

Wave Processes in Films of Superfluid Helium

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Propagation of waves in saturated films and in thin films of helium is considered for the case when some slowing down of the normal component of the liquid occurs. It is shown that surface waves of two types exist. One is related to van der Waals interaction between the film and substrate, and the second to propagation of temperature oscillations and to periodic evaporation and condensation of helium. In both types of waves, both the temperature and thickness of the film oscillate. A peculiar wave dispersion that depends on the film thickness and frequency of the propagating wave is observed. The propagation velocities and absorption coefficients of the waves are calculated.

WAVES of two types propagate inside superfluid helium, namely ordinary acoustic waves connected with the density oscillations (first sound) and waves peculiar to a superfluid liquid and connected with temperature oscillations (second sound). Allowance for the boundary leads to the appearance of a rapidly damped transverse wave (viscous wave) near the wall.

In addition, surface waves analogous to gravitational-capillary waves propagating on the surface of an ordinary liquid can propagate on the free surface of superfluid helium^[1-3]. For a helium layer of thickness d , the square of the propagation velocity of these waves is

$$u^2 = \left(\frac{f}{k} + \frac{\gamma k}{\rho} \right) \text{th } kd,$$

where k is the wave vector, ρ is the density of the liquid, γ is the surface-tension coefficient, and f is the constant of the van der Waals interaction of the superfluid film with the substrate (in helium, owing to the small thickness of the films, the gravitational constant g which determines the velocity of the gravitational waves in an ordinary liquid is replaced by f).

For short waves ($kd \gg 1$) the principal role is played by surface tension, and the wave velocity is $u = \sqrt{\gamma k}$. It should be noted that in this case the perpendicular-to-the-surface velocities v_n and v_s of the normal and superfluid components of the liquid are much larger than the longitudinal velocities. These waves, which are called capillary waves, are therefore in essence transverse waves. In the long-wave case ($kd \ll 1$) the principal role is played by the van der Waals interaction, and the wave velocity is $u = \sqrt{fd}$. In this case the longitudinal components of v_n and v_s are much larger than the perpendicular components. The principal oscillation is along the surface and actually a longitudinal wave propagates through the film.

The propagation velocity of the surface wave is determined by the formula given above under conditions when the normal component of the liquid oscillates freely. With decreasing film thickness, however, the normal component of the liquid is decelerated and the character of the waves propagating over the surface changes. The degree of deceleration of the normal component of the liquid depends on the ratio of the film thickness d to the depth of penetration of the viscous wave λ_v . In the limit when $\delta = d/\lambda_v \ll 1$ the normal component is completely stopped and the surface wave

is due to oscillations of the superfluid component of the liquid only. Atkins^[3] investigated the propagation of the surface waves under these conditions. The long-wave case was considered, i.e., longitudinal oscillations, and attention was paid to the fact that the oscillation of the superfluid component leads to oscillation of the temperature and effects of evaporation of helium from the film become important. These waves were called third sound and were thoroughly investigated^[3-4].

We consider in this paper the propagation of waves in a superfluid film at an arbitrary degree of deceleration of the normal component. We confine ourselves to the long-wave limit ($kd \ll 1$).

To solve our problem it is necessary to solve the complete system of hydrodynamic equations of a superfluid liquid, which is modified somewhat for films^[3-5]. We consider a film that is unbounded in two directions and has an equilibrium thickness d . We choose the z axis along the normal to the film and the x axis along the wave propagation. If we consider waves whose length greatly exceeds the film thickness, then we can set v_{nz} and v_{sz} equal to zero, and the pressure P and the temperature T can be regarded as independent of the coordinate z . The situation here is analogous to that obtained in the case of long gravitational waves in an ordinary liquid^[1].

The mass conservation law for a column of the superfluid film of thickness d , having a unit width (along the y axis) is given by

$$\rho \frac{\partial \zeta}{\partial t} + \rho_s d \frac{\partial v_s}{\partial x} + \rho_n d \frac{\partial v_n}{\partial x} + KT' = 0. \tag{1}$$

where ζ is the deviation of the film thickness from the equilibrium value d , ρ_s and ρ_n are respectively the densities of the superfluid and normal components, T' is the deviation of the temperature from the equilibrium value, v_s is the x -component of the velocity of the superfluid component of the liquid, and v_n is the value of the x -component of the velocity of the normal component averaged over z . v_n depends on the thickness of the film, since it is assumed that the depth of penetration of the viscous wave can be of the order of the film thickness and there is partial deceleration of the normal component of the wave.

