Propagation of Radiation in Optically Thin Anisotropic Media

A. Z. DOLGINOV AND N. A. SILANT'EV

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences

Submitted June 22, 1971

Zh. Eksp. Teor. Fiz. **62**, 100–110 (January, 1972)

The propagation of radiation in optically thin anisotropic noncrystalline media such as interstellar space is considered. Explicit formulas are obtained for the Stokes parameters of transmitted and scattered radiation in media with various types of anisotropy. It is shown that on passage of light through a medium consisting of anisotropic oriented molecules or small particles the plane of polarization may rotate and linear polarization may change into circular or vice versa. Astrophysical phenomena in which the effects considered may be important are discussed.

1. FORMULATION OF PROBLEM

In many problems in physics, and especially astrophysics, one encounters anisotropic non-crystalline media, for example the interstellar medium. The radiation of the stars is polarized on its way to the earth and is scattered by dust-like matter. This indicates that the interstellar particles have an orientation, i.e., that the medium is anisotropic. Different aerosols, streams of colloidal solutions, etc. are also anisotropic non-crystalline media. Although there have been many studies of the propagation of light in such media,[1] explicit formulas for the scattering intensity, and all the more for the polarization, were obtained only in a few particular cases. Yet one can obtain rather simple explicit formulas which are convenient for comparison with experiment and encompass most cases of interest.

In this paper we determine the explicit dependence of the Stokes parameters on such scattering-particle characteristics as the polarizability, the dielectric constant, and the magnetic permeability, and also on the moments of their distribution functions with respect to the orientation for non-resonant radiation passing through an optically thin anisotropic medium. We confine ourselves to Rayleigh scattering by atoms and dust particles (the small parameter is $2\pi a/\lambda$ or $2\pi a|\kappa|/\lambda$, where λ is the wavelength of the radiation in vacuum, κ the refractive index of the dust material, and a the characteristic dimension of the scattering particle), and also consider Rayleigh-Jahns scattering by large dust particles (the small parameters are $|\kappa - 1|$, $|\mu - 1|$, and $2\pi a|\kappa - 1|/\lambda$, where μ is the magnetic permeability of the particle). The dielectric constant and the magnetic permeability can be in tensor form. The concentration of the scattering particles N_0 is assumed to be sufficiently small, $N_0\lambda^3 \ll 1$.

The amplitude of the wave $\mathbf{E}^{(S)}$ scattered by the particle in the direction $\mathbf{n}_1$ is connected with the amplitude of the wave $\mathbf{E}^{(0)}$ incident in the direction $\mathbf{n}_0$ by the well known relation

$$E_i^{(S)} = R^{-1} t_{ik}(\mathbf{n}_1\mathbf{n}_0) E_k^{(0)}, \quad t_{ik}(\mathbf{n}_1\mathbf{n}_0) = k^2(\delta_{im} - n_i^{(1)} n_m^{(1)})\alpha_{mk}(\mathbf{n}_1\mathbf{n}_0). \quad (1)$$

Here R is the distance to the point of observation. For the case of Rayleigh scattering $\alpha_{mk}(\mathbf{n}_1\mathbf{n}_0) \equiv \alpha_{mk}$ denotes the tensor of the polarizability of the molecule or the dust particle as a whole. In the case of Rayleigh-Jeans scattering $\alpha_{mk}(\mathbf{n}_1\mathbf{n}_0)$ denotes the following integral over the volume of the particle:[2]

$$\alpha_{mk}(\mathbf{n}_1\mathbf{n}_0) = \frac{1}{4\pi k^2}\int_V d\mathbf{r}\exp[-ik\mathbf{r}(\mathbf{n}_1 - \mathbf{n}_0)]\{k^2[\varepsilon_{mk}(\mathbf{r}) - \delta_{mk}] + ike_{mpk}e_{pql}\nabla_q[\delta_{ls} - \mu_{ls}^{-1}(\mathbf{r})]n_s^{(0)}\}. \quad (2)$$

Repeated indices mean summation throughout; $k = 2\pi/\lambda = \omega/c$; e_{mpk} is an antisymmetrical unit tensor; $\mu_{ls}^{-1}(\mathbf{r})$ are the components of the inverse of the magnetic permeability tensor $\hat{\mu}(\mathbf{r})$; $\hat{\epsilon}(\mathbf{r})$ is the complex dielectric tensor.

It is convenient to describe the wave $\mathbf{E}^{(S)}$ in a coordinate system with the z axis along $\mathbf{n}_1$, and the wave $\mathbf{E}^{(0)}$ in a system with the z axis along $\mathbf{n}_0$. Using the cyclic components $E_0 = E_z$ and $E_{\pm 1} = \pm(E_x \pm iE_y)/\sqrt{2}$, and specifying the transformation of the vectors by coordinate rotation with the aid of the Wigner matrix $D_{mn}^{(1)}(\Omega)$, we can write

$$E_\alpha^{(S)}(\mathbf{n}_1) = R^{-1} D_{\alpha i}^{(1)+}(\Omega_1) t_{ik}(\mathbf{h}) D_{k\beta}^{(1)}(\Omega_0) E_\beta^{(0)}(\mathbf{n}_0).$$

The symbols $E_\alpha(\mathbf{n}_1)$ etc. indicate that the corresponding components of the vectors or the tensors are taken in the coordinate system with the z axis along $\mathbf{n}_1$, etc. Greek indices take on only two values, $\alpha = \pm 1$, since the field is transverse, while Latin indices take on three values: $i = 0, \pm 1$. The z axis of the system, which is rigidly bound to the medium, is directed along $\mathbf{h}$, one of the physically singled-out directions in the medium. Ω_1 denotes the aggregate of Euler angles φ_1, β_1, and γ_1, which describe a rotation from a coordinate system with z axis along $\mathbf{h}$ to a system with z axis along $\mathbf{n}_1$. Ω_0 has a similar meaning. The quantity

$$t_{\alpha\beta}(\mathbf{n}_1\mathbf{n}_0\mathbf{h}) = D_{\alpha i}^{(1)+}(\Omega_1) k^2 \alpha_{im}(\mathbf{h}) D_{m\beta}^{(1)}(\Omega_0) \quad (3)$$

has the meaning of a scattering matrix.

