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Surface Phenomena in a Superfluid Liquid

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Surface elementary excitations in a superfluid liquid form a surface normal part whose movement along the surface is accompanied by transfer of mass, energy, entropy, etc. In this connection, the boundary conditions at the free surface for the bulk two-velocity hydrodynamics equations are given by the set of surface hydrodynamic equations of motion derived in the paper. The surface normal density and surface thermodynamic quantities, and in particular the surface tension, depend on the relative velocities of the normal and superfluid components. The surface dissipative function is calculated. The surface oscillation spectrum is investigated at low temperatures. Besides capillary waves there exists a surface second sound. In pure He^4 its velocity is proportional to the cube root of temperature. In the presence of very small amounts of He^3 impurities the velocity at first decreases like $T^{1/2}$ on lowering of the temperature, passes through a minimum, and at zero temperature reaches a value on the order of the velocity of first sound in the liquid volume.

THE motion of elementary excitations (phonons, rotons, impurity excitations) inside the volume of superfluid liquid helium is accompanied by transport of mass, energy, entropy, etc. In macroscopic two-velocity hydrodynamics, this motion is described as the motion of the normal component of a liquid. Each type of elementary excitations makes a definite contribution to the density of the normal component. Besides volume excitations, there exist in liquid helium capillary waves, and also surface impurity levels in the case of a superfluid solution.[1, 2] Either of them can be regarded as a surface elementary excitation. Their motion along the surface, just as the motion of the volume excitations, is accompanied by transport of mass, energy, entropy, etc. In this connection, the problem of the motion of a free surface of a liquid in two-velocity hydrodynamics differs significantly from the analogous problem in ordinary hydrodynamics. In the latter case, the boundary conditions on the free surface consist in the vanishing of the volume fluxes (of mass, energy, entropy, etc.) through the surface. Since these quantities are transported in a superfluid liquid by the surface excitations from one place of the surface to another, it is clear that the ordinary conditions do not hold here.

In the present paper we derive a complete system of equations of motion for the surface of a superfluid liquid; this system takes into account the presence of surface excitations. Just as in a volume, the presence of surface excitations leads to the appearance of a normal surface component, and the equations of motion of the surface are a certain system of equations of surface hydrodynamics.

The temperature dependence of the density of the normal surface component turns out to be such that at low temperatures the influence of the volume normal part on the motion of the surface is quite negligible. As a result we arrive, in the particular case of low temperatures, at a unique situation wherein the liquid in the volume is described by single-velocity hydrodynamics, while the normal part exists only on the surface. This phenomenon is most clearly pronounced in superfluid solutions. The point is that in the case of a superfluid solution, at a practically arbitrarily low concentration of impurities in the volume, a dense two-dimensional Fermi liquid of impurity particles is produced on the surface. This is connected with the fact that the impurity part of the density of the surface normal component increases exponentially with decreasing temperature, and then the temperature below which the impurity density on the surface becomes atomic is practically always of the order of several tenths of one degree and depends very weakly (logarithmically) on the impurity concentration in the volume.

We shall see that at low temperatures there exist two types of liquid-surface oscillations. The first type constitutes ordinary capillary waves. The oscillations of the second type can be called surface second sound, since, as we shall see, these oscillations have an acoustic spectrum and are accompanied by oscillations of the temperature at practically constant surface shape. An experimental investigation of surface second sound would be of considerable interest, since it would afford an explanation of the thermodynamic properties and kinetic phenomena in two-dimensional Fermi liquids.

1. SURFACE NORMAL COMPONENT

In this section, we shall calculate the surface thermodynamic functions and normal density in pure helium and in superfluid weak solutions.

For pure He^4, the only form of surface excitations are capillary waves. The temperature dependence of the surface tension or, which is the same, of the surface part of the free energy, was calculated by Atkins:[3]

$$\alpha_4 = \alpha_0 - \frac{T^{7/3}}{4\pi\hbar^{4/3}}\left(\frac{\rho}{\alpha_0}\right)^{2/3}\Gamma\left(\frac{7}{3}\right)\zeta\left(\frac{7}{3}\right) \tag{1}$$

where α_0 is the surface tension at zero temperature, ρ is the density of HeII, and $\Gamma(x)$ and $\zeta(x)$ are respectively the gamma function and the Riemann zeta function.

Calculation of the surface normal density ν_n is analogous to that in the volume case. By defining ν_n as the coefficient of proportionality of the momentum to the velocity of the surface excitations as a whole, we obtain

$$v_n = -\frac{1}{2}\int p^2 \frac{\partial n_0}{\partial \varepsilon}\frac{d^2p}{(2\pi\hbar)^2}, \tag{2}$$

where $\epsilon(p)$ is the excitation energy as a function of the two-dimensional momentum p, and $n_0(\epsilon)$ is the equilibrium distribution function. Substituting in (2) the capillary-wave spectrum $\epsilon = p^{3/2}\sqrt{\alpha/\rho\hbar}$ and the Planck distribution function $n_0(\epsilon)$, we obtain the surface normal density of pure He^4:

$$v_n = \frac{5T^{5/3}}{18\pi\hbar^2}\left(\frac{\rho\hbar}{\alpha}\right)^{4/3}\Gamma\left(\frac{5}{3}\right)\zeta\left(\frac{5}{3}\right). \tag{3}$$

We note that the spectrum of the capillary waves does not satisfy the Landau superfluidity criterion. However, if we take into account the finite character of the forces of gravity, then the resultant spectrum of the gravitation-capillary waves takes the form $\omega^2 = (\alpha/\rho)k^3 + gk$ and satisfies the Landau criterion. The critical superfluid velocity turns out to be finite and equal to

$$v_{cr} = (4\alpha g/\rho)^{1/4} \approx 10 \text{ cm/sec}$$

From the fact that purely capillary waves give, as we have seen, a finite value of the normal density it is clear that the influence of the force of gravity on the normal density is small and can be neglected.

In a solution of He^3 in He^4 there exist, besides capillary waves, also surface impurity excitations.[1, 2] So long as their surface density is low compared with the atomic one and the interaction can be neglected, the spectrum of these excitations is of the form $\epsilon = p^2/2m_S$, where m_S is the effective mass of the impurity on the surface levels. We reckon the energy from the point corresponding to the minimum energy of the impurity on the surface. The minimum impurity energy in the volume is then positive and equal to ϵ_0. According to the latest measurement,[2, 4] $\epsilon_0 \approx 2°K$ and $m_S \approx 2m_3$ (m_3 is the mass of the He^3 atom). The impurity spectrum in the volume is given by

$$\varepsilon = \varepsilon_0 + p^2/2M, \tag{4}$$

where M is the effective mass of the impurity in the volume.