On the free surface of the oscillating film we have $\zeta \approx v_{sz} \approx v_{nz}$, and it is easy to see from (1) that

$v_{sz}, v_{nz} \sim kd(\rho_s v_{sx} + \rho_n v_{nx})$. Since we consider long-wave oscillations ($kd \ll 1$), it is clear that $v_{sz}, v_{nz} \ll v_{sx}, v_{nx}$.

The last term in (1) takes into account the mass lost to evaporation. Following Atkins^[5], we assume that

$$K = \nu(M / 2\pi RT)^{1/2} \beta,$$

where $\beta = (dP/dT)_{sv}$ is the derivative of the saturated-vapor pressure with respect to the temperature, M is the mass of one mole of helium, and ν is a coefficient close to unity.

The energy conservation law yields the equation

$$\rho dC \frac{\partial T'}{\partial t} - d\rho_s \sigma T \frac{\partial v_s}{\partial x} + d\rho_n \sigma T \frac{\partial v_n}{\partial x} + KLT' = 0, \quad (2)$$

where σ , C , and L are respectively the entropy, specific heat, and heat of evaporation of one gram of liquid. The last term in (2) takes into account the energy loss due to evaporation of helium from the film. Experimental data on the temperature dependences of β and L are given in^[6,7].

The equation of motion of the superfluid component has formally the same form as for a free volume

$$\partial v_s / \partial t + \partial \mu / \partial x = 0. \quad (3)$$

In this equation, however, it must be recognized that the chemical potential has an increment connected with the van der Waals interaction of the film with the substrate: $\mu = \mu_0 - \alpha d^{-n}$.

The value of the exponent n depends on the film thickness^[8], and $n = 4$ for thick films and $n = 3$ for thin ones. As shown by estimates^[8,9], films with $d \sim 5 \times 10^{-6}$ cm lie in the transition region between $n = 3$ and $n = 4$. For thin films $\alpha_1 \rho = 2 \times 10^{-14}$ and for thick ones $\alpha_2 \rho = 7.5 \times 10^{-20}$ erg-cm.

Recognizing that the pressure change is determined mainly by the oscillations of the film thickness, we obtain

$$\frac{\partial \mu}{\partial x} = -\sigma \frac{\partial T}{\partial x} + f \frac{\partial \xi}{\partial x},$$

$$f = nad^{-(n+1)}.$$

The equation for the motion of the normal component of the liquid can be obtained from the momentum conservation law. Averaging the x -component of the momentum-conservation law over z (see, for example, ^[10]), we obtain

$$\rho_n \frac{\partial v_n}{\partial t} + \rho_s \frac{\partial v_s}{\partial t} + \frac{\partial P}{\partial x} - \omega \rho r v_n = 0, \quad (4)$$

where $\omega \rho r v_n$ is the average decelerating force due to the finite viscosity of the helium and to the presence of the boundary. The quantity r is a complex function of the frequency of the sound and of the film thickness, and has the same form as the function r introduced in^[11] for a plane-parallel capillary of thickness $2d$:

$$r = -i \frac{\rho_n}{\rho} \frac{\text{tg } k_3 d}{k_3 d - \text{tg } k_3 d} = \frac{\rho_n}{\rho} (m_1 + im_2) = \frac{a + i[a^2 - b(1-b)]}{a^2 + (1-b)^2}, \quad (5)$$

where $k_3 = (i\omega \rho_n / \eta)^{1/2}$ is the wave vector of the viscous wave. The quantities a and b are functions of δ :

$$a = \frac{\text{sh } 2\delta - \sin 2\delta}{4\delta(\cos^2 \delta + \text{sh}^2 \delta)}, \quad b = \frac{\text{sh } 2\delta + \sin 2\delta}{4\delta(\cos^2 \delta + \text{sh}^2 \delta)}.$$

Plots of the dependence of a and b on δ are given in^[11].