In a homogeneous anisotropic medium, plane waves of two types can propagate, corresponding to two eigenvalues of the matrix (3) for forward scattering,

$$\langle t_{\alpha\beta}(\mathbf{nnh})\rangle U_{\beta i}(\mathbf{nh}) = t^{(i)}(\mathbf{nh}) U_{\alpha i}(\mathbf{nh}). \quad (4)$$

The elements of the matrix $U_{\alpha i}(\mathbf{nh})$ are the cyclic components $f_\alpha^{(i)}$ of the unit polarization vectors of these two waves (i = 1, 2). We note that $f_1^{(i)}$ and $f_{-1}^{(i)}$ are connected by the simple relation

$$f_1^{(i)} = f_{-1}^{(i)}[t^{(i)}(\mathbf{nh}) - \langle t_{-1-1}(\mathbf{nnh})\rangle]/\langle t_{-11}(\mathbf{nnh})\rangle = f_{-1}^{(i)} re^{i\delta},$$

where r and δ determine the parameters of the polarization ellipse:

$$a = (r+1)/[2(r^2+1)]^{1/2}, \quad b = |r-1|/[2(r^2+1)]^{1/2}, \quad \alpha = {}^1/_2(\delta \pm \pi). \quad (5)$$

Here a and b are the major and minor semi-axes of the ellipse, and α is the angle between a and the x axis of the right-hand system. The rotation of the electric vector is clockwise when viewed from the end of the wave vector $\mathbf{k}$, if $r > 1$, and counterclockwise if $r < 1$. The case r = 1 corresponds to linear polarization. The angle brackets denote averaging over the initial and summation over the final states of the scattering particle, including also averaging over its possible orientations.

The eigenvalues $t^{(i)}(\mathrm{nh})$ are simply connected with the refractive index of the medium for the corresponding waves:

$$\varkappa_i(\mathbf{nh}) = 1 + 2\pi N_0 t^{(i)}(\mathbf{nh}) k^{-2}, \tag{6}$$

where N_0 is the concentration of the particles.

The field of the scattered radiation will be characterized by a density matrix

$$\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = (c/8\pi) E_\alpha(\mathbf{n}_1\mathbf{r}) E_\beta^*(\mathbf{n}_1\mathbf{r}),$$

which is connected with the Stokes parameters I_n by the relations

$$\rho_{\alpha,\alpha} = {}^1/_2(I_0 - \alpha I_2), \quad \rho_{\alpha,-\alpha} = -{}^1/_2(I_3 + i\alpha I_1), \quad \alpha = \pm 1. \tag{7}$$

Here I_0 is the total radiation intensity, and I_2 is the intensity of the right-hand ($I_2 > 0$) or left-hand ($I_2 < 0$) circular polarization. The degree of linear polarization is equal to $(I_1^2 + I_3^2)^{1/2}/I_0$, and $I_1/I_3 = \tan 2\chi$, where χ is the angle between the plane of the oscillations of the electric vector and the x axis.

An integral equation was previously obtained[3,4] for $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r})$. Its solution in the first iteration approximation, which is convenient for the description of scattering in an optically thin medium, is

$$\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = M_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{r}\mathbf{r}_0\mathbf{h})\, \rho_{\gamma\nu}^{(0)}(\mathbf{n}_1\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} dl M_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{r}\mathbf{r}'\mathbf{h})\, [B_{\gamma\nu}(\mathbf{n}_1\mathbf{r}') + q_{\gamma\nu}(\mathbf{n}_1\mathbf{r}')]. \tag{8}$$

To obtain this solution, we have first changed over to a coordinate system whose axes are the eigenvectors of the polarization $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, solved the equation in this system, and then returned to the initial cyclic coordinates. Therefore the propagation function

$$G_{nm}(\mathbf{n}_1\mathbf{r}\mathbf{r}'\mathbf{h}) = \exp\left[-N_0 g^{(nm)}(\mathbf{n}_1\mathbf{h})\, |\mathbf{r} - \mathbf{r}'|\right] \tag{9}$$

turned out to be summed with the direct and inverse matrices $U_{\alpha l}(\mathbf{n}_1\mathbf{h})$:

$$M_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{r}\mathbf{r}'\mathbf{h}) = U_{\alpha i}(\mathbf{n}_1\mathbf{h})\, U_{i\gamma}^{-1}(\mathbf{n}_1\mathbf{h})\, G_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{r}'\mathbf{h})\, [U^+(\mathbf{n}_1\mathbf{h})]_{\nu k}^{-1} U_{k\beta}^+(\mathbf{n}_1\mathbf{h}). \tag{10}$$

Here $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}_0$ are points lying on the line of sight directed along $\mathbf{n}_1$. The integration is carried out along the line of sight from a certain initial point $\mathbf{r}_0$, where a value $\rho_{\gamma\nu}^{(0)}(\mathbf{n}_1\mathbf{r}_0)$ is specified, to $\mathbf{r}$. The function (9) describes the attenuation of the radiation as a result of scattering and true absorption, and the interference of the waves as a result of the difference between their phase velocities

$$v_i(\mathbf{n}_1\mathbf{h}) = 2c[\varkappa_i(\mathbf{n}_1\mathbf{h}) + \varkappa_i^*(\mathbf{n}_1\mathbf{h})]^{-1}.$$

In addition

$$g^{(nk)}(\mathbf{n}_1\mathbf{h}) = i\lambda[t^{(k)*}(\mathbf{n}_1\mathbf{h}) - t^{(n)}(\mathbf{n}_1\mathbf{h})] = \frac{1}{2}[\sigma_0^{(n)}(\mathbf{n}_1\mathbf{h}) + \sigma_0^{(k)}(\mathbf{n}_1\mathbf{h})] + \frac{i\omega}{N_0}\left(\frac{1}{v_k(\mathbf{n}_1\mathbf{h})} - \frac{1}{v_n(\mathbf{n}_1\mathbf{h})}\right); \tag{11}$$

$\sigma_0^{(n)}$ is the total cross section for the absorption of the wave corresponding to the n-th eigenvalue of (4), $q_{\gamma\nu}(\mathbf{n}_1\mathbf{r}\mathbf{h})$ is the density (erg/cm^3 sec-sr) of the radiation produced by the sources in the medium at the point $\mathbf{r}$, and $B_{\gamma\nu}(\mathbf{n}_1\mathbf{r}\mathbf{h})$ is the analogous quantity for the radiation incident on the point $\mathbf{r}$ from all the other points of the medium.