At very low temperatures $T \ll (\hbar^2/M)(\rho c/m_3)^{2/3}$, where c is the wave concentration of the impurity in the volume, the volume excitations are strongly degenerate and their chemical potential μ can be easily calculated at low concentrations with the aid of the known formulas for the thermodynamic functions of an ideal Fermi gas

$$\mu = \varepsilon_0 + \frac{\hbar^2}{2M}\left(3\pi^2\frac{\rho c}{m_3}\right)^{2/3} - \frac{\pi^2 T^2}{6\hbar^2}M\left(3\pi^2\frac{\rho c}{m_3}\right)^{-2/3} \tag{5}$$

The contribution of the impurities to the surface tension coincides with the surface part of the thermodynamic potential $\Omega_S(\mu, T)$.[5] In the considered region of low temperatures, the second argument of Ω_S can be set equal to zero, since the degeneracy temperature of the impurities is much higher on the surface than in the volume. The impurity part of the surface tension can thus be written in the form

$$\Omega_s(\mu, T) = \Omega_s(\mu, 0) = \Omega_s(\varepsilon_0) + \frac{\partial\Omega}{\partial\mu}(\mu - \varepsilon_0) = \Omega_s(\varepsilon_0) - N_{s0}(\mu - \varepsilon_0). \tag{6}$$

Here N_{S0} is the maximum number of impurities on the surface levels at zero temperature. Substituting the value of the chemical potential (5), we obtain

$$\alpha = \alpha_4 + \Omega_S(\varepsilon_0) - N_{S0}\left\{\frac{\hbar^2}{2M}\left(3\pi^2\frac{\rho c}{m_3}\right)^{2/3} - \frac{\pi^2 T^2 M}{6\hbar^2}\left(3\pi^2\frac{\rho c}{m_3}\right)^{-2/3}\right\} \tag{7}$$

We call attention to the fact that as $c \to 0$ the surface tension differs from α_4. The difference $\alpha - \alpha_4$ as $c \to 0$ is equal to the change of the surface tension resulting from the filling of all the surface levels in the absence of impurities in the volume. In the temperature region under consideration, the surface tension is very sensitive to small amounts of impurities. If $1 \gg N_3/N_4 \gg Sa/V$, where $N_{3,4}$ are the total numbers of He^3 and He^4 atoms in the system, s is the surface area of the solution, V the volume, and a the interatomic distance, then the surface tension is equal to $\alpha_4 + \Omega_S(\epsilon_0)$, and is by no means close to α_4. From the point of view of the value of the surface tension, He^4 can be regarded as pure only if the very stringent inequality $N_3/N_4 \ll Sa/V$ is satisfied.

In the case $T \gg (\hbar^2/M)(\rho c/m_3)^{2/3}$ the impurities in the volume satisfy classical statistics and their chemical potential is equal to

$$\mu = \varepsilon_0 - T\ln\left\{\frac{2m_3}{c\rho}\left(\frac{MT}{2\pi\hbar^2}\right)^{3/2}\right\}. \tag{8}$$

Let T_0 be the temperature at which the chemical potential is equal to zero. We put for brevity

$$L = \ln\left\{\frac{2m_3}{c\rho}\left(\frac{MT_0}{2\pi\hbar^2}\right)^{3/2}\right\}.$$

$L \gg 1$ at low concentrations. If the temperature satisfies the inequality

$$\frac{\varepsilon_0}{L} \gg T \gg \frac{\hbar^2}{M}\left(\frac{\rho c}{m_3}\right)^{2/3},$$

then we can use formulas (6) and (8) in the calculation of the surface tension. We then obtain

$$\alpha = \alpha_4 + \Omega_S(\varepsilon_0) + N_{S0}T\ln\left\{\frac{2m_3}{c\rho}\left(\frac{MT}{2\pi\hbar^2}\right)^{3/2}\right\} \tag{9}$$

If $T < T_0$ and $(T_0 - T)/T_0 \ll 1$, then $\mu = (T_0 - T)L$ and the surface density is much lower than the atomic density. To calculate Ω_S, we can use the formula for the thermodynamic potential of a two-dimensional ideal Fermi gas

$$\Omega_S = -\frac{m_S}{\pi\hbar^2}\int_0^\infty \frac{\varepsilon\, d\varepsilon}{e^{(\varepsilon-\mu)/T} + 1}. \tag{10}$$

At $(T_0 - T)/T_0 \gg 1/L$ this gas is strongly degenerate and from (10) we obtain

$$\alpha - \alpha_4 = \Omega_S = -\frac{m_S}{\pi\hbar^2}\frac{\mu^2}{2} = -\frac{m_S}{2\pi\hbar^2}L^2(T_0 - T)^2$$

$$N_S = -\frac{\partial\Omega}{\partial\mu} = \frac{m_S\mu}{\pi\hbar^2} = \frac{m_S}{\pi\hbar^2}L(T_0 - T). \tag{11}$$

On the other hand, if $|(T_0 - T)/T_0| \ll L^{-1}$, then formula (10) yields

$$\alpha - \alpha_4 = \Omega_S = -\frac{Tm_S}{\pi\hbar^2}\left(\frac{\pi^2 T}{12} + \mu\ln 2\right) - \frac{m_S}{4\pi\hbar^2}\mu^2,$$

$$N_S = \frac{Tm_S}{\pi\hbar^2}\ln 2 + \frac{m_S}{2\pi\hbar^2}\mu. \tag{12}$$

At $T > T_0$ and $(T - T_0)/T_0 \gg 1/L$ (in this case $\mu < 0$ and $|\mu| \ll T$) we can use classical statistics and obtain for Ω_S and N_S the results of [1]:

$$\Omega_s = c\frac{\hbar\rho}{m_3}\frac{m_s}{M}\left(\frac{2\pi T}{M}\right)^{1/2} e^{\varepsilon_0/T},$$

$$N_s = c\frac{\hbar\rho}{m_3}\frac{m_s}{M}\left(\frac{2\pi}{MT}\right)^{1/2} e^{\varepsilon_0/T}. \tag{13}$$

The impurity part of the surface normal density can be calculated when the impurity density on the surface is much lower than the atomic density. Substituting the spectrum $\epsilon = p^2/2m_S$ in (2), we obtain $\nu_n = m_S N_S$. At low temperatures, when the impurities on the surface are strongly degenerate and their density is of the order of atomic, we can write an analogous formula $\nu_n = m_S^*$, N_S with a certain effective mass m_S^*, which differs from m_S because of the Fermi-liquid interaction between the impurities.

As seen from (3), the surface normal density in pure He^4 is proportional to $T^{5/3}$. The volume normal density at low temperatures is proportional, as is well known, to the fourth power of the temperature. At sufficiently low temperatures the influence of the volume normal component on the surface phenomena can therefore be neglected, and we can assume that there is only surface normal density. This means that at low temperatures the liquid can be regarded as incompressible. In an incompressible liquid there are no volume excitations at all, and there are only surface capillary waves.

In the case of a solution, we can obtain with the aid of the second formula of (13) the connection between the concentration c and the temperature corresponding to atomic density of the impurities on the surface:

$$c(\varepsilon_0/T)^{1/2}e^{\varepsilon_0/T} \sim 1.$$

At T = 0.1°K it follows from this that for the surface density of the impurities to be atomic the concentration must be larger than 5×10^{-10}. Such uncontrollably small concentrations of He^3 are practically always present in liquid He^4. Thus, at temperatures on the order of one-tenth of a degree one can in any case neglect the volume normal component in comparison with the surface component.

