General Theory of Transport Processes in Strong Electric Fields

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A quantum transport equation is obtained in a linear approximation with respect to current-carrier concentration and in the one-band model. The equation resembles the kinetic equation, but the probabilities depend on the electric field strength. A new method for solving the equation is proposed, which is also valid for the usual kinetic equation. The method is employed to analyze a number of nontrivial physical situations leading to interesting effects in a strong electric field (negative differential conductivity, nonmonotonic current behavior, nonanalytic dependence of current on field strength).

THE calculation of the electric conductivity σ in a strong electric field $\mathbf{E}$ was based until most recently on solution of the Boltzmann kinetic equation by the method proposed by Davydov back in 1937[1]. On the other hand, in the case of a weak electric field, a convenient mathematical formalism was constructed for the calculation of the kinetic coefficients, suitable also in those cases when the kinetic equation is not applicable. The general expression for σ is given by the Kubo formula[2]. Among the calculation methods, special notice should be taken of the Konstantinov and Perel' graphic technique[3] (the KP method), which could be generalized to include the case of strong electron-phonon coupling[4,5].

The number of investigations devoted to the general theory of electric conductivity of semiconductors in a strong electric field, not based on the kinetic equation, is relatively small. They can be broken up arbitrarily into five groups.

1. Investigations in which the density-matrix method is used to derive the usual kinetic equation in a strong classical electric field (see, for example,[6,7]).

2. Investigations in which a new transport equation in crossed electric (strong) and magnetic (quantizing) fields is obtained[8,9] with a weak electron-phonon coupling. Inasmuch as the quantum-mechanical problem of the behavior of a free electron in crossed electric and magnetic fields can be solved accurately, this uncovers, in principle, a possibility of taking into account the influence of the electric field on the collision act. Incidentally, this effect was not taken into account in[8,9].

3. Investigations in which closed equations are obtained for the current $\mathbf{j}(\mathbf{E})$ in a strong electric field[10,11]. We note that this is only the initial stage in the solution of the problem. To decipher such formulas it is necessary to produce special computation methods even more perfect than those for the deciphering of the Kubo formula.

4. Investigations devoted to the derivation of generalized equations (i.e., equations valid for an arbitrary force of interaction with the scatterers) for the nonequilibrium distribution functions[12-15], in which account is taken of the influence of the electric field on the scattering, but the Coulomb correlations between the carriers are neglected. The following well-known kinetics methods are generalized to include the case of a strong electric field: the Green's function method in[12], the Wigner density matrix method in[13], and the KP method in[14,15].

5. Investigations in which the equation for the transport in the plasma is obtained with allowance for the influence of the electric field on the interaction between the carriers (see, for example,[16], and from among the later papers[17]). For a classical plasma this can be done because of certain properties of the Liouville operator in an external field. For electrons in a solid, analogous possibilities are uncovered because of the Houston theorem[18].

The present paper should be assigned to the fourth group. Among the papers of this group, special notice should be taken of the paper by L. Keldysh[12], where the most general approach, suitable for arbitrary statistics and (formally) for an arbitrary interaction force with the phonons, has been proposed. In this case the nonequilibrium distribution function $f(\mathbf{k})$ must be determined from Dyson's equation, which in the case of the Fermi statistics turns out to be nonlinear. In this case the usual analogy with the transport equation is completely lost (with the exception, of course, of the case when the field can be regarded as quasiclassical and the scattering as weak). Levinson[13] reconstructed this analogy formally, but only in the lowest order in the interaction with the phonons. The influence of the electric field on the scattering has been taken into account in the transition probabilities in[13] (which were calculated in the lowest Born approximation), but no criterion was indicated for the electric field strength at which this influence becomes significant, and it is not indicated in which concrete phenomena this influence can appear.

We consider the case of low carrier density n, when the correlation effects can be neglected and it can be assumed that there is no degeneracy (Boltzmann statistics). This makes it possible to perform all the calculations in the lowest order in n. In addition, we take into account the finite width of the allowed band ΔE, which greatly facilitates the calculations (the calculations in[12,13] correspond to $\Delta E \to \infty$). We impose no limitations whatever on the force of the coupling with the phonons. Thus, our analysis has a more general character than in[13], and is less general than in[12]. In[14,15], the expression for $\mathbf{j}(\mathbf{E})$ was represented in the following form:

$$j_x = en \sum_{\mathbf{k}_\perp'} n(\mathbf{k}_\perp') \sum_{X_m,\, \mathbf{k}_\perp} X_m W_{0,\, X_m}^{0,\, X_m}(\mathbf{k}_\perp', \mathbf{k}_\perp). \qquad (1)$$

Here X_m is the projection on the x axis ($\mathbf{E} \parallel \mathbf{x}$) of the

radius vector of the lattice point $\mathbf{R_m}$. The quantity $\widetilde{W}$ (which we call the effective transition probability) has the dimension sec^{-1} and satisfies the equation[15]

$$\widetilde{W}^{X_{m_1}, X_{m_3}}_{X_{m_2}, X_{m_4}}(\mathbf{k}_\perp', \mathbf{k}_\perp) = W^{X_{m_1}, X_{m_3}}_{X_{m_2}, X_{m_4}}(\mathbf{k}_\perp', \mathbf{k}_\perp) + \sum_{\mathbf{k}_\perp''} \sum_{X_{m_5} \neq X_{m_6}} \widetilde{W}^{X_{m_1}, X_{m_5}}_{X_{m_2}, X_{m_6}}(\mathbf{k}_\perp', \mathbf{k}_\perp'') \frac{\hbar}{ieE(X_{m_6} - X_{m_5})} W^{X_{m_5}, X_{m_3}}_{X_{m_5}, X_{m_4}}(\mathbf{k}_\perp'', \mathbf{k}_\perp), \tag{2}$$

where W is a certain probability (in the general case, not diagonal over the sites) of the transition between the states of the Stark ladder, the graphic recipe for determination of which (for both strong and weak coupling) is given in[14,15]. The function $n(\mathbf{k}_\perp)$ describes the distribution over the transverse momentum $\mathbf{k}_\perp$. It is normalized to unity:

$$\sum_{\mathbf{k}_\perp} n(\mathbf{k}_\perp) = 1,$$

and satisfies an integral equation of the type[14,15]

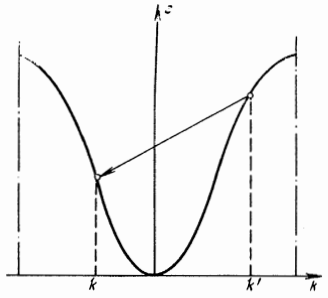
$$\sum_{\mathbf{k}_\perp'} n(\mathbf{k}_\perp') \widetilde{W}(\mathbf{k}_\perp', \mathbf{k}_\perp) = 0, \quad \widetilde{W}(\mathbf{k}_\perp', \mathbf{k}_\perp) \equiv \sum_{X_m} \widetilde{W}^{0, X_m}_{0, X_m}(\mathbf{k}_\perp', \mathbf{k}_\perp). \tag{3}$$

The solution of (2) can be represented formally in the form of a series in powers of W/Ω_E, where $\Omega_E = eEa/\hbar$ (a is the lattice constant). At $W/\Omega_E \ll 1$, the corresponding expansion can be terminated, and the formula obtained thereby for $j(E)$ is somewhat reminescent of the Titeica formula, which describes Umklapp in a strong magnetic field. We present below some new physical results that follow from this formula. In the case of very narrow allowed bands ($\Delta E/kT \ll 1$, where ΔE is the width of the allowed band), we can put $n(\mathbf{k}_\perp) \approx 1$. Then formula (1) becomes much simpler, and Eq. (2) can be solved exactly for arbitrary E. The corresponding results (as applied to the case of a small-radius polaron) are given in[14]. We shall show that in the most general case formula (1), with allowance for relations (2) and (3), is equivalent to a solution with the aid of a certain transport equation, in which the probabilities depend on the electric field (which reflects the influence of the electric field on the scattering act). It is valid for any coupling force with the phonons and at an arbitrary band width. In particular, it is possible to obtain from it all the results given in[14] for small-radius polarons, and also certain new results in the case of a strong electron-phonon coupling.

In the case of a weak electron-phonon coupling, we consider the following concrete situations, which are not described by the Boltzmann equation.

1. Superstrong "quantizing" fields, when $\Delta E/\hbar\Omega_E$ is not too large (i.e., the width of the band spans a small number of Stark levels[19]), but $\Omega_E\tau \gg 1$ (i.e., the levels of the Stark ladder are not smeared out). In this case the experimental current $j(E)$ may experience strong nonmonotonicities in E. This effect was predicted in[14] and was observed independently in[20].

2. Strong (but not necessarily "quantizing") electric fields in semiconductors with narrow allowed bands. For example, if $\Delta E < \hbar\omega_0$ (ω_0 is the frequency of the optical phonons), then single-phonon processes are forbidden as $E \to 0$, and the kinetics is determined by weak two-phonon scattering. However, with increasing E, the single-phonon processes become allowed because of tunneling through the quasiclassical barrier in the strong electric field (in this case the δ functions describing the energy conservation laws become smeared out). The dependence of the corresponding contribution to the current $j(E)$ takes the form $E^{3/2}\exp(-E_0/E)$. We were the first to obtain these results, although they can be determined from the transport equation proposed in[13].