From the condition of compatibility of Eqs. (1)–(4) we obtain the following dispersion equation:

$$k_{||}^4 - k_{||}^2 \left(k_1^2 + k_2^2 + ir \frac{\rho}{\rho_n} \frac{k_2^2 - pk_1^2}{1-p} \right) + k_1^2 k_2^2 \left(1 + ir \frac{\rho}{\rho_n} \right) = 0, \quad (6)$$

where

$$p = -\frac{\rho_n}{\rho_s} \frac{k_2^2 - k_1^2(1+\lambda)}{k_2^2 - k_1^2[1 - (\rho_n/\rho_s)\lambda]}, \quad \lambda = i \frac{Kf}{\rho\omega\sigma} - \frac{\beta}{\rho\sigma};$$

k_1 and k_2 are the wave vectors of the oscillations propagating in a film of sufficient thickness to eliminate deceleration of the normal component ($\delta \gg 1$, $|r| \rightarrow 0$):

$$k_1^2 = \frac{\omega^2}{fd}, \quad k_2^2 = \frac{\rho_n}{\rho_s} \frac{\omega^2 C}{T\sigma^2} \left(1 + \frac{iKL}{C\rho\omega d} \right). \quad (7)$$

The first solution gives the surface van der Waals wave, and the second corresponds to second sound (thermal wave). The second term in k_2^2 is not small and is due to evaporation; it will be shown below that it is important to take it into account.

We note that the dispersion Eq. (6) coincides formally with the equation obtained by Pollack^[5], who, however, introduced r as an independent parameter at fixed values of T , d , and ω , in spite of the fact that r can vary only when either the film thickness d or the sound frequency ω is varied. The parameter r depends also on the temperature, owing to the change of the viscosity of the helium. In the investigation of the dispersion of waves propagating in helium films it is therefore important to take into account the dependence of r on the sound frequency and on the film thickness.

In the region of strong deceleration ($\delta \ll 1$), the inequality $k_1^2 \lesssim k_2^2$ is satisfied simultaneously with the inequality $|r| \gg 1$, so that the solution of Eq. (6) yields the following expressions for the squares of the wave vectors:

$$k_{||1}^2 = \frac{\rho k_1^2 k_2^2}{\rho_s k_2^2 + \rho_n k_1^2 (1+\lambda)}, \quad (8)$$

$$k_{||2}^2 = \left(1 + ir \frac{\rho}{\rho_n} \right) \left(\frac{\rho_s}{\rho} k_2^2 + \frac{\rho_n}{\rho} k_1^2 (1+\lambda) \right).$$

In the case of weak deceleration ($\delta \geq 1$) the condition $k_1^2 > k_2^2$ is satisfied simultaneously with the inequality $|r| \lesssim 1$, and we obtain the following solution of the dispersion equation:

$$k_{||1}^2 = \frac{1 + ir\rho/\rho}{1 + ir} k_2^2, \quad k_{||2}^2 = k_1^2 (1 + ir). \quad (9)$$

As seen from (8), in the region of the strong deceleration of the first mode $k_{||1}^2 = k_1^2 \rho_s / \rho$ (if $k_2^2 \gg k_1^2$), and it is determined mainly by the force of the van der Waals interaction, whereas k_1^2 and k_2^2 become of the same order with increasing δ , and the contribution of the temperature oscillations becomes of the same order as the contribution of the van der Waals waves. At $\delta \geq 1$ the temperature oscillations assume the principal role.

In the second mode, the situation is reversed. In the strong deceleration region ($\delta \ll 1$) the principal role is played by the thermal waves, and at $\delta \geq 1$ the oscillations are determined by the van der Waals waves.

From the system (1)–(4) it is easy to calculate the ratio of the temperature-oscillation amplitudes to the film thickness:

$$\frac{T_0'}{\xi_0'} = \frac{\rho_n}{\rho} \frac{f}{\sigma} \frac{k_{||}^2 - k_1^2 (1 + ir\rho/\rho_n)}{k_{||}^2 - (k_2^2 + \rho_n \omega K / \rho d \sigma) (1 + ir\rho/\rho_n)}. \quad (10)$$

Substituting in this formula the expression for $k_{||1}$ and $k_{||2}$, we obtain the ratio of the oscillation amplitudes in the first and second modes. Numerical estimates show that at $d = 10^{-6}$ cm and $\omega = 2\pi \times 10^3$ sec $^{-1}$ we have $T'_{01}/\xi'_{01} = -1.3 \times 10^{-2} + i4.64$ deg/cm for the first mode and $T'_{02}/\xi'_{02} = 8.78 \times 10^6 + i2.12 \times 10^2$ deg/cm for the second mode. Thus, at small film thicknesses the temperature oscillation in the second mode is much larger than in the first. The fact that the ratio of the amplitudes is complex indicates that there is a phase shift between the oscillations of the temperature and of the film thickness. For the first mode the phase shift is $\approx \pi/2$, and in the second mode the phase shift is negligible. With increasing film thickness the situation changes and, as already noted, in thick films the temperature oscillations in the second mode becomes negligible (van der Waals wave). For the first mode, on the other hand, estimates for $d = 10^{-4}$ cm yield $T'_{01}/\xi'_{01} = -4.19 + i1.03$ deg/cm.