2. EQUATIONS OF MOTION OF THE SURFACE

The sought equations of motion follow uniquely from the conservation laws. Let $z = \zeta(x, y, t)$ be the equation of the free surface of liquid helium occupying the region $z < \zeta$. We note that when surface phenomena are considered the function $\zeta(x, y, t)$ calls for an exact definition. In the thermodynamics of these solutions it is customary to use a definition with the aid of the condition that there be no surface part in the number of particles of the solvent (see [5]). In fact we used this definition in the preceding section. In the hydrodynamics, however, such a definition is inconvenient, owing to the presence of the surface mass. We shall use another definition of the function $\zeta(x, y, t)$, using a condition that calls just for absence of the total surface mass. All other quantities can now be uniquely represented in the form of sums of volume and surface parts. In particular, the total momentum of the system is equal to

$$\int dx\,dy \int_{-\infty}^{\zeta(x,y,t)} dz\,\mathbf{j} + \int dx\,dy\,\mathbf{i},$$

where $\mathbf{j}$ is the momentum of a unit volume of the liquid, $\mathbf{i}$ is the surface momentum per unit area of the projection of the surface on the xy plane and is connected with the momentum $\mathbf{i}_S$ per unit surface area by the relation $\mathbf{i} = \mathbf{i}_S\sqrt{1 + \zeta_\alpha^2}$, where $\zeta_\alpha = \partial\zeta/\partial x_\alpha$, and the indices $\alpha, \beta, \ldots$, run through the values x and y. Here and throughout we assume that the free boundary of the liquid is that with a vacuum, i.e., we assume the saturated-gas pressure to be equal to zero and disregard the influence of the gas on all the phenomena. This is fully justified at not too high temperatures, when the saturated-gas pressure is quite negligible.

Differentiating the total momentum with respect to time and equating the result to zero, we obtain

$$\int dx\,dy\left\{\frac{\partial\zeta}{\partial t}j_k + \frac{\partial i_k}{\partial t} - \Pi_{kl}n_l\right\} = 0, \tag{14}$$

where we used the volume-hydrodynamics equation $\dot{j}_i + \partial\Pi_{ik}/\partial x_k = 0$ and transformed the volume integral into a surface integral. Here Π_{ik} is the tensor of momentum flux in the volume and $\mathbf{n}$ is a vector directed along the normal to the surface and having the components $n_\alpha = -\zeta_\alpha$ and $n_z = 1$. In order for Eq. (14) to be satisfied identically, the integrand must equal a certain two-dimensional divergence. We thus obtain the following differential form of the momentum conservation law:

$$\frac{\partial i_k}{\partial t} + \frac{\partial \pi_{k\alpha}}{\partial x_\alpha} = \Pi_{kl}n_l - j_k\frac{\partial\zeta}{\partial t}, \tag{15}$$

where $\pi_{k\alpha}$ has the meaning of the surface-momentum flux density.

In a perfectly analogous manner we can write the remaining conservation laws, i.e., the conservation laws for the mass and impurity particle number, and the law of entropy growth:

$$\frac{\partial i_\alpha}{\partial x_\alpha} = \mathbf{jn} - \rho\frac{\partial\zeta}{\partial t}, \quad \frac{\partial \nu}{\partial t} + \frac{\partial h_\alpha}{\partial x_\alpha} = \mathbf{Hn} - \rho c\frac{\partial\zeta}{\partial t},$$

$$\frac{\partial\sigma}{\partial t} + \frac{\partial f_\alpha}{\partial x_\alpha} = \mathbf{Fn} - S\frac{\partial\zeta}{\partial t} + \frac{r}{T}. \tag{16}$$

Here S and ρ are the entropy per unit volume and the density of the liquid, **H** and **F** are the flux densities of the impurity particles and of the entropy in the volume, h_α and f_α have the meaning of surface flux of impurity particles and entropy, ν and σ are the impurity mass and the entropy per unit area of the projection of the liquid on the xy plane, and $r > 0$ is the surface dissipation function. We have also taken into account here the fact that there is no surface mass and the fact that the mass flux density coincides with the momentum.

The unknowns $\pi_{i\alpha}$, h_α, and f_α in formulas (15) and (16) are defined in such a way, that (15) and (16) lead automatically to the energy conservation law, i.e., to an equation of the type

$$\frac{\partial\varepsilon}{\partial t} + \frac{\partial\Theta_\alpha}{\partial x_\alpha} = \mathbf{Qn} - E\frac{\partial\zeta}{\partial t}, \tag{17}$$

where **Q** and Θ_α are the flux densities of the volume energy E and of the surface energy ϵ, respectively.

Owing to the absence of a surface mass, the momentum **i** is invariant under Galilean transformations, while the energy ϵ is connected with the energy ϵ_0 in a coordinate system moving at the velocity of the superfluid component $\mathbf{v}_S$ by the relation

$$\varepsilon = \varepsilon_0 + \mathbf{i}\mathbf{v}_s = \varepsilon_0 + \mathbf{i}\mathbf{v}_{st}.$$

We took into account here the fact that the surface momentum $\mathbf{i}$ is tangent to the surface, and therefore $\mathbf{v}_S$ can be replaced by the tangential component

$$\mathbf{v}_{st} = \mathbf{v}_s - \frac{(\mathbf{v}_s\mathbf{n})\,\mathbf{n}}{1+\zeta_\alpha^2}.$$

To derive the thermodynamic equation satisfied by the energy ϵ_0, let us consider first the equation for the total energy $\mathscr{E}_0$ (volume plus surface) of the system in a coordinate system moving with velocity $\mathbf{v}_S$. At a specified surface area we have

$$d\mathscr{E}_0 = Td\mathscr{S} + \mu dN_3 + \mu_4 dN_4 + (\mathbf{v}_n - \mathbf{v}_s,\ d\mathbf{p}), \tag{19}$$

where $\mathscr{S}$ is the total entropy of the system, μ_4 is the chemical potential of He^4, and $\mathbf{p}$ is the momentum in the coordinate system under consideration. We assume that there is no surface mass, i.e., the surface part N'_{4S} of the number of particles of He^4 is connected with the surface number N'_S of the impurity particles by the relation $N'_{4S} = -(m_3/m_4)N'_S$ (m_4 is the mass of the He^4 atom). From (19) it follows here that the energy E'_S per unit surface area satisfies the identity

$$dE_S' = T\,dS_S' + \frac{Z}{\rho}m_3\,dN_S' + (\mathbf{v}_{nt} - \mathbf{v}_{st},\ d\mathbf{p}_S'), \tag{20}$$

where S'_S and $\mathbf{p}'_S$ are the entropy and momentum per unit surface area, and $Z = (\rho/m_3m_4)(\mu_3m_4 - \mu_4m_3)$. Formula (20) contains the tangential component $\mathbf{v}_{nt} - \mathbf{v}_{st}$ of the velocity difference, since it is the only one that can be different from zero in the state of thermodynamic equilibrium. Since

$$\varepsilon_0 = E_S'\sqrt{1+\zeta_\alpha^2}, \quad \sigma = S_S'\sqrt{1+\zeta_\alpha^2},$$
$$\nu = m_3N_S'\sqrt{1+\zeta_\alpha^2}, \quad \mathbf{i} = \mathbf{p}_S'\sqrt{1+\zeta_\alpha^2},$$

we get from (20) the sought thermodynamic relation

$$d\varepsilon_0 = T\,d\sigma + \frac{Z}{\rho}d\nu + (\mathbf{v}_{nt} - \mathbf{v}_{st},\ d\mathbf{i}) + \alpha\frac{\zeta_\alpha\,d\zeta_\alpha}{\sqrt{1+\zeta_\beta^2}}, \tag{21}$$

where $\alpha = E'_S - TS'_S - (Zm_3N'_3/\rho) - (\mathbf{v}_{nt} - \mathbf{v}_{st})\cdot\mathbf{p}'_S$ is a quantity that obviously has the meaning of the surface-tension coefficient.