Motion of electron in the Brillouin zone in the presence of an electric field. The arrow shows the transition following a collision with a phonon. The dash-dot lines limit the first Brillouin zone.

We investigate also the case of a classically strong electric field $\Omega_E\tau \gg 1$, $\Delta E/\hbar\Omega_E \to \infty$, which can be considered with the aid of the ordinary kinetic equation, but we shall solve it by another method, iterating in powers of W/Ω_E. We then arrive at the conclusion that the differential conductivity is negative.

1. NEW FORMULA FOR THE CURRENT IN THE CASE OF WEAK COUPLING AND CLASSICAL ELECTRIC FIELDS ($\Delta E/\hbar\Omega_E \to \infty$)

We shall show below how to obtain from the kinetic equation, for arbitrary $\Omega_E\tau$, a formula of the type (1) for $j(E)$. We present first certain qualitative considerations concerning the form of $j(E)$ when $\Omega_E\tau \gg 1$.

Let us consider for simplicity the case of one-dimensional motion, and then let us generalize the result to the three-dimensional case. The figure shows the dependence of the electron energy ϵ on k within the limits of the first Brillouin zone. At $E \neq 0$, in the absence of scattering ($\Omega_E\tau \to \infty$), the wave vector varies in time like $k(t) = k(0) + eEt/\hbar$, and the velocity v is a periodic function of the time:

$$v = \frac{1}{\hbar}\frac{d\varepsilon}{dk} = \frac{1}{eE}\frac{d\varepsilon[k(0) + eEt/\hbar]}{dt}.$$

Thus, as $\Omega_E\tau \to \infty$ the electron executes oscillations in coordinate (and momentum) space with frequency Ω_E, by virtue of which the average current is equal to zero. If an electron having a momentum $\hbar k'$ collides with a phonon at the instant of time T and changes its momentum jumpwise ($k' \to k$), then at $t > T$ it will again start to move periodically. By the time $T + \Delta t$, when its momentum has again become equal to $\hbar k'$, it has moved in coordinate space through a distance Δx:

$$\Delta x(k'\ k) = \int_T^{T+\Delta t} v(t)\,dt = \int_k^{k'} dk'' \frac{dt}{dk''}\frac{1}{\hbar}\frac{d\varepsilon(k'')}{dk''} = \frac{\varepsilon(k') - \varepsilon(k)}{eE}. \tag{4}$$

Multiplying Δx by the collision probability $W(k', k)$, summing over k, and averaging over k', we obtain the drift velocity $\bar{v}$, where $j = en\bar{v}$:

$$j_x = en\sum_{\mathbf{k},\mathbf{k}'} n(\mathbf{k}_\perp')\Delta x(\mathbf{k}', \mathbf{k}) W(\mathbf{k}', \mathbf{k}) = \frac{n}{E}\sum_{\mathbf{k}',\mathbf{k}} n(\mathbf{k}_\perp')[\varepsilon(\mathbf{k}') - \varepsilon(\mathbf{k})] W(\mathbf{k}', \mathbf{k}). \tag{5}$$

Differentiating (11) with respect to k_x with allowance for (15) and (13), we obtain

$$\frac{\partial f(\mathbf{k})}{\partial k_x} = \frac{\hbar}{eE}\frac{1}{\Lambda}\int_{(A)} d^3k' n(\mathbf{k}_\perp')\widetilde{W}(\mathbf{k}', \mathbf{k}). \tag{18}$$

According to (11) and (14) we have

$$f(\mathbf{k}+\mathbf{G}) = f(\mathbf{k}),\ \widetilde{W}(\mathbf{k}'+\mathbf{G}',\ \mathbf{k}+\mathbf{G}) = \widetilde{W}(\mathbf{k}', \mathbf{k}),\ n(\mathbf{k}_\perp+\mathbf{G}_\perp) = n(\mathbf{k}_\perp).$$

We change over in (17) to integration with respect to $\mathbf{k}$ only in the first Brillouin zone. Substituting (18) in (17), we obtain ultimately

$$j_x = \frac{n}{E}\frac{v_a^2}{(2\pi)^6}\int d^3k \int d^3k' n(\mathbf{k}_\perp')[\varepsilon(\mathbf{k}')-\varepsilon(\mathbf{k})]\widetilde{W}(\mathbf{k}', \mathbf{k}). \tag{19}$$

We have taken into account here the fact that, according to (14), $\int d^3k\widetilde{W}(\mathbf{k}', \mathbf{k}) = 0$. This has made it possible to add in (19) a term containing $\epsilon(\mathbf{k}')$ and identically equal to zero. However, such a way of writing the equation is convenient, since it makes it possible to omit the contributions made to $\widetilde{W}$ by the departure terms.

Thus, (19) coincides with (5) apart from the substitution $W \to \widetilde{W}$. If the integral term in (16) is small (i.e., $\Omega_E\tau > 1$), then the first iteration is $\widetilde{W}^{(1)} = W$. The next iteration term makes a contribution from the processes with two successive collisions during the period of the oscillation in the electric field, etc. We present below several concrete examples of the use of the formulas obtained in the present section.

2. QUANTUM TRANSPORT EQUATION IN ARBITRARY ELECTRIC FIELDS

In the preceding section it was shown that in nonquantizing electric fields and in the case of weak coupling with the phonons, a formula of the type (19) and the traditional expression for the current (17) are equivalent. In the present section we shall show that the expression (1) for the current in an arbitrary field $\mathbf{E}$ and at an arbitrary force of interaction with the phonons can be written in the form

$$j_x(\mathbf{E}) = en\sum_{\mathbf{k}} f(\mathbf{k})\, v_x^{\text{eff}}. \tag{20}$$

The summation over $\mathbf{k}$ is carried out here within the limits of the first Brillouin zone, and $f(\mathbf{k})$ is a distribution function normalized to unity and satisfying the quantum transport equation

$$\frac{eE}{\hbar}\frac{\partial f(\mathbf{k})}{\partial k_x} = \sum_{\mathbf{k}'}[W(\mathbf{k}', \mathbf{k})f(\mathbf{k}') - f(\mathbf{k})W(\mathbf{k}, \mathbf{k}')]. \tag{21}$$

The transition probabilities $W(\mathbf{k}', \mathbf{k})$ are defined as follows:

$$W(\mathbf{k}', \mathbf{k}) = \lim_{\varkappa\to 0} W(\mathbf{k}', \mathbf{k}, \varkappa),$$

$$W(\mathbf{k}', \mathbf{k}, \varkappa) = \frac{1}{N_x^2}\sum_{X_m} W^{X_{m_1}, X_{m_3}}_{X_{m_2}, X_{m_4}}(\mathbf{k}_\perp', \mathbf{k}_\perp)\exp\{ik_x'(X_{m_1}-X_{m_3}) - ik_x \times (X_{m_3}-X_{m_4}) - i\frac{\varkappa_x}{2}(X_{m_1}+X_{m_2}-X_{m_3}-X_{m_4}) - i\chi\left(\mathbf{k}'+\frac{\varkappa}{2}\right) \tag{22}$$

$$+ i\chi\left(\mathbf{k}'-\frac{\varkappa}{2}\right) + i\chi\left(\mathbf{k}+\frac{\varkappa}{2}\right) - i\chi\left(\mathbf{k}-\frac{\varkappa}{2}\right)\Big\}. \tag{23}$$

The quantity $v_x^{\text{eff}}(\mathbf{k})$ has the dimension of velocity and is equal to

$$v_x^{\text{eff}}(\mathbf{k}) = \frac{1}{\hbar}\frac{\partial\varepsilon(\mathbf{k})}{\partial k_x} - i\sum_{\mathbf{k}'} W_1(\mathbf{k}, \mathbf{k}'),\quad W_1(\mathbf{k}, \mathbf{k}') = \lim_{\varkappa\to 0}\frac{\partial}{\partial\varkappa_x}W(\mathbf{k}, \mathbf{k}', \varkappa); \tag{24}$$

N_X is the normalization number for the summation over X_m, which of the order of the number of atoms in the crystal along the x axis[15], and $\chi(\mathbf{k})$ is[15]

$$eE\frac{\partial\chi(\mathbf{k})}{\partial k_x} = \varepsilon(\mathbf{k}) - \varepsilon(\mathbf{k}_\perp),\quad \varepsilon(\mathbf{k}_\perp) = \frac{1}{N_x}\sum_{k_x}\varepsilon(\mathbf{k}). \tag{25}$$