If we are interested only in the attenuation of the primary beam incident along $\mathbf{n}_1$, then the integral term of (8) can be omitted. On the other hand, if one specifies not the incident beam but the radiation source inside the medium, then it suffices to retain only the term with $q_{\gamma\nu}(\mathbf{n}_1\mathbf{r}\mathbf{h})$. To determine the Stokes parameters in either case, it suffices to calculate the function (10). We note that these formulas describe the attenuation of the beam also in an optically thick medium. We have therefore not expanded the exponential in (9). On the other hand, if we are interested in the problem of scattering, in the direction of $\mathbf{n}_1$, of radiation initially incident along n_0 and characterized by the matrix $\rho_{\gamma\nu}^{(0)}(\mathbf{n}_0\mathbf{r})$, then it is necessary to calculate also

$$B_{\gamma\nu}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = N_0\langle t_{\gamma\alpha}(\mathbf{n}_1\mathbf{n}_0\mathbf{h})\, \rho_{\alpha\beta}^{(0)}(\mathbf{n}_0\mathbf{r})\, t_{\beta\nu}^+(\mathbf{n}_1\mathbf{n}_0\mathbf{h})\rangle = N_0 k^4 \langle \alpha_{nm}(\mathbf{h})\, \alpha_{ps}^+(\mathbf{h})\rangle\, \rho_{\alpha\beta}^{(0)}(\mathbf{n}_0\mathbf{r})\, D_{\gamma n}^{(1)+}(\Omega_1)\, D_{s\nu}^{(1)}(\Omega_1)\, D_{m\alpha}^{(1)}(\Omega_0)\, D_{\beta p}^{(1)+}(\Omega_0) \tag{12}$$

Both (10) and (12) are expressed in terms of different combinations of the tensors α_{nm}. Let us calculate these quantities for the cases of interest to us. The tensor $\alpha_{nm}(\mathbf{h})$ in a coordinate system (h) that is fixed relative to the medium is connected with its value $\alpha_{nm}(\mathbf{n}_c)$ in the coordinate system rigidly connected with the particle (the vector $\mathbf{n}_c$ lies along its z axis) by the relation

$$\alpha_{ik}(\mathbf{h}) = D_{in}^{(1)}(\Omega_c)\, \alpha_{nm}(\mathbf{n}_c)\, D_{mk}^{(1)+}(\Omega_c). \tag{13}$$

Ω_c denotes the aggregate of the Euler angles φ_c, β_c, and γ_c, which determine the orientation of the particle relative to the medium. Since different orientations are possible, it is necessary to average all the combinations of α_{ik} over the orientations and take into account their different probabilities. For example,

$$\langle \alpha_{ik}(\mathbf{h})\rangle = \int_0^{2\pi} d\gamma_c \int_0^{2\pi} d\varphi_c \int_0^{\pi} d\beta_c \sin\beta_c W(\Omega_c)\, \alpha_{ik}(\mathbf{h}), \tag{14}$$

$W(\Omega_c)$ is the probability, normalized to unity, that the particle has an orientation given by the definite angles φ_c, β_c, and γ_c. Quantities of the type $\langle \alpha_{kn}(\mathbf{h})\, \alpha_{ml}^*(\mathbf{h})\rangle$ are averaged in similar fashion.

The solution (8) is the most general one. In the sections that follow we shall present only concrete formulas suitable for comparison with experiment for the most widespread types of anisotropic media.

2. PHOTON SCATTERING CROSS SECTIONS

The total cross section $\sigma_0 = \sigma_s + \sigma_a$ is the sum of the cross sections for elastic scattering σ_s and true absorption σ_a. We denote by $\sigma_0^{(m)}(\mathbf{n}_0\mathbf{h})$ the total cross section for the case when the initial wave propagated along $\mathbf{n}_0$, and its polarization was described by a unit vector $\mathbf{e}_m$, which generally speaking is complex:

$$\sigma_0^{(m)}(\mathbf{n}_0\mathbf{h}) = 4\pi k\, \mathrm{Im}[U_{m\alpha}^{-1}(\mathbf{n}_0\mathbf{h})\, D_{\alpha\beta}^{(1)+}(\Omega_0)\, \langle \alpha_{pq}(\mathbf{h})\rangle\, D_{q\beta}^{(1)}(\Omega_0)\, U_{\beta m}(\mathbf{n}_0\mathbf{h})]. \tag{15}$$

The elements of the matrix $U_{\alpha m}(\mathbf{n}_0\mathbf{h}) = \mathbf{e}_m\mathbf{e}^*_\alpha$ are cyclic components of the polarization unit vectors $\mathbf{e}_m$. If $\mathbf{e}^*_1\cdot\mathbf{e}_2 = 0$, then $U^{-1} = U^+$ and only the anti-Hermitian part of the tensor $\langle\alpha_{pq}(\mathbf{h})\rangle$ contributes to (15):

$$iA_{pq}(\mathbf{h}) = {}^1/_2[\langle\alpha_{pq}(\mathbf{h})\rangle - \langle\alpha_{qp}(\mathbf{h})\rangle^*]. \tag{16}$$

The Hermitian tensor $A_{pq}(\mathbf{h})$ has nine independent real components. In terms of the cyclic coordinates ($p, q = 0, \pm 1$) it is convenient to introduce the notation

$$A_{pp} \equiv A_p,\quad A_{-11} \equiv A_2 + iA_3,\quad A_{-10} \equiv A_4 + iA_5,\quad A_{01} \equiv A_6 + iA_7. \tag{17}$$