If we define the surface with the aid of a condition according to which there is no surface number of particles of He^4, then this leads to a redefinition of the surface quantities. Namely, there is added to each of them a corresponding volume quantity multiplied by a certain constant λ which is common to all quantities:

$$N_S = N_S' + \rho c\lambda/m_3, \quad S_S = S_S' + S\lambda, \quad E_S = E_S' + E_0\lambda,$$
$$\mathbf{p}_S = \mathbf{p}_S' + \mathbf{j}_0\lambda, \quad N_{4S} = N_{4S}' + \rho(1-c)\lambda/m_4.$$

Here E_0 and $\mathbf{j}_0$ are the energy and momentum per unit volume in a coordinate system moving with velocity $\mathbf{v}_S$ and are connected with the energy E in (17) by the relation $E = \rho v_S^2/2 + \mathbf{j}_0\cdot\mathbf{v}_S + E_0$; the differential of E_0 is equal to (see [6])

$$dE_0 = TdS + \Phi d\rho + Zdc + (\mathbf{v}_n - \mathbf{v}_s,\ d\mathbf{j}_0), \tag{22}$$

where $\Phi = \mu/m_3 - (1-c)(Z/\rho)$.

From the condition $N_{4S} = 0$ we obtain

$$\lambda = \frac{m_3}{\rho(1-c)}N_S'$$

and thus

$$N_S' = N_S(1-c), \quad S_S' = S_S - Sm_3N_S/\rho, \quad E_S' = E_S - E_0m_3N_S/\rho,$$
$$\mathbf{p}_S' = \mathbf{p}_S - \mathbf{j}_0m_3N_S/\rho. \tag{23}$$

Formulas (23) establish the connection between the surface thermodynamic quantities calculated in the preceding section and the quantities that enter in formulas (15), (16), and (17).

Using the identities (20) and (22), formula (23), and also the condition that the equilibrium pressure vanish, $P = -E_0 + TS + \Phi\rho + (\mathbf{v}_n - \mathbf{v}_S)\cdot\mathbf{j}_0 = 0$, we get

$$dE_S = TdS_S + \{\Phi + Z(1-c)/\rho\}m_3dN_S + (\mathbf{v}_{nt} - \mathbf{v}_{st},\ d\mathbf{p}_S), \tag{24}$$

which, of course, could have been obtained also from the general identity (19) under the condition $N_{4S} = 0$. In the same manner as used above for the primed quantities, we obtain from the identity (24) an expression for the surface tension in terms of the unprimed quantities:

$$\alpha = E_S - TS_S - \{\Phi + Z(1-c)/\rho\}m_3N_S - (\mathbf{v}_{nt} - \mathbf{v}_{st},\ \mathbf{p}_S).$$

With the aid of (23) we can easily verify that the surface tension is independent, as it should be, of the method of defining the surface. The differential of α is equal to

$$d\alpha = -S_SdT - N_Sd\mu - \mathbf{p}_Sd(\mathbf{v}_{nt} - \mathbf{v}_{st}). \tag{25}$$

Like all the thermodynamic functions, the surface tension of a superfluid liquid depends on the relative velocity of the normal and superfluid components. Since $\mathbf{p}_S = \nu_n(\mathbf{v}_{nt} - \mathbf{v}_{st})$, where ν_n is the surface normal density introduced above, it follows that at low velocities we obtain from (25) the following dependence of the surface tension on the relative velocity of the normal and superfluid motions:

$$\alpha = \alpha_0(T,\mu) - \nu_n(\mathbf{v}_{nt} - \mathbf{v}_{st})^2/2.$$

Here α_0 is the surface tension in the liquid at rest at a specified temperature and a specified chemical potential of the impurities.

Differentiating (18) with respect to time and using (21), (15), (16), and the well known (see [6]) explicit form of the volume fluxes Π_{ik}, $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{Q}$, we obtain after some transformations

$$\begin{aligned}\dot{\varepsilon} - E\dot{\zeta} - \mathbf{Q}\mathbf{n} &= r - T\frac{\partial f_\alpha}{\partial x_\alpha} - \frac{Z}{\rho}\frac{\partial h_\alpha}{\partial x_\alpha} + \frac{\alpha}{\sqrt{1+\zeta_\beta^2}}\zeta_\alpha\dot{\zeta}_\alpha \\ &- \frac{(\mathbf{v}_n\mathbf{n})}{1+\zeta_\beta^2}i_\alpha\dot{\zeta}_\alpha - v_{ni}\frac{\partial\pi_{i\alpha}}{\partial x_\alpha} + (\mathbf{j} - \rho\mathbf{v}_n,\ \mathbf{n})\,[\zeta_3\operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) \\ &+ \zeta_4\operatorname{div}\mathbf{v}_n] + P(\mathbf{v}_n\mathbf{n} - \dot{\zeta}) - \frac{\partial i_\alpha}{\partial x_\alpha}\left(\Phi - \frac{Zc}{\rho} + \frac{v_s^2}{2} - \mathbf{v}_s\mathbf{v}_n\right) + \mathbf{i}\dot{\mathbf{v}}_{st},\end{aligned} \tag{26}$$

where ζ_3 and ζ_4 are the coefficients of the second viscosity in the volume of the liquid, and P is the pressure.

The velocity $\mathbf{v}_S$ of the superfluid component satisfies the equation of volume hydrodynamics

$$\frac{\partial\mathbf{v}_s}{\partial t} + \nabla\psi = 0,$$

$$\psi = \Phi - \frac{Z}{\rho}c + \frac{v_s^2}{2} - \zeta_3\operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) - \zeta_4\operatorname{div}\mathbf{v}_n.$$

When using these formulas it is necessary to remember, however, that in the volume equations the differentiation with respect to time is carried out with all three spatial coordinates fixed. In formula (26), on the other hand, the dot denotes differentiation with respect to time at fixed x and y and at $z = \zeta(x, y, t)$. Thus, the dot denotes the total derivative with respect to time d/dt, which is connected with the derivative $\partial/\partial t$ by the relation $d/dt = \partial/\partial t + \dot{\zeta}\partial/\partial z$. We should introduce analogously two symbols for differentiation with respect to the coordinate x_α: the partial derivative $\partial/\partial x_\alpha$, which enters in the equations of volume hydrodynamics, and the total derivative $d/dx_\alpha = \partial/\partial x_\alpha + \zeta_\alpha\partial/\partial z$. For pure surface quantities (such as ζ, σ, $\pi_{i\alpha}$, etc.), differentiation always has only one meaning and it is convenient to assume that both differentiation symbols coincide when applied to such quantities.