The probabilities $W^{X_{m_1}, X_{m_3}}_{X_{m_2}, X_{m_4}}(\mathbf{k}_\perp', \mathbf{k}_\perp)$ were defined in[15], where a graphic recipe was given for the calculation for both weak coupling with the phonons and for the model of small-radius polarons (see the Appendices in[15]). We present here an expression only for the diagonal probability

$$W^{X_{m'}, X_m}_{X_{m'}, X_m}(\mathbf{k}_\perp', \mathbf{k}_\perp) = \frac{2\pi}{\hbar}\sum_{\mathbf{q}}|c_{\mathbf{q}}|^2\{\delta_{\mathbf{k}_\perp', \mathbf{k}_\perp-\mathbf{q}_\perp}[(1+N_{\mathbf{q}})\Pi^+(\mathbf{q}, \mathbf{k}_\perp', X_m - X_{m'}) + N_{\mathbf{q}}\Pi^-(\mathbf{q}, \mathbf{k}_\perp', X_m - X_{m'})] - \delta_{X_{m'}, X_m}\delta_{\mathbf{k}_\perp', \mathbf{k}_\perp}\sum_{X_{m''}}[(1+N_{\mathbf{q}})\Pi^+ \times (\mathbf{q}, \mathbf{k}_\perp', X_{m''}) + N_{\mathbf{q}}\Pi^-(\mathbf{q}, \mathbf{k}_\perp', X_{m''})]\}, \tag{26}$$

in which

$$\Pi^\pm(\mathbf{q}, \mathbf{k}_\perp, X_m) = \left|\frac{1}{N_x}\sum_{k_x}\Theta(\mathbf{k}, \mathbf{k}+\mathbf{q}, X_m)\right|^2\delta(\varepsilon(\mathbf{k}_\perp+\mathbf{q}_\perp) - \varepsilon(\mathbf{k}_\perp) - eEX_m \pm\hbar\omega_{\mathbf{q}}),$$

$$\Theta(\mathbf{k}, \mathbf{k}', X_m) = \exp\{-ik_xX_m + i\chi(\mathbf{k}) - i\chi(\mathbf{k}')\}. \tag{27}$$

Equation (23) is outwardly similar to the kinetic equation (6), but does not coincide with it, since $W(\mathbf{k}', \mathbf{k})$ take into account the influence of the electric field on the collision act. Only at not too strong a field and in the first Born approximation in the scattering does (21) coincide with (6) (see below).

We go over to the $\mathbf{k}$ representation in the expression for the current (1) with the aid of (23), which obtains in the case of homogeneous space (compare with (6) in[24]), when the following relation is satisfied:

$$W^{X_{m_1}+X_m,\ X_{m_3}+X_m}_{X_{m_2}+X_m,\ X_{m_4}+X_m}(\mathbf{k}_\perp', \mathbf{k}_\perp) = W^{X_{m_1},\ X_{m_3}}_{X_{m_2},\ X_{m_4}}(\mathbf{k}_\perp', \mathbf{k}_\perp). \tag{28}$$

After simple algebraic transformations we obtain

$$j_x^{(1)} = -\frac{en}{N_x}\sum_{\mathbf{k}, \mathbf{k}'} n(\mathbf{k}_\perp')\widetilde{W}(\mathbf{k}', \mathbf{k})\frac{\varepsilon(\mathbf{k})}{eE},\quad j_x^{(2)} = -i\frac{en}{N_x}\sum_{\mathbf{k}, \mathbf{k}'} n(\mathbf{k}_\perp')\widetilde{W}_1(\mathbf{k}', \mathbf{k}), \tag{29}$$

with $j_X = j_X^{(1)} + j_X^{(2)}$. In the derivation of (29) we used sum rules for $\widetilde{W}$ in the $\mathbf{k}$ representation:

$$\sum_{\mathbf{k}}\widetilde{W}(\mathbf{k}', \mathbf{k}) = 0,\quad \sum_{\mathbf{k}', k_x} n(\mathbf{k}_\perp')\widetilde{W}(\mathbf{k}', \mathbf{k}) = 0. \tag{30}$$

Formulas (30) coincide with (3) of the present paper and (35) of[15], written out in the $\mathbf{k}$ representation. We change over likewise to the $\mathbf{k}$ representation in Eq. (2) for $\widetilde{W}$:

$$\widetilde{W}(\mathbf{k}', \mathbf{k}, \varkappa) = W(\mathbf{k}', \mathbf{k}, \varkappa) + \sum_{\mathbf{k}''}\widetilde{F}(\mathbf{k}', \mathbf{k}'', \varkappa)W(\mathbf{k}'', \mathbf{k}, \varkappa); \tag{31}$$

Here

$$\widetilde{F}(\mathbf{k}', \mathbf{k}, \varkappa) = \sum_{\mathbf{k}''}\widetilde{W}(\mathbf{k}', \mathbf{k}'', \varkappa)\delta_{\mathbf{k}_\perp'', \mathbf{k}_\perp}\frac{1}{N_x}\sum_{X_m\neq 0}\frac{\hbar}{ieEX_m}\exp\Big\{i(k_x - k_x'')X_m - i\chi\left(\mathbf{k}''+\frac{\varkappa}{2}\right) + i\chi\left(\mathbf{k}''-\frac{\varkappa}{2}\right) + i\chi\left(\mathbf{k}+\frac{\varkappa}{2}\right) - i\chi\left(\mathbf{k}-\frac{\varkappa}{2}\right)\Big\}. \tag{32}$$

It is now convenient to introduce the function $f(\mathbf{k})$ in accordance with the equation

$$f(\mathbf{k}) = N_x^{-1}\left[n(\mathbf{k}_\perp) + \sum_{\mathbf{k}'} n(\mathbf{k}_\perp')\widetilde{F}(\mathbf{k}', \mathbf{k})\right], \tag{33}$$

$$\widetilde{F}(\mathbf{k}', \mathbf{k}) = \lim_{\varkappa\to 0}\widetilde{F}(\mathbf{k}', \mathbf{k}, \varkappa).$$

The function $n(\mathbf{k}_\perp)$ introduced here is a reflection of the specific character of the three-dimensional case (in the one-dimensional case $n(\mathbf{k}_\perp) = 1$) and describes the distribution with respect to the momentum perpendicular to the field, i.e., $n(\mathbf{k}_\perp)$ is the distribution function $f(\mathbf{k})$ averaged over $k_x(\mathbf{E} \parallel \mathbf{x})$:

$$n(\mathbf{k}_\perp) = \sum_{k_x} f(\mathbf{k}).$$

It is possible to take into account analogously the probability of successive (in time) collisions of an electron with two phonons within one period of oscillations in a field, etc. This would lead to a renormalization of the probability W, i.e., to a substitution $W \to \widetilde{W}$ (see below). However, when $\Omega_E \tau \gg 1$, we can confine ourselves to allowance for only one collision per period, and use formula (5). The case of arbitrary $\Omega_E \tau$ is more conveniently considered with the aid of the kinetic equation

$$\frac{eE}{\hbar}\frac{\partial f(\mathbf{k})}{\partial k_x} = \frac{1}{\Lambda}\int_{(\Lambda)} d^3k' f(\mathbf{k}') W(\mathbf{k}', \mathbf{k}). \tag{6}$$

The probabilities W include here both the arrival terms and the departure terms, i.e., in the usual notation

$$W(\mathbf{k}', \mathbf{k}) \to W(\mathbf{k}', \mathbf{k}) - \delta_{\mathbf{k}'\mathbf{k}} \sum_{\mathbf{k}''} W(\mathbf{k}, \mathbf{k}'').$$

If $\mathbf{E}$ makes an irrational angle with the crystallographic axes, then in the absence of scattering the trajectory of the electron's motion in the first Brillouin zone does not close, and its motion is quasiperiodic. When the direction of $\mathbf{E}$ coincides with a high-symmetry axis of the crystal, the electron motion becomes periodic. In the general case it is convenient to consider the motion of the electron in complete $\mathbf{k}$ space with a normalization volume Λ, without confining ourselves to the first Brillouin zone. This makes it possible to find a formula for the current at an arbitrary orientation of $\mathbf{E}$ relative to the crystallographic axes. Indeed, for any rational angle between $\mathbf{E}$ and the crystallographic axes there exists a finite region in $\mathbf{k}$ space, within which the electron executes periodic motion along x with a period K_x. We choose in $\mathbf{k}$ space a normalization length Λ_x in the x direction, equal to an integer number of periods K_x. This makes it possible to formulate boundary conditions for the differential equation (6), which we choose to be a condition of cyclicity in x:

$$f(k_x + \Lambda_x, \mathbf{k}_\perp) = f(k_x, \mathbf{k}_\perp). \tag{7}$$

When solving the kinetic equation under ordinary conditions, when the moving electron does not reach the edge of the Brillouin zone, we use as the boundary condition $f(\epsilon) \to 0$ as $\epsilon \to \infty$. The condition (7) is a generalization to the case of arbitrary motion in the zone. In the next section this condition will actually be derived.