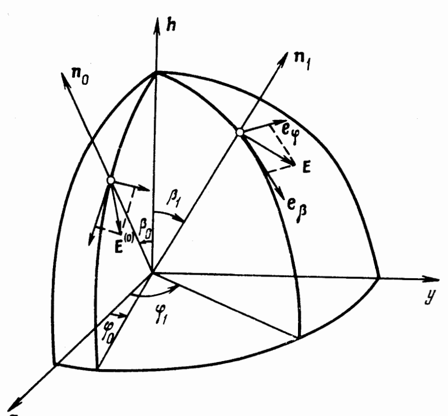
The total cross section $\sigma_0^{(-)}$ and $\sigma_0^{(+)}$ for photons with right-hand and left-hand polarization, respectively, are obtained from (15) in the form

$$\begin{aligned}\sigma_0^{(\pm)}(\mathbf{n}_0\mathbf{h}) = 4\pi k\, \{{}^1/_4(A_1 + A_{-1})(1+\cos^2\beta_0) + {}^1/_2\sin^2\beta_0[A_0 \\ + A_2\cos 2\varphi_0 - A_3\sin 2\varphi_0] + 2^{-1/2}\sin\beta_0\cos\beta_0[(A_4 - A_6)\cos\varphi_0 \\ - (A_5 - A_7)\sin\varphi_0] \mp {}^1/_2(A_{-1} - A_1)\cos\beta_0 \\ \mp 2^{-1/2}\sin\beta_0[(A_4 + A_6)\cos\varphi_0 - (A_5 + A_7)\sin\varphi_0]\},\end{aligned} \tag{18}$$

where β_0 and φ_0 are the angles of the vector $\mathbf{n}_0$ in the system with z axis parallel to $\mathbf{h}$. Obviously, the total cross section for the unpolarized radiation is

$$\sigma_0^{\text{н.п.}} = {}^1/_2(\sigma_0^{(-)} + \sigma_0^{(+)}).$$

If the initial radiation was linearly polarized at an angle θ_0 to the x axis in the system where the z axis



lies along $\mathbf{n}_0$ (see the figure), then the total cross section is

$$\begin{aligned}\sigma_0^{(l)}(\mathbf{n}_0\mathbf{h}) = 4\pi k\{{}^1/_4(A_1 + A_{-1})(1+\cos^2\beta_0) + {}^1/_4\sin^2\beta_0[(2A_0 - A_1 \\ - A_{-1})\cos 2\theta_0 + 2A_0] + {}^1/_2(A_2\cos 2\varphi_0 - A_3\sin 2\varphi_0)[\sin^2\beta_0 \\ - (1+\cos^2\beta_0)\cos 2\theta_0] + \cos\beta_0\sin 2\theta_0[A_2\sin 2\varphi_0 + A_3\cos 2\varphi_0] \\ + 2^{-1/2}\sin\beta_0\cos\beta_0(1+\cos 2\theta_0)[(A_4 - A_6)\cos\varphi_0 - (A_5 - A_7)\sin\varphi_0] \\ - 2^{-1/2}\sin\beta_0\sin 2\theta_0[(A_4 - A_6)\sin\varphi_0 + (A_5 - A_7)\cos\varphi_0]\}.\end{aligned} \tag{19}$$

The elastic-scattering cross section of radiation propagating along $\mathbf{n}_0$ and described by the normalized density matrix $\rho^{(0)}_{\alpha\beta}$ is equal to

$$\sigma_s(\mathbf{n}_0\mathbf{h}) = {}^8/_3\pi k^4 T_{mn}(\mathbf{n}_0|\mathbf{h})D^{(1)}_{n\gamma}(\Omega_0)\rho^{(0)}_{\gamma\nu}D^{(1)+}_{\nu m}(\Omega_0), \tag{20}$$

where the Hermitian tensor $T_{mn}(\mathbf{n}_0|\mathbf{h})$ has the following explicit form:

$$T_{mn}(\mathbf{n}_0|\mathbf{h}) = \frac{3}{8\pi}\int d\mathbf{n}_1 D^{(1)+}_{\alpha p}(\Omega_1)D^{(1)}_{q\alpha}(\Omega_1)\langle\alpha_{pn}(\mathbf{n}_1\mathbf{n}_0|\mathbf{h})\alpha_{mq}^+(\mathbf{n}_1\mathbf{n}_0|\mathbf{h})\rangle. \tag{21}$$

By using the notation $\alpha_{pn}(\mathbf{n}_1\mathbf{n}_0|\mathbf{h})$ we have emphasized here the explicit dependence of the components of the tensor $\alpha_{pn}(\mathbf{h})$ on the directions $\mathbf{n}_1$ and $\mathbf{n}_0$. The tensor $T_{mn}(\mathbf{n}_0|\mathbf{h})$ has nine independent components, which we shall designate in analogy with (17). The elastic-scattering cross sections of circularly or linearly polarized photons can be obtained from (18) and (19) by replacing there $4\pi k A_{mn}(\mathbf{h})$ by ${}^8/_3\pi k^2 T_{mn}(\mathbf{n}_0|\mathbf{h})$. We shall henceforth present explicit expressions for the tensor T_{mn} only in the case of Rayleigh scattering, when $T_{mn}(\mathbf{n}_0|\mathbf{h}) \equiv T_{mn}(\mathbf{h})$.

3. AXIALLY SYMMETRICAL ORIENTATION OF THE MEDIUM

Let us consider a medium with axial orientation of the scattering particles, when $W(\Omega) = (2\pi)^{-1}W(\beta;\gamma)$. In this case $\langle\alpha_{nm}(\mathbf{h})\rangle = \delta_{nm}\langle\alpha_n(\mathbf{h})\rangle$, and the polarization unit vectors of the ordinary and extraordinary waves ($\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$) will be orthogonal if the following condition is satisfied:

$$\operatorname{Im}\langle\alpha_{-1}(\mathbf{h})\rangle^*\langle\alpha_1(\mathbf{h})\rangle + \langle\alpha_0(\mathbf{h})\rangle(\langle\alpha_1(\mathbf{h})\rangle^* - \langle\alpha_{-1}(\mathbf{h})\rangle^*)] = 0.$$