Taking all the foregoing into account we obtain

$$\mathbf{i}\mathbf{v}_{st} = -\frac{d}{\partial x_\alpha}(i_\alpha\psi) + \frac{\partial i_\alpha}{\partial x_\alpha}\psi + \zeta i_\alpha\frac{dv_{sz}}{dx_\alpha} + \frac{\mathbf{v}_s\mathbf{n}}{1+\zeta_\beta^2}i_\alpha\frac{\partial\zeta}{\partial x_\alpha}$$

where we have used the condition $\mathbf{i}\cdot\mathbf{n} = 0$ and the fact that the superfluid motion is potential, curl $\mathbf{v}_s = 0$. Formula (26) can now be transformed into

$$\begin{aligned}\varepsilon + E\dot{\zeta} - \mathbf{Q}\mathbf{n} &= r - T\frac{\partial f_\alpha}{\partial x_\alpha} - \frac{Z}{\rho}\frac{\partial h_\alpha}{\partial x_\alpha} + \frac{\alpha}{\sqrt{1+\zeta_\beta^2}}\zeta_\alpha\dot{\zeta}_\alpha \\ &- v_{ni}\frac{\partial\pi_{i\alpha}}{\partial x_\alpha} - \frac{d}{dx_\alpha}(i_\alpha\psi) + \dot{\zeta}i_\alpha\frac{dv_{sz}}{dx_\alpha} + \frac{\mathbf{v}_s\mathbf{n}-\dot{\zeta}}{1+\zeta_\beta^2}i_\alpha\frac{\partial\zeta}{\partial x_\alpha} + \frac{\partial i_\alpha}{\partial x_\alpha}\mathbf{v}_s\mathbf{v}_n \\ &+ (\mathbf{v}_n\mathbf{n} - \dot{\zeta})\left\{P - \rho\left[\zeta_3\operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) + \zeta_4\operatorname{div}\mathbf{v}_n\right] - \frac{i_\alpha\dot{\zeta}_\alpha}{1+\zeta_\beta^2}\right\}\end{aligned} \quad (27)$$

Notice should be taken of the following. In hydrodynamics it is always assumed that all quantities vary slowly, and the expansion is carried out with respect to gradients. In the hydrodynamic approximation, the volume fluxes Π_{ik}, **F**, **H**, and **Q** contain terms up to the first order in the gradients, inclusive. Second-order terms are already discarded. In the surface equations (15) and (16), the volume fluxes enter as such, and the surface fluxes $\Pi_{i\alpha}$, f_α, and h_α enter under the sign of spatial derivatives. Therefore in the hydrodynamic approximation the surface fluxes should contain only terms of zeroth order in the derivatives. Accordingly, we should retain in the right-hand side of (27) only terms up to first order in the derivatives. It should be borne in mind here that the derivatives $\dot{\zeta}$ and ζ_α are quantities of zeroth order, and the quantities $\mathbf{v}_s\cdot\mathbf{n} - \dot{\zeta}$ and $\mathbf{v}_n\cdot\mathbf{n} - \dot{\zeta}$ are of first order.

There exists, however, a very important case when it is possible to take into account in the surface fluxes the dissipative terms proportional to the spatial derivatives, and use volume hydrodynamics at the same time. This is the already-mentioned case of low temperatures, when the volume normal component can be neglected. Expansion in terms of the gradients in the volume is in this case equivalent to expansion in the parameter a/L (a is the interatomic distance and L is the distance over which an appreciable change of all the quantities takes place). The expansion parameter on the surface is the ratio l/L, where l is the mean free path of the surface excitations. Since $l \gg a$, we should take into account terms of order $(l/L)^2$ and disregard the corrections to volume hydrodynamics, i.e., terms of order $(a/L)^2$. The mean free path l greatly exceeds the interatomic distances even under conditions when the density of the surface Fermi liquid is of the order of atomic. This is connected with the fact that, as we have seen in the preceding section, the impurities on the surface are already strongly degenerate when they reach atomic density with decreasing temperature. In a degenerate Fermi liquid, as is well known, the mean free path is much larger than the interatomic distance.

We shall perform the calculations bearing in mind simultaneously both high and low temperatures, i.e., we discard only terms that are negligible in both cases. It should be recalled here that at high temperatures it is necessary to discard the surface dissipative terms from the results. At low temperatures, the normal component exists only on the surface, and therefore the velocity $\mathbf{v}_n$ describes the motion along the surface. In other words, the difference $\mathbf{v}_n\cdot\mathbf{n} - \dot{\zeta}$ is in this case identically equal to zero, since it is proportional to the difference between the normal and surface components of $\mathbf{v}_n$ and the surface velocity.

Let us consider the last term in the right-hand side of (27). At low temperatures it is equal to zero. At high temperatures the expression in the curly brackets should be calculated accurate to terms of zeroth order in the gradients, i.e., it is necessary to retain only the pressure in the curly brackets. However, the pressure also vanishes in the zeroth approximation, as can be verified by differentiating the equation $\mathbf{i}\cdot\mathbf{n} = 0$ with respect to time and using (15). Thus, the last term in the right-hand side of (27) can be omitted in both cases.

We put

$$f_\alpha = \sigma v_{n\alpha} + \varphi_\alpha, \qquad h_\alpha = \nu v_{n\alpha} + \mu_\alpha, \qquad \pi_{i\alpha} = \pi_{i\alpha}^{(0)} + \tau_{i\alpha}, \quad (28)$$

where

$$\begin{aligned}\pi_{\beta\alpha}^{(0)} &= v_{nt\alpha}i_\beta + i_\alpha v_{st\beta} - \frac{\dot{\zeta}}{1+\zeta_\gamma^2}(\zeta_\alpha i_\beta + \zeta_\beta i_\alpha) \\ &- \alpha\sqrt{1+\zeta_\gamma^2}\left(\delta_{\alpha\beta} - \frac{\zeta_\alpha\zeta_\beta}{1+\zeta_\gamma^2}\right),\end{aligned}$$

$$\pi_{z\alpha}^{(0)} = \pi_{\alpha\beta}^{(0)}\zeta_\beta + \dot{\zeta}i_\alpha.$$

Substituting (28) in (27) and carrying out the necessary transformation with the required accuracy, using the identities

$$d\alpha\sqrt{1+\zeta_\alpha^2} = -\sigma dT - \nu d(Z/\rho) - (\mathbf{i}, d\mathbf{v}_{nt} - d\mathbf{v}_{st}),$$

we obtain

$$\begin{aligned}\dot{\varepsilon} + E\dot{\zeta} - \mathbf{Q}\mathbf{n} + \frac{d}{dx_\alpha}\Big(Tf_\alpha + \frac{Z}{\rho}h_\alpha + \pi_{i\alpha}v_{ni} + i_\alpha\psi + v_{n\alpha}\sqrt{1+\zeta_\beta^2}\,\alpha \\ - i_\alpha\mathbf{v}_s\mathbf{v}_n\Big) = r + \varphi_\alpha\frac{dT}{dx_\alpha} + \mu_\alpha\frac{d}{dx_\alpha}\left(\frac{Z}{\rho}\right) + \tau_{i\alpha}\frac{dv_{ni}}{dx_\alpha}.\end{aligned} \quad (29)$$

By comparing (29) with (17) we obtain the following expressions for the surface energy flux Θ_α and for the surface dissipative function r:

$$\Theta_\alpha = Tf_\alpha + \frac{Z}{\rho}h_\alpha + \pi_{i\alpha}v_{ni} + i_\alpha\psi + v_{n\alpha}\alpha\sqrt{1+\zeta_\beta^2} - i_\alpha\mathbf{v}_s\mathbf{v}_n,$$

$$r = -\varphi_\alpha\frac{dT}{dx_\alpha} - \mu_\alpha\frac{d}{dx_\alpha}\left(\frac{Z}{\rho}\right) - \tau_{i\alpha}\frac{dv_{ni}}{dx_\alpha}. \quad (30)$$

Equations (15) and (16) are the sought complete system of equations of motion of the surface of a superfluid liquid. At high temperatures it is necessary to neglect the surface dissipative terms, i.e., to substitute in (15) and (16) $f_\alpha = \sigma v_{n\alpha}$, $h_\alpha = \nu v_{n\alpha}$, $\pi_{i\alpha} = \pi_{i\alpha}^{(0)}$, and $r = 0$. At low temperatures it is necessary also to know the values of φ_α, μ_α, and $\tau_{i\alpha}$.