Any irrational direction of $\mathbf{E}$ can be obtained by means of an infinitesimally small rotation relative to a certain rational direction. It is clear that in the presence of scattering, such a rotation should not change $f(\mathbf{k})$ jumpwise. Therefore any irrational direction of $\mathbf{E}$ can be regarded as the limit of a sequence of rational directions, by the same token generalizing (7) to the case of arbitrary angles. In all the final results it is necessary to make the transition $\Lambda_x \to \infty$.

We rewrite Eq. (6) in integral form

$$f(\mathbf{k}) = n(\mathbf{k}_\perp) + \frac{1}{\Lambda}\int_{(\Lambda)} d^3k' f(\mathbf{k}') F(\mathbf{k}', \mathbf{k}) + \frac{\hbar}{eE} k_x \int_{(\Lambda)} d^3k' f(\mathbf{k}') W(\mathbf{k}', \mathbf{k}_\perp). \tag{8}$$

Here

$$W(\mathbf{k}', \mathbf{k}_\perp) = \frac{1}{\Lambda_x}\int_{(\Lambda_x)} dk_x W(\mathbf{k}', \mathbf{k}),$$

and $F(\mathbf{k}', \mathbf{k})$ is defined by the equations

$$\frac{\partial F(\mathbf{k}', \mathbf{k})}{\partial k_x} = \frac{\hbar}{eE}\{W(\mathbf{k}', \mathbf{k}) - W(\mathbf{k}', \mathbf{k}_\perp)\}, \quad \int_{(\Lambda_x)} F(\mathbf{k}', \mathbf{k})\, dk_x = 0. \tag{9}$$

Solving (9), we obtain

$$F(\mathbf{k}', \mathbf{k}) = \frac{\hbar}{eE}\Big\{\int_{-\Lambda_x/2}^{k_x} dk_x''[W(\mathbf{k}'; k_x'', \mathbf{k}_\perp) - W(\mathbf{k}', \mathbf{k}_\perp)] + \frac{1}{\Lambda_x}\int_{(\Lambda_x)} dk_x'' k_x''[W(\mathbf{k}'; k_x'', \mathbf{k}_\perp) - W(\mathbf{k}', \mathbf{k}_\perp)]\Big\}. \tag{9a}$$

It is easy to verify that the function $F(\mathbf{k}', \mathbf{k})$ defined in accordance with (9) and (9a) is periodic: $F(\mathbf{k}' + \mathbf{G}', \mathbf{k} + \mathbf{G}) = F(\mathbf{k}', \mathbf{k})$ ($\mathbf{G}$ and $\mathbf{G}'$ are the reciprocal-lattice vectors). Since Λ_x is one of the reciprocal-lattice vectors ($\Lambda_x = G_x^{(0)}$, $G_\perp^{(0)} = 0$), we get, in particular, $F(\mathbf{k}'; k_x + \Lambda_x, \mathbf{k}_\perp) = F(\mathbf{k}', \mathbf{k})$. The condition of periodicity of (7) requires that the last term in the right-hand side of (8) vanish:

$$\int_{(\Lambda)} d^3k' f(\mathbf{k}') W(\mathbf{k}', \mathbf{k}_\perp) = 0. \tag{10}$$

Relation (10) makes it possible to determine the function $n(\mathbf{k}_\perp)$.

We can now write the formal solution of Eq. (8):

$$f(\mathbf{k}) = n(\mathbf{k}_\perp) + \frac{1}{\Lambda}\int_{(\Lambda)} d^3k' n(\mathbf{k}_\perp') \widetilde{F}(\mathbf{k}', \mathbf{k}). \tag{11}$$

The function $\widetilde{F}(\mathbf{k}', \mathbf{k})$ is determined from the equation

$$\widetilde{F}(\mathbf{k}', \mathbf{k}) = F(\mathbf{k}', \mathbf{k}) + \frac{1}{\Lambda}\int_{(\Lambda)} d^3k'' \widetilde{F}(\mathbf{k}', \mathbf{k}'') F(\mathbf{k}'', \mathbf{k}) = F(\mathbf{k}', \mathbf{k}) + \frac{1}{\Lambda}\int_{(\Lambda)} d^3k'' F(\mathbf{k}', \mathbf{k}'') \widetilde{F}(\mathbf{k}'', \mathbf{k}). \tag{12}$$

To determine $n(\mathbf{k}_\perp)$ we substitute the solution (11) for $f(\mathbf{k})$ into relation (10). As a result we obtain an equation for $n(\mathbf{k}_\perp)$:

$$\int_{(\Lambda)} d^3k' \int_{(\Lambda_x)} dk_x n(\mathbf{k}_\perp') \widetilde{W}(\mathbf{k}', \mathbf{k}) = 0, \tag{13}$$

$$\widetilde{W}(\mathbf{k}', \mathbf{k}) = W(\mathbf{k}', \mathbf{k}) + \frac{1}{\Lambda}\int_{(\Lambda)} d^3k'' \widetilde{F}(\mathbf{k}', \mathbf{k}'') W(\mathbf{k}'', \mathbf{k}). \tag{14}$$

Comparing (14) with Eq. (12) differentiated with respect to k_x, we have

$$\frac{\partial \widetilde{F}(\mathbf{k}', \mathbf{k})}{\partial k_x} = \frac{\hbar}{eE}[\widetilde{W}(\mathbf{k}', \mathbf{k}) - \widetilde{W}(\mathbf{k}', \mathbf{k}_\perp)], \quad \int_{(\Lambda_x)} dk_x \widetilde{F}(\mathbf{k}', \mathbf{k}) = 0, \tag{15}$$

i.e., $\widetilde{F}$ is expressed in terms of $\widetilde{W}$ in exactly the same manner as F in terms of W (see (9a)). In addition, $\widetilde{W}$ satisfies the equation

$$\widetilde{W}(\mathbf{k}', \mathbf{k}) = W(\mathbf{k}', \mathbf{k}) + \frac{1}{\Lambda}\int_{(\Lambda)} d^3k'' F(\mathbf{k}', \mathbf{k}'') \widetilde{W}(\mathbf{k}'', \mathbf{k}). \tag{16}$$

Formulas (11)–(14) give the exact solution of the kinetic equation. It is possible to express with their aid the current in terms of $\widetilde{W}$:

$$j_x = \frac{en}{\hbar}\frac{1}{\Lambda}\int_{(\Lambda)} d^3k f(\mathbf{k}) \frac{\partial \varepsilon(\mathbf{k})}{\partial k_x} = -\frac{en}{\hbar}\frac{1}{\Lambda}\int_{(\Lambda)} d^3k\, \varepsilon(\mathbf{k}) \frac{\partial f(\mathbf{k})}{\partial k}. \tag{17}$$

From the definition (33) it follows that

$$\sum_{k_x} f(\mathbf{k}) = n(\mathbf{k}_\perp), \quad \sum_{\mathbf{k}} f(\mathbf{k}) = 1,$$

since, according to (32),

$$\sum_{k_x} F(\mathbf{k}', \mathbf{k}) = 0.$$

We shall now show that the function f(**k**) introduced by us actually satisfies the quantum transport equation (21), i.e., we can assign to it the meaning of a distribution function. To this end we multiply both parts of (33) by W(**k**, **k**″) and sum over **k**. Using (31), we obtain

$$\sum_{\mathbf{k}} f(\mathbf{k}) W(\mathbf{k}, \mathbf{k}'') = \frac{1}{N_x} \sum_{\mathbf{k}} n(\mathbf{k}_\perp) \bar{W}(\mathbf{k}, \mathbf{k}''). \tag{34}$$

On the other hand, differentiating (33) with respect to k_x and recognizing that

$$\frac{\partial}{\partial k_x} F(\mathbf{k}', \mathbf{k}) = \frac{\hbar}{eE} \left\{ \bar{W}(\mathbf{k}', \mathbf{k}) - \frac{1}{N_x} \sum_{k_x} \bar{W}(\mathbf{k}', \mathbf{k}) \right\},$$

we get

$$\frac{\partial f(\mathbf{k})}{\partial k_x} = \frac{\hbar}{eE} \frac{1}{N_x} \sum_{\mathbf{k}'} n(\mathbf{k}_\perp') \bar{W}(\mathbf{k}', \mathbf{k}). \tag{35}$$

Combining (34) and (35) and taking into account the sum rule

$$\sum_{\mathbf{k}} W(\mathbf{k}', \mathbf{k}) = 0,$$

which follows from (31), we arrive at the sought equation (21) for f(**k**).

We can now express the current in terms of f(**k**). According to (29) and (34)

$$j_x^{(1)} = -\frac{en}{\hbar} \sum_{\mathbf{k}} \varepsilon(\mathbf{k}) \frac{\partial f(\mathbf{k})}{\partial k_x} = en \sum_{\mathbf{k}} f(\mathbf{k}) \frac{1}{\hbar} \frac{\partial \varepsilon(\mathbf{k})}{\partial k_x}. \tag{36}$$

In order to express $j_x^{(2)}$ in terms of f(**k**), we differentiate Eq. (31) with respect to κ_x, after which we put $\boldsymbol{\kappa} = 0$ and sum over **k**:

$$\sum_{\mathbf{k}} \bar{W}_1(\mathbf{k}', \mathbf{k}) = \sum_{\mathbf{k}} W_1(\mathbf{k}', \mathbf{k}) + \sum_{\mathbf{k}, \mathbf{k}_1} F(\mathbf{k}', \mathbf{k}_1) W_1(\mathbf{k}_1, \mathbf{k}).$$

Comparing this expression with (29), we obtain

$$j_x^{(2)} = -ien \sum_{\mathbf{k}'} f(\mathbf{k}') \sum_{\mathbf{k}} W_1(\mathbf{k}', \mathbf{k}). \tag{37}$$

Adding (36) and (37), we obtain the sought formula (20) for the current.