This condition is satisfied, for example, when the tensor $\alpha_{ik}(\mathbf{n}_c)$ is diagonal in a Cartesian coordinate system connected with the particle (i, k = x, y, z). Then, according to (14)

$$\begin{aligned}\langle\alpha_{-1}(\mathbf{h})\rangle = \langle\alpha_1(\mathbf{h})\rangle = {}^1/_2(1+R_2)\alpha_1(\mathbf{n}_c) + {}^1/_2(1-R_2)\alpha_0(\mathbf{n}_c) + {}^1/_2S_2\alpha_2(\mathbf{n}_c), \\ \langle\alpha_0(\mathbf{h})\rangle = (1-R_2)\alpha_1(\mathbf{n}_c) + R_2\alpha_0(\mathbf{n}_c) - S_2\alpha_2(\mathbf{n}_c), \\ \alpha_0 = \alpha_{zz},\quad \alpha_1 = {}^1/_2(\alpha_{xx} + \alpha_{yy}),\quad \alpha_2 = {}^1/_2(\alpha_{yy} - \alpha_{xx}), \\ R_n = \int d\Omega W(\Omega)\cos^n\beta,\quad S_2 = \int d\Omega W(\Omega)\sin^2\beta\cos 2\gamma.\end{aligned} \tag{22}$$

Here $A_{pq} = \delta_{pq}A_p$, $A_p = \operatorname{Im}\langle\alpha_p(\mathbf{h})\rangle$, and $T_{pq}(\mathbf{h}) = \delta_{pq}T_p$, where

$$\begin{aligned}T_0 = (1-R_2)(|\alpha_1|^2 + |\alpha_2|^2) + R_2|\alpha_0|^2 - S_2(\alpha_2\alpha_1^* + \alpha_1\alpha_2^*), \\ T_1 = T_{-1} = {}^1/_2(1+R_2)(|\alpha_1|^2 + |\alpha_2|^2) + {}^1/_2(1-R_2)|\alpha_0|^2 \\ + {}^1/_2S_2(\alpha_2\alpha_1^* + \alpha_1\alpha_2^*).\end{aligned} \tag{23}$$

The matrix $U_{\alpha i}$, made up of the components $f_\alpha^{(i)}$ that can be chosen in this case along $\mathbf{e}_\beta$ and $\mathbf{e}_\varphi$ (see the figure), is equal to $U_{\alpha 1} = -i\alpha U_{\alpha 2} = \alpha/2^{1/2}$, where $\alpha = \pm 1$. Therefore, taking (4) into account, we obtain

$$t^{(1)}(\mathbf{n}_0\mathbf{h}) = k^2[\langle\alpha_1(\mathbf{h})\rangle\cos^2\beta_0 + \langle\alpha_0(\mathbf{h})\rangle\sin^2\beta_0],\quad t^{(2)}(\mathbf{n}_0\mathbf{h}) = k^2\langle\alpha_1(\mathbf{h})\rangle. \tag{24}$$

Substituting then (24) in (11), we obtain explicit expressions for $g^{(ik)}(\mathbf{n}_0\mathbf{h})$.

The elastic-scattering cross sections $\sigma_s^{(m)}$ of waves with proper polarizations $\mathbf{e}_\beta$ and $\mathbf{e}_\varphi$ are respectively given by

$$\sigma_s^{(1)}(\mathbf{n}_0\mathbf{h}) = {}^8/_3\pi k^4[T_1(\mathbf{h})\cos^2\beta_0 + T_0(\mathbf{h})\sin^2\beta_0],\quad \sigma_s^{(2)}(\mathbf{n}_0\mathbf{h}) = {}^8/_3\pi k^4 T_1(\mathbf{h}). \tag{25}$$

The expressions for $\sigma_0^{(m)}$ differ in that ${}^8/_3\pi k^4 T_m$ is replaced by $4\pi k A_m$.

If $W(\beta;\gamma)$ depends only on β and is an even function of β, then the condition $\langle\alpha_{-1}(\mathbf{h})\rangle = \langle\alpha_1(\mathbf{h})\rangle$ is satisfied also for a non-diagonal tensor $\alpha_{ik}(\mathbf{n}_c)$ in the proper system. The eigenvalues $t^{(i)}$ and the scattering cross sections coincide with (24) and (25), but now

$$\begin{aligned}T_0 = {}^1/_2(1-R_2)(L_1 + L_{-1}) + R_2L_0, \\ T_1 = T_{-1} = {}^1/_4(1+R_2)(L_1 + L_{-1}) + {}^1/_2(1-R_2)L_0, \\ L_i = \sum_k |\alpha_{ki}(\mathbf{n}_c)|^2, \\ \langle\alpha_1(\mathbf{h})\rangle = \langle\alpha_{-1}(\mathbf{h})\rangle = {}^1/_4(1+R_2)[\alpha_{11}(\mathbf{n}_c) + \alpha_{-1-1}(\mathbf{n}_c)] \\ + {}^1/_2(1-R_2)\alpha_{00}(\mathbf{n}_c),\end{aligned} \tag{26}$$

$$\langle \alpha_0(\mathbf{h})\rangle = {}^1\!/_2(1-R_2)\left[\alpha_{11}(\mathbf{n}_c)+\alpha_{-1-1}(\mathbf{n}_c)\right]+R_2\alpha_{00}(\mathbf{n}_c).$$

4. MEDIUM THAT IS NOT AXIALLY SYMMETRICAL

We consider a case when the orientation of the particles depends both on β and on φ, but the Fourier expansion of the distribution function $W(\varphi;\beta)$ in $\cos n\varphi$ and $\sin n\varphi$ does not contain terms with $\cos\varphi$ and $\sin\varphi$. Then, when (14) is averaged, all the terms containing the non-diagonal elements of the polarizability tensor $\alpha_{ik}(n_c)$ drop out in the system that is rigidly bound to the particle:

$$\begin{gathered}\langle \alpha_{10}(\mathbf{h})\rangle = \langle \alpha_{-10}(\mathbf{h})\rangle = \langle \alpha_{01}(\mathbf{h})\rangle = \langle \alpha_{0-1}(\mathbf{h})\rangle = 0,\\ \langle \alpha_{-1-1}(\mathbf{h})\rangle = {}^1\!/_2(1+R_2)a_1 + {}^1\!/_2(1-R_2)a_0 + R_2a_{-1}\\ = \langle \alpha_{11}(\mathbf{h})\rangle + 2R_2a_{-1},\\ \langle \alpha_{00}(\mathbf{h})\rangle = (1-R_2)a_1 + R_2a_0,\end{gathered} \tag{27}$$

$$\begin{gathered}\langle \alpha_{-11}(\mathbf{h})\rangle = {}^1\!/_2(a_1-a_0)(N_2-iM_2),\\ \langle \alpha_{1-1}(\mathbf{h})\rangle = {}^1\!/_2(a_1-a_0)(N_2+iM_2);\end{gathered}$$

$$a_{\pm1} = {}^1\!/_2[\alpha_{11}(\mathbf{n}_c)\pm\alpha_{-1-1}(\mathbf{n}_c)], \qquad a_0 = \alpha_{00}(\mathbf{n}_c); \tag{28}$$

$$N_2 = \int d\Omega W(\Omega)\sin^2\beta\cos 2\varphi, \qquad M_2 = \int d\Omega W(\Omega)\sin^2\beta\sin 2\varphi. \tag{29}$$

