter at high density. Such a phase transition could occur in neutron stars.

Nuclear matter produces a scalar well for mesons, whose depth in the gas approximation is equal to

$$v_0 = 4\pi nf \equiv 4\pi n(f_{\pi^+} + f_{\pi^-})/2,$$

where n is the nucleon density and f(k) the amplitude of zero-angle pion scattering.

It follows from (5) that under the condition

$$v_0 = 4\pi f(k)n > \varepsilon_c^2 = \mu^2 + k^2$$

mesons with zero energy appear, i.e., an instability of vacuum sets in. The instability arises for mesons with momentum k, at which f(k) has a maximum, and this yields $\epsilon_c \sim 2\mu$.

The results obtained in the preceding section pertained to the case of a narrow well, when an important role was played by one dangerous level. In the case of a broad well, a large number of levels approach immediately the critical value, but the problem of the meson field can be easily solved by confining oneself to the transcritical region, when a sufficiently strong classical meson field is produced. This field is determined by the energy minimum

$$E_{\rm cl}^{(\pi)} = \frac{\varepsilon_c^2 - 4\pi nf}{2}\varphi_{\rm cl}^2 + \frac{\lambda}{4}\varphi_{\rm cl}^4,$$

i.e., it equals

$$\varphi_{\rm cl}^2 = (4\pi nf - \varepsilon_c^2)/\lambda. \tag{40}$$

The corresponding energy is

$$E_{\rm cl}^{(\pi)} = -(4\pi nf - \varepsilon_c)^2/4\lambda. \tag{41}$$

At sufficiently large n, this energy will be much larger than the quantum part connected with the change of the zero-point oscillation energy.

Formula (41) is perfectly analogous to formula (16).

Thus, at $n = n_c = \epsilon_c^2/4\pi f$, a second-order phase transition takes place (a jump of dP/dn occurs, where $P = -dE/dV$ is the pressure). The new equation of state is determined by the expression

$$E = E^{(n)}(n) - (4\pi nf - \varepsilon_c^2)^2/4\lambda, \tag{42}$$

where $E^{(10)}$ is the energy density of the nucleons.

2. Possible Existence or Superdense Stars

The phase transition described above can apparently be realized in a finite system, i.e., in an ordinary nucleus. However, as we shall see, the superdense state, if it does exist, is separated from the usual one by a colassal energy barrier.

Let us consider a sufficiently heavy nucleus, when it is possible to neglect the term $(\nabla\varphi)^2 \sim \varphi^2/R^2$ in comparison with $4n\pi f\varphi^2$, and also neglect the surface effects ($A^3 \gg 1$, $R \gg 1/\mu$). Then the additional energy of the mesic field is

$$E^{(\pi)} = -\frac{(4\pi nf - \varepsilon_c^2)^2}{4\lambda}\mathcal{V} = -\frac{(4\pi nf - \varepsilon_c^2)^2}{4\lambda}A,$$

where $\mathcal{V}$ is the value of the nucleus. At $4\pi nf < \epsilon_c^2$ the value of $E^{(\pi)}$ is equal to zero ($\varphi_{cl} = 0$).

In order for a phase transition to occur it is necessary to go through a potential barrier whose height is determined by the condition

$$dE^{(\pi)}/dn + dE^{(n)}/dn = 0.$$

As an estimate for $E^{(n)}$ at high density ($n \gg n_{nuc}$) we can take the energy of a Fermi gas of nucleons

$$E^{(n)} \sim p_F^5 A/n_A \sim n^{2/3}A;$$

p_F is the Fermi momentum and A the number of particles. The order of magnitude of the barrier is determined by the expression

$$(4\pi nf - \varepsilon_c^2)/\lambda n \sim 1/n^{1/3},$$

or neglecting μ^2

$$n^{1/3} \sim \lambda/4\pi f, \quad 4\pi nf > \varepsilon_c^2.$$

The second condition is apparently more stringent and corresponds to $n \sim (3-4)n_{nuc}$. The height of the barrier at such values of n is

$$E_{max} \sim (n/n_{nuc})^{2/3}\varepsilon_F A.$$

An estimate of the penetrability of such a barrier yields $\ln P^{-1} \sim 20A^{4/3}$. Therefore if superdense nuclei do not appear during the process of element formation, they cannot occur in practice from ordinary nuclei without external actions. It is possible that among the heavy nuclei of cosmic rays there are superdense nuclei with a charge-to-mass ratio much different from that of ordinary nuclei. Who knows?

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