Formula (20) for the current differs from the traditional formula (17) in that v is replaced by v^{eff}. In the usual case (weak coupling with phonons), the operator of the coordinate $\hat{\mathbf{R}}$ commutes with the Froehlich interaction Hamiltonian. Actually, in crystals with an inversion center, the Wannier functions $\varphi(\mathbf{r})$ are symmetrical with respect to the substitution $\mathbf{r} \to -\mathbf{r}$, and in the single-band approximation we have

$$\hat{\mathbf{R}} = \sum_{\mathbf{m}} \mathbf{R}_{\mathbf{m}} a_{\mathbf{m}}^+ a_{\mathbf{m}},$$

i.e., $\hat{\mathbf{R}}$ is diagonal in the **m** representation and commutes with the Froehlich Hamiltonian which is diagonal in the **m** representation. In this case we have the relation[1)]

$$\sum_{\mathbf{k}} W_1(\mathbf{k}', \mathbf{k}) = 0$$

and the formula for the current assumes the traditional form. In the small-radius polaron model, $\hat{\mathbf{R}}$ does not commute with the interaction Hamiltonian, and the contribution from W_1 differs from zero (W_1 gives rise to an Umklapp contribution to the mobility).

In[15] are given expressions for W in the case of weak coupling with phonons in the Stark-ladder representation (see (A.4) and (A.5) in[15]). By using formula (23) it is possible to obtain with their aid an expression for W(**k**′, **k**), in the lowest (Born) approximation in the interaction:

$$W(\mathbf{k}', \mathbf{k}) = \sum_{\mathbf{q}} [w_{\mathbf{q}}^+(\mathbf{k}', \mathbf{k}+\mathbf{q}) - w_{\mathbf{q}}^+(\mathbf{k}', \mathbf{k}) + w_{\mathbf{q}}^-(\mathbf{k}', \mathbf{k}+\mathbf{q}) - w_{\mathbf{q}}^-(\mathbf{k}', \mathbf{k})],$$

$$w_{\mathbf{q}}^{\pm}(\mathbf{k}', \mathbf{k}) = \frac{2}{\hbar^2} |c_{\mathbf{q}}|^2 \left(N_{\mathbf{q}} + \frac{1}{2} \pm \frac{1}{2}\right) \mathrm{Re} \int_0^\infty dt\, \delta_{\mathbf{k}', \mathbf{k} - eEt/\hbar} \exp\{-(s \pm i\omega_{\mathbf{q}})t - \frac{i}{\hbar} \int_0^t dt' \left[\varepsilon\left(\mathbf{k} - \mathbf{q} - \frac{e\mathbf{E}}{\hbar} t'\right) - \varepsilon\left(\mathbf{k} - \frac{e\mathbf{E}}{\hbar} t'\right)\right]\}; \tag{38}$$

here $\omega_{\mathbf{q}}$ is the frequency of a phonon with momentum **q**, s is the adiabatic parameter ($s \to +0$), $N_{\mathbf{q}}$ is the distribution function for the phonons, and $|c_{\mathbf{q}}|^2$ is the square of the amplitude of the electron-phonon interaction. We note that W(**k**′, **k**), written out in the lowest approximation in the interaction, actually coincides with the probability from[13] (where, however, the case of an infinitely broad electron band is considered), i.e., in this particular case the quantum transport equation (21) obtained by us goes over into the corresponding equation of[13]. If we put in (38) **E** = 0, then W(**k**′, **k**) goes over into the classical expression for the probability without allowance for the influence of the field **E**.

The expression (20) for the current and the transport equation (21) in the **k** representation are valid for any field direction, whereas formula (1) for the current was valid only when the field direction coincided with that of a high-symmetry axis of the crystal. Indeed, W(**k**′, **k**) is the probability of the transition between the electronic states in the Houston representation[18], which is valid for an arbitrary field direction.

We note that if W(**k**′, **k**) and f(**k**) are linearized with respect to **E** in the transport equation (21), the latter goes over into the kinetic equation obtained in[3], and the linear-in-E corrections to W(**k**′, **k**) renormalize the left-hand side of the equation and constitute its vertex part r (see (8) in[3]).

3. DISTRIBUTION FUNCTION WITH RESPECT TO THE PERPENDICULAR MOMENTA $n(\mathbf{k}_\perp)$

It was shown above that $n(\mathbf{k}_\perp)$ is the symmetrical part, averaged over k_x, of the distribution function f(**k**). Therefore in the classical limit, but $n(\mathbf{k}_\perp)$ for $\Omega_E \tau > 1$ should be "smeared out" over the entire Brillouin zone. Let us find the form of $n(\mathbf{k}_\perp)$ upon interaction with nonpolarization optical phonons ($|c_{\mathbf{q}}|^2 \equiv |c|^2$, $\omega_{\mathbf{q}} \equiv \omega_0$), assuming for simplicity that $\epsilon(\mathbf{k}) = \epsilon(k_x) + \epsilon(\mathbf{k}_\perp)$. Then the equation for $n(\mathbf{k}_\perp)$ (13) assumes in the classical limit the form

$$\int d^3q \int dk_x [n(\mathbf{q}_\perp)\{(N_{\omega_0}+1)D^-(\mathbf{q}, \mathbf{k}) + N_{\omega_0} D^+(\mathbf{q}, \mathbf{k})\} - n(\mathbf{k}_\perp)\{(N_{\omega_0}+1)D^+(\mathbf{q}, \mathbf{k}) + N_{\omega_0} D^-(\mathbf{q}, \mathbf{k})\}] = 0, \tag{39}$$

$$D^{\pm}(\mathbf{q}, \mathbf{k}) = \delta(\varepsilon(\mathbf{q}_\perp) - \varepsilon(\mathbf{k}_\perp) + \varepsilon(q_x) - \varepsilon(k_x) \pm \hbar\omega_0).$$

[1)] This relation can be obtained within the framework of the graphic techniaue proposed in[15], by pairwise adding of graphs that differ from one another by the transfer of the extreme right-hand point from the upper axis to the lower one.

The integration is carried out within the limits of the first Brillouin zone, the volume of which is $v_c = K_x K_\perp^2$, where K_x is the length of the zone in the x direction, and $K_\perp^2$ is the area of its cross section in the yz plane. As seen from the structure (39), $n(\mathbf{k}_\perp) = n(\epsilon(\mathbf{k}_\perp))$ (we put for brevity $\epsilon(\mathbf{k}_\perp) = \zeta$). Now (39) can be simplified using the "quasielasticity" condition $\hbar\omega_0 \ll \Delta E_x, \Delta E_\perp$, where ΔE_x and $\Delta E_\perp$ are the widths of the band in the longitudinal and transverse directions, respectively. Expanding $D^\pm$ in powers of $\hbar\omega_0$ and retaining the first and zeroth terms, we obtain

$$\int d^3q \int dk_x \Big\{ \operatorname{cth}\Big(\frac{\hbar\omega_0}{2kT}\Big) [n(\varepsilon(\mathbf{q}_\perp)) - n(\zeta)] D + \hbar\omega_0 [n(\varepsilon(\mathbf{q}_\perp)) + n(\zeta)] D' \Big\} = 0, \quad (40)$$

$$D = \delta(\varepsilon(\mathbf{q}_\perp) - \zeta + \varepsilon(q_x) - \varepsilon(k_x)).$$

The prime denotes differentiation with respect to ζ.

We consider first the case $\Delta E_x \sim \Delta E_\perp \sim \Delta E$. It is convenient to represent $n(\zeta)$ in the form

$$n(\zeta) = C\Big[1 - \frac{\hbar\omega_0}{\Delta E}\varphi(\zeta)\Big],$$

where C is a constant determined from the normalization condition. For $\varphi(\zeta)$ in the lowest order in $\hbar\omega_0/\Delta E$ we obtain from (40):

$$\operatorname{cth}\Big(\frac{\hbar\omega_0}{2kT}\Big) \int d^3q \int dk_x \{\varphi(\varepsilon(\mathbf{q}_\perp)) - \varphi(\zeta)\} D = \Delta E \int d^3q \int dk_x D'. \quad (41)$$

We see therefore that $\varphi(\zeta) \approx 1$, i.e., $n(\mathbf{k}_\perp) \approx C$ accurate to $\hbar\omega_0/\Delta E$.