Substituting then (1) in (3) and averaging, we obtain

$$\begin{gathered}\langle t_{\alpha\alpha}(\mathbf{nnh})\rangle = k^2(Q_1+Q_2), \quad \langle t_{\alpha-\alpha}(\mathbf{nnh})\rangle = {}^1\!/_2k^2(a_1-a_0)(K-i\alpha L),\\ Q_1 = {}^1\!/_2(1+R_2)a_1 + {}^1\!/_2(1-R_2)a_0 + {}^1\!/_2\sin^2\beta[{}^1\!/_2(3R_2-1)(a_0-a_1)\\ + {}^1\!/_2(a_1-a_0)(N_2\cos 2\varphi + M_2\sin 2\varphi)], \quad (30)\\ Q_2 = -R_2a_{-1}\cos\beta, \qquad L = (N_2\sin 2\varphi - M_2\cos 2\varphi)\cos\beta,\\ K = {}^1\!/_2(3R_2-1)\sin^2\beta + {}^1\!/_2(1+\cos^2\beta)(N_2\cos 2\varphi + M_2\sin 2\varphi).\end{gathered} \tag{31}$$

Since in this case $\langle \hat{t}^+\rangle\langle\hat{t}\rangle \neq \langle\hat{t}\rangle\langle\hat{t}^+\rangle$, the eigenvectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are not orthogonal. They can be determined by using formulas (30) and (5). The eigenvalues of (4) are

$$t^{(1,2)}(\mathbf{nh}) = k^2\{Q_1 \mp {}^1\!/_2(a_1-a_0)[K^2+L^2+4Q_2^2(a_1-a_0)^{-2}]^{1/2}\}. \tag{32}$$

Substituting (32) in (11), we easily determine $g^{(ik)}(\mathbf{nh})$. We note that the inequality $\alpha_{11}(n_c) \neq \alpha_{-1-1}(n_c)$ leads to different scattering of photons with right- and left-hand circular polarization.

If the distribution function $W(\beta;\varphi)$ depends on φ in an arbitrary manner, but is an even function of β, and the tensor $\alpha_{ik}(n_c)$ is diagonal, then the expressions (27)–(32) remain valid, and in this case

$$\alpha_{11}(\mathbf{n}_c) = \alpha_{-1-1}(\mathbf{n}_c), \quad Q_2 = 0, \quad \langle t^+\rangle\langle t\rangle = \langle t\rangle\langle t^+\rangle,$$

while the vectors $\mathbf{f}^{(i)}$ are orthogonal:

$$\mathbf{f}^{(1)} = \cos\theta\mathbf{e}_\beta + \sin\theta\mathbf{e}_\varphi, \qquad \mathbf{f}^{(2)} = -\sin\theta\mathbf{e}_\beta + \cos\theta\mathbf{e}_\varphi, \tag{33}$$

where θ is the angle that determines the direction of $\mathbf{f}^{(i)}$ in the $(e_\beta e_\varphi)$ plane, with $\tan 2\theta = L/K$. The matrix $U_{\sigma l}$, made up of the components (33), is unitary, $U_{\sigma 1} = -i\sigma U_{\sigma 2} = 2^{-1/2}\sigma e^{i\sigma\theta}$.

In both cases under consideration, the matrix $T_{mn}(h)$ can be obtained from the matrix $\langle a_{mn}(h)\rangle$ by replacing $\alpha_{nn}(n_c)$ in it by $\sum_m |\alpha_{mn}(n_c)|^2$.

5. DIRECTLY TRANSMITTED RADIATION

After determining $g^{(ik)}(nh)$ and obtaining from (9) the propagation functions G_{ik}, we need to find the value of (10) in order to calculate the density matrix (8) and the Stokes parameters (7). If the eigenvectors of the linear polarization can be chosen orthogonal, then using (10) and the explicit form of $U_{\sigma k}$, we obtain

$$M_{\alpha\sigma\eta\beta}(\mathbf{nrr'h}) = {}^1\!/_4e^{-i\theta(\sigma-\alpha+\beta-\eta)}[\alpha\sigma\eta\beta G_{11} + \alpha\sigma G_{12} + \eta\beta G_{12}^* + G_{22}], \tag{34}$$

where the angle θ is defined by (33). For an axially-symmetrical medium we can take $\theta = 0$.

To describe the radiation passing directly in the forward direction it suffices to calculate the free term in (8). Taking (34) into account, we obtain the following expressions for the Stokes parameters of radiation that has covered a distance l in the medium:

$$I_0 = {}^1\!/_2 I_0^{(0)}(e^{-\tau_1}+e^{-\tau_2}) + {}^1\!/_2 I_3^{(0)}(e^{-\tau_1}-e^{-\tau_2}). \tag{35}$$

If we interchange the Stokes parameters of the incident radiation in (35), replacing $I_0^{(0)}$ by $I_3^{(0)}$ and $I_3^{(0)}$ by $I_0^{(0)}$, then we obtain an expression for I_3:

$$I_2 = [I_2^{(0)}\cos\Phi + I_1^{(0)}\sin\Phi]\exp\left(-\frac{\tau_1+\tau_2}{2}\right). \tag{36}$$

We obtain I_1 from (36) by replacing $I_2^{(0)}$ by $I_1^{(0)}$ and $I_1^{(0)}$ by $-I_2^{(0)}$. Here $\tau_i = N_0 l\sigma_0^{(i)}(nh)$ is the optical path of the wave characterized by the polarization $f^{(i)}$, and $\Phi = 2\pi l/\lambda^{-1}[c/v_2 - c/v_1]$ is the phase shift of the ordinary and extraordinary waves along the path l. The Stokes parameters in formulas (35) and (36) are measured in a coordinate system in which the x and y axes are directed along the vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ of the proper linear polarization.