3. SURFACE KINETIC COEFFICIENTS

Formula (30) for the dissipative function is completely analogous to the corresponding formula in volume hydrodynamics. Just as in the volume case, it follows from the form of the dissipative function that the dissipative parts of the fluxes φ_α, μ_α, and $\tau_{i\alpha}$ should equal to certain linear forms relative to the derivatives dT/dX_α, $d(Z/\rho)/dx_\alpha$, and dv_{ni}/dx_α. An important role in volume hydrodynamics is played here by the condition of symmetry of the momentum flux, which follows from the angular-momentum conservation law. Let us ascertain the limitations imposed on the form of the tensor $\tau_{i\alpha}$ by the conservation of the angular momentum in our case. The total angular momentum of the system is

$$M_{ik}=\int dx\,dy\int_{-\infty}^{\zeta(x_\alpha,t)} dz\,(x_i j_k-x_k j_i)+\int dx\,dy\,(x_i i_k-x_k i_i).$$

Differentiating M_{ik} with respect to time and using the equations for the conservation of the volume and surface momenta we obtain, after transforming the volume integral into a surface integral,

$$\frac{dM_{ik}}{dt}=\int dx\,dy\left\{x_k\frac{\partial\pi_{i\alpha}}{\partial x_\alpha}-x_i\frac{\partial\pi_{k\alpha}}{\partial x_\alpha}+\dot{x}_i i_k-\dot{x}_k i_i\right\}.$$

Integrating the first two terms by parts, we obtain

$$\frac{dM_{ik}}{dt}=\int dx\,dy\left\{\pi_{k\alpha}\frac{dx_i}{dx_\alpha}-\pi_{i\alpha}\frac{dx_k}{dx_\alpha}+\dot{x}_i i_k-\dot{x}_k i_i\right\}.$$

From this we see that the angular momentum will be conserved only if the flux $\pi_{i\alpha}$ satisfies the equations

$$\pi_{\alpha\beta}=\pi_{\beta\alpha},\ \pi_{z\alpha}-\pi_{\alpha\beta}\frac{\partial\zeta}{\partial x_\beta}-i_\alpha\dot{\zeta}=0.$$

Since the equilibrium part of the flux $\pi_{i\alpha}^{(0)}$ automatically satisfies the foregoing equations, as it should, we obtain the following conditions for the dissipative part of the flux:

$$\tau_{\alpha\beta}=\tau_{\beta\alpha},\qquad \tau_{z\alpha}=\tau_{\alpha\beta}\zeta_\beta. \tag{31}$$

The dissipative function can now be written in the form

$$r=-\varphi_\alpha\frac{dT}{dx_\alpha}-\mu_\alpha\frac{d}{dx_\alpha}\left(\frac{Z}{\rho}\right)-\tau_{\alpha\beta}\left(\frac{dv_{n\beta}}{dx_\alpha}+\zeta_\beta\frac{dv_{nz}}{dx_\alpha}\right), \tag{32}$$

from which we conclude in the usual manner that the dissipative fluxes are determined by the following equations:

$$\varphi_\alpha=-a_{\alpha\beta}\frac{dT}{dx_\beta}-b_{\alpha\beta}\frac{d\nu}{dx_\beta},\ \mu_\alpha=-c_{\alpha\beta}\frac{dT}{dx_\beta}-d_{\alpha\beta}\frac{d\nu}{dx_\beta},$$

$$\tau_{\alpha\beta}=-\eta_{\alpha\beta\gamma\delta}\left(\frac{dv_{n\gamma}}{dx_\delta}+\zeta_\gamma\frac{dv_{nz}}{dx_\delta}\right), \tag{33}$$

where we took into account the fact that at a given (zero) pressure the potential (Z/ρ) can be regarded as a function of the temperature and of the impurity surface density ν, and we can write

$$\frac{d(Z/\rho)}{dx_\alpha}=\left(\frac{\partial}{\partial\nu}\frac{Z}{\rho}\right)\frac{d\nu}{dx_\alpha}+\left(\frac{\partial}{\partial T}\frac{Z}{\rho}\right)\frac{dT}{dx_\alpha}.$$

The quantities $\eta_{\alpha\beta\gamma\delta}$, $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, and $d_{\alpha\beta}$ depend on the form of the surface of the liquid, i.e., on the derivatives of ζ_α. This dependence is obviously due to purely geometrical factors, and to determine it it therefore suffices to write the formulas in covariant form.

The quantities $x_\alpha = x^\alpha$ can be regarded as the coordinates of a point on a surface. The metric tensor $g_{\alpha\beta}$ in such a system of coordinates can be easily determined by writing the element of length on the surface

$$ds^2=dx_i^2=\frac{dx_i}{dx^\alpha}\frac{dx_i}{dx^\beta}dx^\alpha\,dx^\beta=g_{\alpha\beta}dx^\alpha\,dx^\beta.$$

The metric tensor is thus equal to

$$g_{\alpha\beta}=\frac{dx_i}{dx^\alpha}\frac{dx_i}{dx^\beta}=\delta_{\alpha\beta}+\zeta_\alpha\zeta_\beta,$$

from which we readily obtain the corresponding contravariant tensor

$$g^{\alpha\beta}=\delta_{\alpha\beta}-\zeta_\alpha\zeta_\beta/(1+\zeta_\gamma^2) \tag{34}$$

and the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma}=\frac{\zeta_\alpha\,\partial\zeta_\gamma/\partial x_\beta}{1+\zeta_\delta^2}.$$

By virtue of the Galilean invariants, the kinetic phenomena do not depend on the surface velocity, and we therefore cannot choose in a special manner the coordinate system in such a way that the derivative $\dot{\zeta}$ be equal to zero. In this system, the quantities $v_{n\alpha}$ are two components of a three-dimensional vector satisfying the condition $\mathbf{v}_n\cdot\mathbf{n}=0$, i.e., $v_{nz}=v_{n\alpha}\zeta_\alpha$. From the components of the three-dimensional vectors we can make up a covariant two-dimensional vector in the following manner:

$$\tilde{v}_{n\alpha}=v_{ni}\frac{dx_i}{dx^\alpha}=v_{n\alpha}+v_{nz}\zeta_\alpha=(\delta_{\alpha\beta}+\zeta_\alpha\zeta_\beta)v_{n\beta}=g_{\alpha\beta}v_{n\beta},$$

from which we see that the quantities $v_{n\alpha}$ coincide with the contravariant components $\tilde{v}_n^\alpha$. At $\dot{\zeta}=0$ we get the equation

$$\frac{dv_{n\beta}}{dx_\alpha}+\zeta_\beta\frac{dv_{nz}}{dx_\alpha}=g_{\beta\gamma}\left\{\frac{d\tilde{v}_n{}^\gamma}{dx^\alpha}+\Gamma_{\alpha\beta}{}^\gamma\tilde{v}_n{}^\delta\right\}=g_{\beta\gamma}\frac{D\tilde{v}_n{}^\gamma}{Dx^\alpha}, \tag{35}$$

where the symbol D denotes the covariant derivative. The derivative with respect to the time of the total entropy is equal to the integral $\int r\,dx\,dy$. Since this quantity is obviously scalar and the invariant surface element is equal to the product $\sqrt{g}\,dx\,dy$ (g is the determinant of the metric tensor), it is clear that $R_S = r/\sqrt{g}$ is a scalar. Formula (32) with allowance for (35) can be written in the following covariant form:

$$R_s=-\tilde{\varphi}^\alpha\frac{dT}{dx^\alpha}-\tilde{\mu}^\alpha\frac{d}{dx^\alpha}\left(\frac{Z}{\rho}\right)-\tilde{\tau}^{\alpha\beta}g_{\beta\gamma}\frac{D\tilde{v}_n{}^\gamma}{Dx^\alpha},$$

where

$$\tilde{\varphi}^\alpha=\varphi_\alpha/\sqrt{g},\quad \tilde{\mu}^\alpha=\mu_\alpha/\sqrt{g},\quad \tilde{\tau}^{\alpha\beta}=\tau_{\alpha\beta}/\sqrt{g}.$$