Let $\Delta E_\perp \gg \Delta E_x$. Expanding D in (40) in powers of $\epsilon(q_x) - \epsilon(k_x)$ and confining ourselves to terms linear in $\Delta E_x/\Delta E_\perp$, we obtain

$$n(\zeta) = C \exp\Big\{-\zeta \frac{\hbar\omega_0}{(\Delta\bar{E}_x)^2} \operatorname{th}\Big(\frac{\hbar\omega_0}{2kT}\Big)\Big\}, \quad (\Delta\bar{E}_x)^2 = \frac{1}{K_x}\int dk_x \varepsilon^2(k_x). \quad (42)$$

Thus, with the exception of an extremely asymmetrical band, when

$$\frac{\Delta E_\perp \hbar\omega_0}{(\Delta E_x)^2} \operatorname{th}\Big(\frac{\hbar\omega_0}{2kT}\Big) \gtrsim 1,$$

we can put in classical fields $n(\mathbf{k}_\perp) = C$.

In the limit of quantizing fields, in the single-band approximation, the current tends to zero, the system becomes nondissipative, and

$$n(\mathbf{k}_\perp) = C \exp\{-\varepsilon(\mathbf{k}_\perp)/kT\}.$$

Such a distribution corresponds to the case of scattering of an electron inside a single Stark subband, and it can be obtained from (3) if in the sum over X_m we confine ourselves to the term with $X_m = 0$ (see formula (26)). We note that in the quantum case the probability of scattering between different Stark subbands is small compared with the intraband scattering with respect to the parameter $\Delta E_x/\hbar\Omega_E$. As shown by Levinson (private communication), when $\hbar\Omega_E > \Delta E_x \ll \hbar\omega_0$, allowance for weak interband scattering causes a Maxwellian distribution to be obtained only at $\epsilon(\mathbf{k}_\perp) \lesssim \hbar\omega_0$, and in the interval from $\hbar\omega_0$ to $\hbar\Omega_E$ there is a small smooth nonexponential tail.

4. ELECTRIC CONDUCTIVITY IN A STRONG FIELD IN THE PRESENCE OF TOTAL HEATING

In the preceding section it was shown that in nonquantizing fields ($\Delta E/\hbar\Omega_E \gg 1$) there is complete heating of the carriers and $n(\mathbf{k}_\perp) = C$ if $\Omega_E\tau \gg 1$. Let us examine the current under such conditions.

We begin with the case of classically strong fields ($\Delta E/\hbar\Omega_E \to \infty$, $\Omega_E\tau \gg 1$). Substituting $n(\mathbf{k}_\perp) = C$ in (19) and making the substitution $\tilde{W} \to W(\Omega_E\tau \gg 1)$, we obtain

$$j_x = \frac{n}{E}\frac{v_a^2}{(2\pi)^3}\int d^3k \int d^3q |c_q|^2 \cdot 2\pi\omega_q \delta(\varepsilon(\mathbf{k}+\mathbf{q}) - \varepsilon(\mathbf{k}) + \hbar\omega_q). \quad (43)$$

It follows from (43) that when $n(\mathbf{k}_\perp) = C$ and $\Omega_E\tau \gg 1$, the current does not depend on the lattice temperature, which, generally speaking, contradicts the conclusions of Bychkov and Dykhne[21]. Incidentally, in scattering by unpolarized optical phonons only ($|c_q|^2 \equiv |c|^2 \equiv \gamma(\hbar\omega_0)^2$, $\omega_q \equiv \omega_0$), we obtain

$$j_x = \frac{n}{E}\int_{\hbar\omega_0}^{\Delta E} d\varepsilon \, \frac{\hbar\omega_0}{\tau_0(\varepsilon)}\frac{dN(\varepsilon)}{d\varepsilon}, \quad \frac{dN(\varepsilon)}{d\varepsilon} = \frac{v_a}{(2\pi)^3}\int d^3k\, \delta(\varepsilon - \varepsilon(\mathbf{k})), \quad (44)$$

which coincides with[21] if we put in place of (12) in[21]

$$\tau_0^{-1}(\varepsilon) = 2\pi\omega_0\gamma\hbar\omega_0 dN(\varepsilon - \hbar\omega_0)/d\varepsilon. \quad (45)$$

Formula (44) is valid for any temperature, and not only at $kT < \hbar\omega_0$ as in[21]. Another important difference between (43) and the results of[21] is that the contributions to the current from different scattering mechanisms are additive.

In scattering by polarization optical phonons we have in order of magnitude, according to (43)

$$j_x \approx env_{max}\alpha\frac{\hbar\omega_l}{eEa}\Big(\frac{\hbar\omega_l}{\Delta E}z\Big)^{1/2}, \quad (46)$$

where

$$v_{max} = \frac{\Delta E}{\hbar z}a \approx \Big[5\cdot 10^6 - 1\cdot 10^7 \frac{\text{cm}}{\text{sec}}\Big]\frac{\Delta E}{1\text{eV}},$$

a is the lattice constant, α is the Froehlich coupling constant, z is the number of nearest neighbors in the lattice, and ω_l is the frequency of the polarization phonons.

In the case of scattering by acoustic phonons we have

$$j_x \approx env_{max}\Big(\frac{E_0}{\Delta E}\Big)^2\frac{\hbar\omega_D}{Ms^2}\frac{\hbar\omega_D}{eEa}\cdot 2\pi z^2, \quad (47)$$

where ω_D is the Debye frequency, M is the reduced mass of the cell, s is the speed of sound, and E_0 is the constant of the deformation potential.

The condition $\Omega_E\tau > 1$ can be rewritten in the form $E > 2\times 10^4\,(\text{sec}/10^{12}\tau)$. By τ^{-1} is meant the frequency of collisions with phonons, averaged over the entire width of the electron band, and in order of magnitude $\tau \sim \tau_{tr}(\hbar\bar{\omega}/\Delta E)^{1/2}$, where τ_{tr} is the transport relaxation time, $\bar{\omega}$ is the characteristic phonon frequency: $\bar{\omega} = \omega_D$ for acoustic phonons and $\bar{\omega} = \omega_0$ or $\bar{\omega} = \omega_l$ for optical phonons.

Thus, in the absence of cascade ionization, negative differential conductivity takes place at $\Omega_E\tau \gg 1$, and the current is inversely proportional to the field. This conclusion coincides with the results of[21].

We now obtain for the current a formula that is valid in quantizing fields in the case of total heating ($n(\mathbf{k}_\perp) \equiv C$). To this end we substitute in (20) $n(\mathbf{k}_\perp) = C$, $\tilde{W} \to W$, and we use W in the form (38). Then in the model where the coupling with the phonons is weak, as noted in Sec. 2, we have $v^{\text{eff}} = \hbar^{-1}\nabla\epsilon(\mathbf{k})$. Integrating by parts (with respect to (t) in the obtained expression, we get

$$j_x = \frac{n}{\hbar E}\frac{v_a^2}{(2\pi)^6}\int d^3k \int d^3q |c_q|^2 \omega_q \cdot 2\mathrm{Re}\int_0^\infty dt\, e^{-st}\exp[-iF(\mathbf{k},\mathbf{q},t)], \quad (48)$$

where

$$F(\mathbf{k},\mathbf{q};t) = \omega_q t + \frac{1}{\hbar}\int_{-t/2}^{t/2} dt' \left\{\varepsilon\left(\mathbf{k}-\mathbf{q}+\frac{e\mathbf{E}}{\hbar}t'\right) - \varepsilon\left(\mathbf{k}+\frac{e\mathbf{E}}{\hbar}t'\right)\right\}. \quad (49)$$

According to (48) in the quantum case, just as in the classical one, the current is independent of the temperature when $n(\mathbf{k}_\perp) = C$. Putting $\mathbf{E} = 0$ in (49), we arrive at formula (43), which has been obtained from the classical kinetic equation.

In the case when $\mathbf{E}$ coincides with a rational direction in the lattice, it is convenient to go over in (48) to the Fourier representation

$$j_x = \frac{n}{\hbar E}\frac{v_a^2}{(2\pi)^6}\int d^3k \int d^3q |c_q|^2 \times 2\pi\omega_q \int_{-\pi/\Omega_E}^{\pi/\Omega_E} dt \exp[-iF(\mathbf{k},\mathbf{q};t)] \sum_{n=-\infty}^{\infty}\delta(2\pi n - \nu(\mathbf{k}_\perp,\mathbf{q})), \quad (50)$$

where

$$\nu(\mathbf{k}_\perp,\mathbf{q}) = \frac{2\pi}{\Omega_E}\left\{\omega_q + \frac{1}{\hbar}[\varepsilon(\mathbf{k}_\perp-\mathbf{q}_\perp)-\varepsilon(\mathbf{k}_\perp)]\right\}, \quad \varepsilon(\mathbf{k}_\perp) = \frac{1}{G_0}\int_0^{a_0} dk_x \varepsilon(\mathbf{k}). \quad (51)$$

Here G_0 is the smallest of the reciprocal-lattice vectors coinciding with the field direction. For example, in a cubic crystal with a lattice constant a, for $\mathbf{E}$ directed along the axes [100], [110], and [111], G_0 is equal to $2\pi/a$, $2\pi\sqrt{2}/a$, and $2\pi\sqrt{3}/a$, respectively. The quantity Ω_E, as before, is the frequency of the electron oscillations in the field, $\Omega_E = 2\pi eE/\hbar G_0$.