Expressions (35) and (36) remain in force also for an optically thick medium, if the incident radiation has one definite direction that coincides with the observation line (e.g., scattering of radiation from a remote source by a layer of matter), for in this case the probability of rescattering into a narrow forward angle is small.

We note that the radiation acquires a circular polarization $I_2 \neq 0$ even if it had no polarization at first, provided $I_1^{(0)} \neq 0$, i.e., if the direction of the preferred oscillations of the electric vector of the initial beam did not coincide with $\mathbf{f}^{(1)}$ or $\mathbf{f}^{(2)}$. The reason for this is the interference of waves propagating with different phase velocities. At $I^{(0)} = 0$ there is no such interference, owing to the statistical independence of the beams polarized along $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$. If the initial beam was polarized, then the beam becomes polarized, owing to the different absorption and scattering of waves with different polarizations. We note that the phase plates used in optics[5] are based on analogous effects.

It is also interesting to note that when light propagates in an interstellar medium, and particularly when it passes through dust clouds, effects described by formulas (35) and (36) may be observed. Rotation of the plane of linear polarization of light traveling from stars is possible. Conversion of linear polarization into circular or vice-versa is also possible. The observed complicated dependence of the circular and linear polarizations on the wavelength in white dwarfs[6] may be due to scattering in an anisotropic medium surrounding these stars. These phenomena were considered in detail in [7].

6. STOKES PARAMETERS OF RADIATION SCATTERED IN AN AXIALLY-SYMMETRICAL MEDIUM

To determine the Stokes parameters of the scattered radiation, it remains to find the quantity (12). We confine ourselves to its calculation in explicit form only for isotropic media such that the polarizability tensor of the particles in their own Cartesian system is diagonal, and the distribution function $W(\Omega)$ depends only on the angle β. In this case

$$\begin{gathered}\langle \alpha_{nm}(\mathbf{h})\alpha_{rs}^{+}(\mathbf{h})\rangle = \delta_{nm}\delta_{rs}H_{ns} + \delta_{ns}\delta_{rm}C_{nm},\\ H_{11} = H_{-1-1} = H_{1-1} = H_{-11} \equiv b_1, \quad H_{00} \equiv b_5, \quad C_{nn} = 0,\\ H_{10} = H_{-10} = H_{01}^{*} = H_{0-1}^{*} \equiv b_4 + ib_6, \quad C_{-11} = C_{1-1} \equiv b_3,\\ C_{10} = C_{01} = C_{0-1} = C_{-10} \equiv b_2. \end{gathered} \tag{37}$$

The quantities b_n are real and are given by

$$\begin{gathered} b_1 = {}^1/_4(1+2R_2+R_4)|\alpha_1|^2 + {}^1/_4(1-R_4)(\alpha_0\alpha_1^{*} + \alpha_0^{*}\alpha_1)\\ + {}^1/_8(1-2R_2+R_4)(|\alpha_2|^2 + 2|\alpha_0|^2),\\ b_2 = {}^1/_2(R_2-R_4)|\alpha_1-\alpha_0|^2 + {}^1/_4(1-R_4)|\alpha_2|^2,\\ b_3 = {}^1/_4(1-2R_2+R_4)|\alpha_1-\alpha_0|^2 + {}^1/_8(1+6R_2+R_4)|\alpha_2|^2,\\ b_4 = {}^1/_2(1-R_2)|\alpha_1|^2 + {}^1/_2(R_2-R_4)|\alpha_1-\alpha_0|^2 + {}^1/_4(1-2R_2+R_4)|\alpha_2|^2\\ + {}^1/_4(1+R_2)(\alpha_0\alpha_1^{*} + \alpha_0^{*}\alpha_1),\\ b_5 = {}^1/_2(1-2R_2+R_4)(2|\alpha_1|^2 + |\alpha_2|^2) + R_4|\alpha_0|^2\\ + (R_2-R_4)(\alpha_0\alpha_1^{*} + \alpha_1\alpha_0^{*}),\\ b_6 = {}^1/_4 i(1-3R_2)(\alpha_1\alpha_0^{*} - \alpha_0\alpha_1^{*}), \qquad \alpha_n \equiv \alpha_n(\mathbf{n}_c). \end{gathered}$$

Substituting then (37) in (12) we obtain after straightforward but quite prolonged calculations the value of $B_{\gamma\nu}$. To calculate the Stokes parameters it is more convenient to use not $B_{\gamma\nu}$ itself, but the quantities B_n ($n = 0, 1, 2, 3$), which are connected with $B_{\gamma\nu}$ by the relations (see (7)):

$$B_{\sigma\sigma} = {}^1/_2(B_0 - \sigma B_2), \qquad B_{\sigma-\sigma} = -{}^1/_2(B_3 + i\sigma B_1).$$

We obtain for them

$$B_n = k^4 N_0 \sum_m F_{nm} I_m^{(0)}; \tag{38}$$

$$\begin{gathered} F_{10} = {}^1/_2 b_4 \sin 2\beta_0 \sin\beta_1 \sin(\varphi_0-\varphi_1) - {}^1/_2 b_1 \sin^2\beta_0 \cos\beta_1 \sin 2(\varphi_0-\varphi_1),\\ F_{13} = {}^1/_2 b_4 \sin 2\beta_0 \sin\beta_1 \sin(\varphi_0-\varphi_1) + {}^1/_2 b_1(1+\cos^2\beta_0)\cos\beta_1 \sin 2(\varphi_0-\varphi_1),\\ F_{30} = b_2\cos^2\beta_0 \sin^2\beta_1 - {}^1/_4(b_1+b_3)(1+\cos^2\beta_0)\sin^2\beta_1\\ + {}^1/_2 b_5 \sin^2\beta_0 \sin^2\beta_1 + {}^1/_4 b_4 \sin 2\beta_0 \sin 2\beta_1 \cos(\varphi_0-\varphi_1)\\ - {}^1/_4 b_1 \sin^2\beta_0(1+\cos^2\beta_1)\cos 2(\varphi_0-\varphi_1). \end{gathered} \tag{39}$$