In a local-Cartesian coordinate system on the surface, the difference between the covariant and contravariant components vanishes and we obtain

$$R_s=-\tilde{\varphi}_\alpha\frac{dT}{dx_\alpha}-\tilde{\mu}_\alpha\frac{d}{dx_\alpha}\left(\frac{Z}{\rho}\right)-\tilde{\tau}_{\alpha\beta}\frac{\partial\tilde{v}_{n\beta}}{\partial x_\alpha},)$$

from which it follows in the usual manner that

$$\tilde{\tau}_{\alpha\beta}=-\eta_s\left(\frac{\partial\tilde{v}_{n\alpha}}{\partial x_\beta}+\frac{\partial\tilde{v}_{n\beta}}{\partial x_\alpha}-\delta_{\alpha\beta}\frac{\partial\tilde{v}_{n\gamma}}{\partial x_\gamma}\right)-\xi_s\delta_{\alpha\beta}\frac{\partial\tilde{v}_{n\gamma}}{\partial x_\gamma},$$

$$\tilde{\varphi}_\alpha = -\frac{\varkappa_s}{T}\frac{dT}{dx_\alpha} + \left(\beta_s\frac{\partial}{\partial\nu}\frac{Z}{\rho} - \frac{\partial}{\partial T}\frac{Z}{\rho}\right)\tilde{\mu}_\alpha,$$

$$\tilde{\mu}_\alpha = -D_s\left(\frac{d\nu}{dx_\alpha} + \beta_s\frac{dT}{dx_\alpha}\right),$$

where η_S and ξ_S have the meaning of the first and second surface viscosity, κ_S is the coefficient of surface thermal conductivity, and D_S and $\beta_S D_S$ are the coefficients of surface diffusion and thermal diffusion. The latter relations can be written in the following covariant form:

$$\tau^{\alpha\beta} = -\eta_s(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})\frac{D\tilde{v}_{n\gamma}}{Dx^\delta} - (\xi_s - \eta_s)g^{\alpha\beta}g^{\gamma\delta}\frac{D\tilde{v}_{n\gamma}}{Dx^\delta},$$

$$\tilde{\mu}^\alpha = -D_s g^{\alpha\beta}\left(\frac{d\nu}{dx^\beta} + \beta_s\frac{dT}{dx^\beta}\right),$$

$$\tilde{\varphi}^\alpha = -\frac{\varkappa_s}{T}g^{\alpha\beta}\frac{dT}{dx^\beta} + \left(\beta_s\frac{\partial}{\partial\nu}\frac{Z}{\rho} - \frac{\partial}{\partial T}\frac{Z}{\rho}\right)\tilde{\mu}^\alpha,$$

from which we obtain by comparing with (33) and taking (35) into account

$$\eta_{\alpha\beta\gamma\delta} = \eta_s\sqrt{g}(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}) + (\xi_s - \eta_s)g^{\alpha\beta}g^{\gamma\delta}\sqrt{g},$$

$$a_{\alpha\beta} = \frac{\varkappa_s}{T}g^{\alpha\beta}\sqrt{g} + \left(\beta_s\frac{\partial}{\partial\nu}\frac{Z}{\rho} - \frac{\partial}{\partial T}\frac{Z}{\rho}\right)D_s\beta_s g^{\alpha\beta}\sqrt{g},$$

$$b_{\alpha\beta} = \left(\beta_s\frac{\partial}{\partial\nu}\frac{Z}{\rho} - \frac{\partial}{\partial T}\frac{Z}{\rho}\right)D_s g^{\alpha\beta}\sqrt{g},$$

$$c_{\alpha\beta} = D_s\beta_s g^{\alpha\beta}\sqrt{g}, \qquad d_{\alpha\beta} = D_s g^{\alpha\beta}\sqrt{g},$$

where the quantities $g^{\alpha\beta}$ are determined by (34), $g = 1 + \zeta_\alpha^2$, and the kinetic coefficients η_S, ξ_S, κ_S, β_S, and D_S do not depend on the form of the surface. We call attention to the fact that the number of independent surface kinetic coefficients in a superfluid liqiud coincides with the number of kinetic coefficients in volume hydrodynamics.

4. SURFACE SECOND SOUND

Let us apply the obtained equations to an investigation of the spectrum of small oscillations of a plane surface of a superfluid liquid, and consider the most interesting case of low temperatures.

We choose the coordinate system such that the xy plane coincides with the unperturbed surface and the x axis coincides with the direction of the wave vector **k** of the oscillations. The linearized system of equations (15) and (16) can then be written as follows:

$$ik\nu_n(v_{nx} - v_{sx}) = \rho v_{sz} + i\omega\rho\zeta, \qquad -i\omega\delta\sigma + ik\sigma v_{nx} = 0, \tag{36}$$
$$-i\omega\nu_n(v_{nx} - v_{sx}) - ik\delta\alpha = 0, \qquad -i\omega\delta\nu + ik\nu v_{nx} = 0, \quad \alpha k^2\zeta = P,$$

where ω is the oscillation frequency and δ denotes the deviation of a quantity from its equilibrium value.

The potential φ_S of the superfluid velocity in the liquid volume satisfies the Laplace equation $\nabla^2\varphi_S = 0$, the solution of which is $\varphi_S = b\exp(ikx + kz - i\omega t)$ (b is a constant). We can therefore substitute in (36) $v_{sx} = ikb$, $v_{sz} = kb$, and $P = -\rho\partial\varphi_S/\partial t = i\omega\rho b$. Substituting in (36)

$$\delta\alpha = \frac{\partial\alpha}{\partial\nu}\delta\nu + \frac{\partial\alpha}{\partial\sigma}\delta\sigma$$

and eliminating $\delta\nu$, $\delta\sigma$, and b, we obtain

$$i\nu_n v_{nx} + i\frac{\alpha k^2}{\omega}\left(1 - \frac{\omega^2\rho}{\alpha k^3}\right)\zeta = 0,$$

$$iv_{nx}\left\{\omega\nu_n + \frac{k^2}{\omega}\left(\frac{\partial\alpha}{\partial\sigma}\sigma + \frac{\partial\alpha}{\partial\nu}\nu\right)\right\} - i\frac{\alpha\nu_n}{\rho}k^3\zeta = 0, \tag{37}$$

where we have neglected the quantities that are known to be small at small ω and k.

The system (37) describes two types of oscillations. In the first case, as can be readily verified by using the result given below, one can neglect the term with v_{nx} in the first equation of (37). The spectrum of the oscillations of the first type is thus determined by the formula

$$\omega_1^2 = \alpha k^3/\rho,$$

i.e., it is spectrum of ordinary capillary waves.