The formula for the current (50) can readily be obtained from (1), since the transition to the Fourier representation with respect to time connects the Houston representation with the Stark-ladder representation.

In some applications, when the field is quasiclassical ($\Delta E/\hbar\Omega_E \gg 1$), it is convenient to use for the current a formula in somewhat different form, summing over n in (50) by the Poisson method:

$$j_x = \frac{n}{\hbar E}\frac{v_a^2}{(2\pi)^6}\int d^3k \int d^3q |c_q|^2 \omega_q \int_{-\pi/\Omega_E}^{\pi/\Omega_E} dt \cdot \times \exp[-iF(\mathbf{k},\mathbf{q};t)] \sum_{l=-\infty}^{\infty}\exp\{il\nu(\mathbf{k}_\perp,\mathbf{q})\}. \quad (52)$$

In the succeeding sections we present examples of current calculation by (52) in quasiclassical electric fields.

5. NONMONOTONIC DEPENDENCE OF j ON E IN STRONG FIELDS

With the aid of (52) we can find the current in the system when the resonant transitions between the Stark subbands lead to a nonmonotonic dependence of the current on the field[14,15,20,22]. We shall consider here the quasiclassical case $\Delta E/\hbar\Omega_E \gg 1$. By virtue of the fact that integration with respect to t in (52) is carried out within the limits of one period of the oscillations in the field, the quasiclassical corrections from this integral have a monotonic character. Neglecting these corrections, we can put $\mathbf{E} = 0$ in the integrand:

$$\int_{-\pi/\Omega_E}^{\pi/\Omega_E} dt \exp\{-iF(\mathbf{k},\mathbf{q};t)\} \approx 2\pi\hbar\delta(\hbar\omega_q + \varepsilon(\mathbf{k}-\mathbf{q}) - \varepsilon(\mathbf{k})). \quad (53)$$

The term with $l = 0$ in (52) coincides with the expression for the current in the classical limit (43).

Let us calculate the current resulting from the interaction with unpolarized optical phonons ($|c_q|^2 = \gamma(\hbar\omega_0)^2$, γ is the dimensionless coupling constant, and $\omega_q \equiv \omega_0$). In this case (52) becomes much simpler

$$j_x = 2\frac{n}{E}\gamma(\hbar\omega_0)^2\pi\omega_0 \sum_{l=-\infty}^{\infty}\int_{\hbar\omega_0}^{\Delta E} d\varepsilon \frac{dN_l^*(\varepsilon)}{d\varepsilon}\exp\left\{2\pi i l\frac{\omega_0}{\Omega_E}\right\}\frac{dN_l(\varepsilon-\hbar\omega_0)}{d\varepsilon}, \quad (54)$$

where

$$\frac{dN_l(\varepsilon)}{d\varepsilon} = \int_0^{\Delta E} d\xi \exp\left\{2\pi i l\frac{\xi}{\hbar\Omega_E}\right\}\varphi(\varepsilon,\xi), \quad \varphi(\varepsilon,\xi) = \frac{v_a}{(2\pi)^3}\int d^3k\,\delta(\varepsilon-\varepsilon(\mathbf{k}))\delta(\xi-\varepsilon(\mathbf{k}_\perp)). \quad (55)$$

The functions $dN_l(\epsilon)/d\epsilon$ at $l = 0$, which we have introduced here, coincide with the density of states in the band $dN(\epsilon)/d\epsilon$ (44), and therefore the term with $l = 0$ in (54) gives the classical contribution to the current (44). If $\Delta E_\perp/\hbar\Omega_E \gg 1$ and the density of states in the two-dimensional zone has no singularities, then

$$\frac{dN_l(\varepsilon)}{d\varepsilon} \approx i\frac{\hbar\Omega_E}{2\pi l}\left[\varphi(\varepsilon,0) - \varphi(\varepsilon,\Delta E_\perp)\exp\left\{2\pi i l\frac{\Delta E_\perp}{\hbar\Omega_E}\right\}\right]\Big|_{l\neq 0}. \quad (56)$$

Substituting (56) in (54) and summing over l, we obtain a final expression for the current

$$j_x = j_x^{cl} + en\gamma\frac{2\pi\omega_0}{G_0}\pi\frac{\hbar\Omega_E}{\Delta E}\left(\frac{\hbar\omega_0}{\Delta E_\perp}\right)^2\sum_{i=1}^{3} a_i\left(\frac{1}{6} - \mu_i + \mu_i^2\right); \quad (57)$$

here j_x^{cl} is the classical monotonic contribution to the current (44)

$$a_1 = \Delta E(\Delta E_\perp)^2\int_{\hbar\omega_0}^{\Delta E} d\varepsilon\,[\varphi(\varepsilon,0)\varphi(\varepsilon-\hbar\omega_0,0) + \varphi(\varepsilon,\Delta E_\perp)\varphi(\varepsilon-\hbar\omega_0,\Delta E_\perp)],$$

$$a_2 = -\Delta E(\Delta E_\perp)^2\int_{\hbar\omega_0}^{\Delta E} d\varepsilon\,\varphi(\varepsilon,0)\varphi(\varepsilon-\hbar\omega_0,\Delta E_\perp),$$

$$a_3 = -\Delta E(\Delta E_\perp)^2\int_{\hbar\omega_0}^{\Delta E} d\varepsilon\,\varphi(\varepsilon-\hbar\omega_0,0)\varphi(\varepsilon,\Delta E_\perp) \quad (58)$$

and

$$\mu_1 = \frac{\omega_0}{\Omega_E} - N_1, \quad \mu_2 = \frac{1}{\Omega_E}\left(\frac{\Delta E_\perp}{\hbar} + \omega_0\right) - N_2, \quad \mu_3 = \frac{1}{\Omega_E}\left(\frac{\Delta E_\perp}{\hbar} - \omega_0\right) \quad (59)$$

where N_i are integers chosen such that $0 \le \mu_i < 1$.

It is easy to see that the nonmonotonic contribution to the current is smaller than j_x^{cl} by a factor $(\hbar\Omega_E/\Delta E_\perp)^2$. In the case when this parameter is not small, their ratio is of the order of unity. We note that at certain field directions in a crystal of definite symmetry, the quasiclassical condition $\Delta E/\hbar\Omega_E \gg 1$ does not automatically imply the relation $\Delta E_\perp/\hbar\Omega_E \gg 1$. For example, in a cubic lattice with strong coupling, where

$$\varepsilon(\mathbf{k}) = J\sum_{\mathbf{g}} e^{i\mathbf{k}\mathbf{g}}$$

(J is the overlap integral between the nearest neighbors, $\mathbf{g}$ is the vector drawn to the site of the nearest neighbor, and $|\mathbf{g}| = a$), when the field is directed along [111], we have $\epsilon(\mathbf{k}_\perp) = 0$ in accordance with (51), and a contribution to $\epsilon(\mathbf{k}_\perp)$ is made by the overlap integrals with the succeeding neighbors $J' \ll J$. In the same crystal, but with the field directed along the [100] axis, we get $\Delta E_\perp = (2/3)\Delta E \sim \Delta E$. Thus, in cubic crystals it is most con-

venient to direct the field along [111] when observations are made of the described nonmonotonicities.

A detailed discussion of the described effect was published in[22,23]. All we mention here is that the current in the quasiclassical case has a discontinuity of the derivative at the points $\omega_0 = N_1\Omega_E$ and $\hbar^{-1}\Delta E_\perp \pm \omega_0 = N_{2,3}\Omega_E$. However, the nonmonotonicities at the points $\hbar^{-1}\Delta E_\perp \pm \omega_0$ should be less manifest in experiment in the quasiclassical limit, since the resonant points lie very close to one another in terms of E. The nonmonotonicities at $\omega_0 = N_1\Omega_E$ were observed experimentally in ZnS with the field directed along [111] in[20], and were independently predicted theoretically in[15].

6. NONANALYTIC DEPENDENCE OF j ON E IN THE CASE OF NARROW BANDS ($\Delta E \lesssim \hbar\omega_0$)

In certain semiconductors there can be realized the case of very narrow electron bands, such that $\Delta E \lesssim \hbar\omega_0 \sim 0.01$—$0.1$ eV. Examples of such materials are polaron semiconductors (such as TiO_2, in which the carriers are small-radius polarons), or organic semiconductors. Without taking the influence of the field on the electron-phonon collisions into account, the transition processes with participation of one optical phonon are forbidden, and contributions are made to the current only by processes in which either one acoustical or two optical phonons take part. The presence of a field leads to the smearing of the δ function that describes the energy conservation law in the electron-phonon collision, and makes the single-phonon processes allowed for the optical phonons. This situation recalls the Franz-Keldysh effect in optics and leads to a nonanalytic dependence of the current on the field; this dependence can be found with the aid of (52).