In order to obtain F_{01}, F_{31}, and F_{03} it suffices to interchange the angles, $\beta_0 \rightleftarrows \beta_1$ and $\varphi_0 \rightleftarrows \varphi_1$, in the expressions for F_{10}, F_{13}, and F_{30}. Next,

$$F_{20} = F_{23} = {}^1/_2 b_6 \sin 2\beta_0 \sin\beta_1 \sin(\varphi_0-\varphi_1), \tag{40}$$

and $F_{02} = F_{32}$ differs from F_{20} only in the interchange $\beta_0 \rightleftarrows \beta_1$. Finally,

$$\begin{gathered} F_{21} = -F_{12} = b_6 \sin\beta_0 \sin\beta_1 \cos(\varphi_0-\varphi_1),\\ F_{11} = b_4 \sin\beta_0 \sin\beta_1 \cos(\varphi_0-\varphi_1) + b_1 \cos\beta_0 \cos\beta_1 \cos 2(\varphi_0-\varphi_1),\\ F_{22} = b_4 \sin\beta_0 \sin\beta_1 \cos(\varphi_0-\varphi_1) + (b_1-b_3)\cos\beta_0\cos\beta_1,\\ F_{00} = {}^1/_4(b_1+b_3)(1+\cos^2\beta_0)(1+\cos^2\beta^1) + {}^1/_2 b_5 \sin^2\beta_0 \sin^2\beta_1\\ + b_2(1-\cos^2\beta_0\cos^2\beta_1) + {}^1/_4 b_1 \sin^2\beta_0 \sin^2\beta_1 \cos 2(\varphi_0-\varphi_1)\\ + {}^1/_4 b_4 \sin 2\beta_0 \sin 2\beta_1 \cos(\varphi_0-\varphi_1),\\ F_{33} = {}^1/_4(b_1+b_3-4b_2)\sin^2\beta_0\sin^2\beta_1 + {}^1/_2 b_5 \sin^2\beta_0 \sin^2\beta_1\\ + {}^1/_4 b_1(1+\cos^2\beta_0)(1+\cos^2\beta_1)\cos 2(\varphi_0-\varphi_1)\\ + {}^1/_4 b_4 \sin 2\beta_0 \sin 2\beta_1 \cos(\varphi_0-\varphi_1). \end{gathered} \tag{41}$$

If the incident radiation is not polarized, then all $I_m^{(0)} = 0$ with the exception of $I_0^{(0)}$, and the expressions in (39) become much simpler. As a limiting case we can obtain from (38) the well known formulas for scattering in an isotropic medium. In this case $W(\Omega) = (8\pi^2)^{-1}$, $R_2 = 1/3$, and $R_4 = 1/5$, with all $b_i = 0$ with the exception of b_1 and b_2. In addition, it is possible to choose the z axis along $\mathbf{n}_0$, to make $\beta_0 = \varphi_0 = 0$. The results make it possible to determine the Stokes parameters for different concrete cases. Let us consider by way of an example the scattering of an unpolarized plane wave by an axially-anisotropic plane layer of thickness L. Using (8), (34), and (38) we obtain for the radiation passing through the layer

$$I_n = I_0^{(0)} F_{n0} k^4 N_0 L \sec\beta_1. \tag{42}$$

For the particular case when the radiation is incident on the layer in the direction h, i.e., $\beta_0 = 0$, we obtain

$$\begin{gathered} I_0 = {}^1/_2 I_0^{(0)} k^4 N_0 L \sec\beta_1 (b_1+b_3)[2+(y-1)\sin^2\beta_1],\\ I_3 = {}^1/_2 I_0^{(0)} k^4 N_0 L \sec\beta_1 (b_1+b_3)(y-1)\sin^2\beta_1, \qquad I_1 = I_2 = 0,\\ y = 2b_2(b_1+b_3)^{-1}. \end{gathered} \tag{43}$$

The case $y = 0$ corresponds to complete orientation of the particles along the direction of the incident radiation ($R_2 = R_4 = 1$), and also at $\alpha_{xx}(\mathbf{n}_c) = \alpha_{yy}(\mathbf{n}_c)$, to complete orientation in the perpendicular direction ($R_2 = R_4 = 0$). In this case the electric vector $\mathbf{E}^{(0)} \perp \mathbf{h}$ cannot induce a dipole moment along h, and no axial anisotropy of the particles comes into play. Therefore (43) at $y = 0$ coincides with the expression for Rayleigh scattering. At $y = 1$, the scattered radiation is isotropic and unpolarized, and at $y > 1$ polarization of opposite sign appears. By way of an example we indicate that when light of $\gamma = 5 \times 10^{-5}$ cm is scattered by dust particles with 50% orientation ($R_2 = 2/3$, $R_4 = 3/5$) in the form of ellipsoids of revolution and made of iron graphite, with an axis ratio 0.1, the parameters y are equal to 0.15 and 0.26, respectively.

[1]V. V. Zheleznyakov and V. I. Tatarskiĭ, Usp. Fiz. Nauk **102**, (9), (1970) [sic !].

[2]R. C. Newton, Scattering Theory of Waves and Particles, McGraw, 1966.

[3]A. Z. Dolginov, Yu. N. Gnedin, and N. A. Silant'ev, JQSRT **10**, 707 (1970).

[4]Yu. N. Gnedin, A. Z. Dolginov, and N. A. Silant'ev, Zh. Eksp. Teor. Fiz. **58**, 706 (1970) [Sov. Phys. JETP **31**, 378 (1970)].

[5]M. Born and E. Wolf, Principles of Optics, Pergamon, 1966.

[6]J. C. Kemp, Astrophys. J. **162**, 69 (1970).

[7]Yu. N. Gnedin, A. Z. Dolginov, and N. A. Silant'ev, Astron. Zh. (1971) (in press).

Translated by J. G. Adashko
14.