The oscillations of the second type occur at a practically immobile boundary, and therefore we can neglect the term with ζ in the second equation of (37). The spectrum of the oscillations takes the form $\omega_2 = uk$, where

$$u^2 = -\frac{1}{\nu_n}\left\{\frac{\partial\alpha}{\partial\nu}\nu + \frac{\partial\alpha}{\partial\sigma}\sigma\right\}.$$

The last formula can be rewritten in two equivalent forms:

$$u^2 = -\frac{\sigma}{\nu_n}\left(\frac{\partial\alpha}{\partial\sigma}\right)_{\nu/\sigma} = \frac{\nu}{\nu_n}\left(\frac{\partial\alpha}{\partial\nu}\right)_{\sigma/\nu}. \tag{38}$$

The oscillations of the second type are analogous to volume second sound and we shall call them surface second sound.

By using formulas (1), (3), and the equality $\sigma = -d\alpha/dT$, we obtain for the velocity of the surface second sound in pure He^4 the following expression:

$$u^2 = \frac{63}{40}\frac{\Gamma(7/3)\zeta(7/3)}{\Gamma(5/3)\zeta(5/3)}\left(\frac{\alpha T}{\hbar\rho}\right)^{2/3}, \tag{39}$$

i.e., the velocity is proportional to the cube root of the temperature. It should be noted that the applicability of formula (39) is quite limited, in view of the already noted unusual sensitivity of the surface phenomena to small amounts of He^3 impurity.

Using the surface thermodynamic quantities calculated in Sec. 1 for solutions of He^3 in He^4, we obtained from (38) the velocity of surface second sound in different temperature intervals. In the classical region, i.e., at $T > T_0$ and $(T - T_0)/T_0 \gg 1/L$, we have

$$u = \sqrt{2T/m_s}, \tag{40}$$

which corresponds to the velocity of sound in a two-dimensional monatomic ideal gas.

If $|(T - T_0)/T_0| \ll 1/L$, then we obtain a formula similar to (40), but with a different numerical coefficient:

$$u = \pi\sqrt{T/(m_s\cdot 6\ln 2)}.$$

At $T < T_0$ and $(T_0 - T)/T_0 \gg 1/L$, we have

$$u = \sqrt{(T_0 - T)L/m_s}.$$

In the region of the lowest temperature, the velocity of the surface second sound is expressed in terms of the thermodynamic quantities of a two-dimensional Fermi liquid:

$$u = \sqrt{\frac{N_s}{m_s^*}\frac{\partial\mu}{\partial N_s}}.$$

At very low temperatures, the velocity of the surface second sound was the same order of magnitude as the velocity of ordinary sound in liquid helium. From this and from (40) we see that the function u(T) has a minimum at $T \sim T_0$. We note that at low temperatures and frequencies larger than the reciprocal time between the collisions of impurities on the surface, the surface second sound should go over into oscillations of the zero-sound type in a two-dimensional Fermi liquid.

The main mechanism of damping of surface second sound at not too high frequencies is interaction with the volume normal part. For an estimate of the value of this damping we should substitute in the right-hand side of the second formula of (37) the force of friction between the surface and volume normal parts. This force has an order of magnitude $\eta \partial v_{nx}/\partial z \sim \eta v_{nx}\sqrt{\omega\rho_n/\eta}$. When the friction force is taken into account, an imaginary addition appears in the oscillation frequency and determines the damping:

$$\gamma = \left|\frac{\operatorname{Im}\omega}{\omega}\right| \sim \eta\sqrt{\frac{\omega\rho_n}{\eta}}\frac{1}{\omega\nu_n} \sim \sqrt{\frac{\eta\rho_n}{\omega\nu_n^2}}.$$

We note that there exists another damping mechanism, which is connected with the Cerenkov radiation of volume second sound. Simple estimates show, however, that its contribution is small compared with viscous damping. Since the impurity concentration in the volume is small, it follows that $\eta \sim \rho_n\sqrt{T/Ml}$, where $l \sim a_0/c$ is the mean free path of the impurity and a_0 is the interatomic distance. Substituting this into the expression for γ, we find that the second sound attenuates weakly under the condition

$$\omega \gg \frac{cM^2}{\nu_n^2 a_0^5}\sqrt{\frac{T}{M}}. \qquad (41)$$

At very small impurity concentrations c, an important role may be assumed by the interaction between the surface second sound and the volume phonons. The force of friction of the phonons against the surface is of the order of $\rho_{nph} swv_{nx}$, where ρ_{nph} is the phonon normal density, s the velocity of first sound, and w the coefficient of diffuseness of reflection of the phonons from the surface. The damping connected with the interaction with the phonons is thus equal to $\gamma \sim \rho_{nph} sw/\omega\nu_n$. The diffuseness of the phonon reflection is due mainly to the presence of capillary waves with wavelength exceeding the phonon length. Under these conditions, as is well known, we have $w \sim (k_{ph}\zeta)^2$, where k_{ph} is the wave vector of the phonon and ζ is the average amplitude of the surface roughness due to the capillary waves. To estimate ζ, we note that the energy of the curved surface is of the order of

$$S\sum_{\mathbf{k}} \alpha k^2 |\zeta_{\mathbf{k}}|^2,$$

where ζ_k are the Fourier components of the function $\zeta(x, y)$, which defines the shape of the surface, and S is the area of the surface. It is clear therefore that

$$\overline{|\zeta_{\mathbf{k}}|^2} \sim T/S\alpha k^2,$$

where the square of the amplitude ζ is equal to the sum $\Sigma\,\overline{|\zeta_k|^2}$ taken over all possible values of **k**, so that we obtain for the diffuseness coefficient

$$w = \left(\frac{T}{\hbar s}\right)^2 \sum_{\mathbf{k}} \frac{T}{S\alpha k^2} \sim \left(\frac{T}{\hbar s}\right)^2 \frac{T}{\alpha}\int\frac{dk}{k}.$$

The upper limit of the integral in this equation should be the value of the wave vector of the phonon $k_{ph} \sim T/\hbar s$, and the lower limit the value of the wave vector k $\sim (\rho g/\alpha)^{1/2}$, corresponding to the boundary between the capillary and gravitational waves. The diffuseness coefficient is thus equal to

$$(T^3/\hbar^2 s^2\alpha)\ln\{(T/\hbar s)(\alpha/\rho g)^{1/2}\}.$$

Substituting this into the expression for the damping, we obtain a limitation on the frequency of the surface second sound in the caes when the principal role in the damping is played by the phonons:

$$\omega \gg \frac{\Theta}{\hbar}\left(\frac{M}{\nu_n a_0^2}\right)\left(\frac{T}{\Theta}\right)^7 \ln\left\{\frac{T}{\hbar s}\left(\frac{\alpha}{\rho g}\right)^{1/2}\right\}, \qquad (42)$$

where Θ is the Debye temperature of liquid helium.

At $T \sim 0.1°K$ and concentrations $c \sim 10^{-6}-10^{-8}$ the surface normal density can be assumed to have its atomic value, i.e., $\nu_n \sim M/a_0^2$, and formulas (41) and (42) lead to the conditions $\omega \gg 10^3-10^5\,\text{sec}^{-1}$ and $\omega \gg 10^{-1}\,\text{sec}^{-1}$, respectively (we have assumed $\Theta \approx 10°K$).

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