We shall calculate the current using the following relations between the parameters:

$$\Omega_E\tau \gg 1, \tag{60a}$$

$$\hbar\Omega_E / \Delta E \ll 1, \tag{60b}$$

$$\eta_1^2 = (\hbar\omega_0 - \Delta E) / \Delta E \ll 1, \tag{60c}$$

$$(\hbar\omega_0 - \Delta E) / \eta_1\hbar\Omega_E \gg 1. \tag{60d}$$

Here $\Delta\widetilde{E}$ is an energy on the order of ΔE, a rigorous definition for which will be given below. The inequality (60a) makes it possible to use formula (52), while (60b) is the condition under which the field is quasiclassical and the integral with respect to t in (52) can be taken by the saddle-point method. To this end, we expand F(k, q; t) in (52) in a series in t about the point $t = -iT$:

$$T(\mathbf{k},\mathbf{q}) = \frac{\hbar}{eE}\left\{8[\hbar\omega_{\mathbf{q}} + \varepsilon(\mathbf{k}-\mathbf{q}) - \varepsilon(\mathbf{k})] \Big/ \left[\frac{\partial^2\varepsilon(\mathbf{k}-\mathbf{q})}{\partial k_x^2} - \frac{\partial^2\varepsilon(\mathbf{k})}{\partial k_x^2}\right]\right\}^{1/2} \tag{61}$$

Confining ourselves to the terms $(t + iT)^2$ inclusive, we obtain

$$j_x \approx \frac{n}{eE^2}\frac{v_a^2}{(2\pi)^6}\int d^3k \int d^3q\,|c_{\mathbf{q}}|^2\omega_{\mathbf{q}}\left\{\frac{T(\mathbf{k},\mathbf{q})}{8\pi\hbar}\left[\frac{\partial^2\varepsilon(\mathbf{k}-\mathbf{q})}{\partial k_x^2} - \frac{\partial^2\varepsilon(\mathbf{k})}{\partial k_x^2}\right]\right\}^{-1/2}$$
$$\times \exp\left\{-\frac{2T(\mathbf{k},\mathbf{q})}{3\hbar}[\hbar\omega_{\mathbf{q}} + \varepsilon(\mathbf{k}-\mathbf{q}) - \varepsilon(\mathbf{k})]\right\}\sum_{l=-\infty}^{\infty}\exp\{il\nu(\mathbf{k}_\perp,\mathbf{q})\}. \tag{62}$$

Expression (61) for T is valid if $(\Omega_E T)^2 \ll 1$. On the other hand, the integrand in (62) decreases rapidly with increasing T, the minimum value of which is $T_{min} \sim \eta_1/\hbar\Omega_E$, i.e., (61) is valid if the inequality (60c) is satisfied. The argument of the exponential in (62) reaches its lowest value at $\mathbf{k} = \boldsymbol{\kappa}_1$ and $\mathbf{q} = \boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2$, where $\boldsymbol{\kappa}_1(\boldsymbol{\kappa}_2)$ is the wave vector corresponding to the largest (smallest) value of $\epsilon(\mathbf{k})$, i.e., $\epsilon(\boldsymbol{\kappa}_1) - \epsilon(\boldsymbol{\kappa}_2) = \Delta E$, $d\epsilon(\boldsymbol{\kappa}_i)/d\boldsymbol{\kappa}_i = 0$. For simplicity we assume that at $\mathbf{k}_\perp = \boldsymbol{\kappa}_{i\perp}$ the transverse band $\epsilon(\mathbf{k}_\perp)$ is also extremal, i.e., $d\epsilon(\boldsymbol{\kappa}_{i\perp})/d\boldsymbol{\kappa}_{i\perp} = 0$, $\epsilon(\boldsymbol{\kappa}_{1\perp}) - \epsilon(\boldsymbol{\kappa}_{2\perp}) = \Delta E_1$, and near $\mathbf{k} = \boldsymbol{\kappa}_i$ the band is spherical:

$$|d^2\varepsilon(\varkappa_i)/d\varkappa_i^2| = \hbar^2/m_i, \quad |d^2\varepsilon(\varkappa_{i\perp})/d\varkappa_{i\perp}^2| = \hbar^2/m_{i\perp}. \tag{63}$$

Here $m_2(m_1)$ is the effective mass near the lower (upper) edge of the band $\epsilon(\mathbf{k})$, and $m_{2\perp}$ and $m_{1\perp}$ are the same parameters for the transverse band $\epsilon(\mathbf{k}_\perp)$. Expanding the argument of the exponential in (62) in powers of $\mathbf{k} - \boldsymbol{\kappa}_1$ and $\mathbf{q} - \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2$[2)] and confining ourselves to quadratic terms, we obtain

$$j_x = j_x^{(0)}\left[1 - 8\pi^2\eta_1^2\frac{m_{1\perp}m_{2\perp}}{m_1m_2}\left(\frac{1}{6} - \mu + \mu^2\right)\right], \tag{64}$$

where

$$j_x^{(0)} = enE\,\frac{e}{\hbar}\left(\frac{2\pi}{G_0}\right)^3\gamma_{\varkappa_1-\varkappa_2}\frac{\pi}{4}\left[v_a\left(\frac{G_0}{2\pi}\right)^3\right]^2\frac{(m_1+m_2)^3}{8(m_1m_2)^{3/2}}\left(\frac{\hbar\Omega_E}{\hbar\omega_0 - \Delta E}\right)^{1/2}$$
$$\times\frac{\eta_2^3}{\eta_1^{7/2}}\exp\left\{-\frac{8\pi\eta_1}{3}\frac{\hbar\omega_0 - \Delta E}{\hbar\Omega_E}\right\}. \tag{65}$$

In deriving (64) we have neglected the dispersion of the phonons $\omega_{\mathbf{q}} \equiv \omega_0$; $\gamma_{\mathbf{q}} = |c_{\mathbf{q}}|^2/(\hbar\omega_0)^2$ is the dimensionless coupling constant with the phonons, $\eta_2 = \hbar\omega_0/\Delta\widetilde{E}$, and $\Delta\widetilde{E} = \frac{1}{2}\hbar^2G_0^2(1/m_1 + 1/m_2)$. The parameter $\mu = \hbar\omega_0 - \Delta E_\perp)/\hbar\Omega_E - N$ is chosen with the aid of the integer N such as to make $0 \le \mu < 1$.

In ionic crystals, where the coupling with the optical phonons is stronger than with the acoustical ones, we can obtain an N-shaped current-voltage characteristic. In weak fields, the current increases with the field, and then at $\Omega_E\tau > 1$, owing to the interaction with the acoustic phonons, a decrease of j begins with increasing E ($j \sim 1/E$), and when the argument of the exponential in (65) $\frac{8}{3}\pi\eta_1(\hbar\omega_0 - \Delta E)/\hbar\Omega_E$ becomes not too large, the current increases like $E^{3/2}\exp(-E_0/E)$. Just as in optics, the current oscillates against the monotonic background, with a period $2\pi eG_0^{-1}(\hbar\omega_0 - \Delta E_\perp)^{-1}$, and with an amplitude that is small compared with the background in terms of the parameter $\eta_1^2m_{1\perp}m_{2\perp}/m_1m_2$.

Apparently the nonanalytic dependence of j on E at $\hbar\omega_0 \gtrsim \Delta E$ also remains valid at $\Omega_E\tau < 1$, when the closed formula (52) for the current cannot be used and it becomes necessary to solve the transport equation (21) with probabilities W that have a nonanalytic dependence on the field.

CONCLUSION

Let us discuss briefly the main premises on which the theory described above is based. The most stringent among them are a) the single-band approximation, b) neglect of correlations between the carriers. The

[2)] We can disregard here the corrections due to the expansion of $\partial^2[\epsilon(\mathbf{k}-\mathbf{q}) - \epsilon(\mathbf{k})]/\partial k_x^2$ in the expression for T (61), since they are smaller by a factor η_1^2 than the corrections to the numerator $\hbar\omega_0 + \epsilon(\mathbf{k}-\mathbf{q}) - \epsilon(\mathbf{k})$ in (61).

single-band approximation imposes an upper bound on E: the probability of tunneling between bands should be small. The second assumption does not permit us to ascertain when the electron-electron collisions begin to influence processes such as heating, runaway, etc.

We disregard the influence of the field on the Hamiltonian of the electron-phonon interaction. In the small-radius polaron problem or for interaction with piezoacoustic phonons, this influence can be large. Incidentally, it is easy to forgo this assumption.

It is assumed that the electron density n does not vary with the field. A distinction should be made between two cases: 1) the total number of carriers in the band varies with the field, but this does not affect their distribution with respect to energy (the Frenkel-Pool effect); 2) the change of n with increasing E affects their energy distribution (cascade ionization tunneling breakdown). In case (1), n(E) can be obtained from independent considerations and substituted in (20) in place of n. In case (2) it is necessary to add in (21) a recombination-generation term in a strong electric field.

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