In the case of sound propagation in a system of narrow capillaries, the dispersion equation that determines the oscillation modes has formally the same form (6), but k_1^2 and k_2^2 are respectively the squares of the wave vectors of the first and second sound, for which the inequality $k_2^2 \gg k_1^2$ is always satisfied. In the propagation of sound through capillaries^[11], unlike in a film, there is therefore no transition from one oscillation regime to another. A similar result was obtained by Pollack^[5] for a film, since he did not take into account the dependence of r on the film thickness and on the sound frequency.

From (8) and (9) we obtain for the propagation velocity of the first mode

$$v_{||1} = \begin{cases} \left[2 \frac{\rho_s}{\rho} f d \right]^{1/2} [(q^2 + g^2)(\sqrt{q^2 + g^2} + q)^{-1}]^{1/2}, & k_1^2 \leq k_2^2, \\ \left[2 \frac{\rho_s}{\rho} g f d \left(1 - \frac{\rho_s b}{\rho}\right) \right]^{1/2} \left[1 + \frac{a \rho_s}{2(\rho - \rho_s b)} \right], & k_1^2 > k_2^2, \end{cases} \quad (11)$$

where

$$q = 1 + T\sigma/L, \quad g = \rho\omega\sigma^2 T/KLf.$$

Figure 1 shows plots of the sound velocity against the film thickness d and the sound frequency ω . Up to thicknesses $d \sim 10^{-5}$ cm we obtain the result of Atkins^[3-4] (third sound), after which the temperature wave and the wave connected with the van der Waals forces make equal contributions, corresponding to the minimum region on Fig. 1a. At $d > 10^{-5}$ cm the temperature oscillations assume the principal role and a certain increase of the velocity is observed. As to the dependence of the sound velocity on ω (Fig. 1b), at low frequencies, when there is strong deceleration of the normal component of

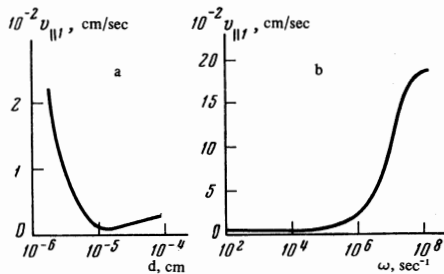


FIG. 1. Dependence of the propagation velocity of the first mode on the film thickness (a) at $\omega = 2\pi \times 10^3$ sec $^{-1}$ and on the sound frequency (b) at $d = 5 \times 10^{-6}$ cm for $T = 1.8^\circ\text{K}$.

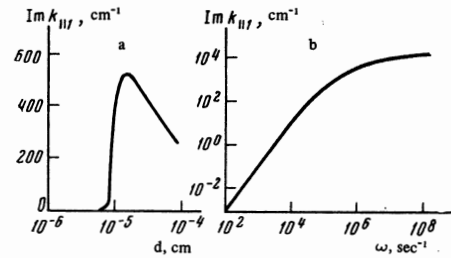


FIG. 2. Dependence of the absorption coefficient of the first mode on the film thickness (a) at $\omega = 2\pi \times 10^3$ sec $^{-1}$ and on the sound frequency (b) at $d = 5 \times 10^{-6}$ cm for $T = 1.8^\circ\text{K}$.

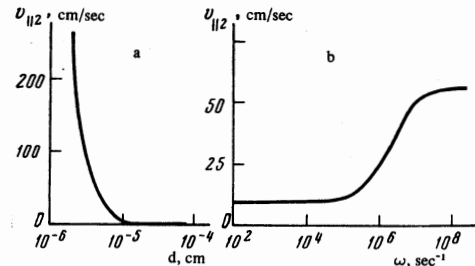


FIG. 3. Dependence of the propagation velocity of the second mode on the film thickness (a) at $\omega = 2\pi \times 10^3$ sec $^{-1}$ and on the sound frequency (b) at $d = 5 \times 10^{-6}$ cm for $T = 1.8^\circ\text{K}$.

the liquid, the sound velocity is practically independent of ω (third sound), and with increasing frequency, at $\omega > 10^5$ sec $^{-1}$, the temperature oscillations come into play, and the velocity increases and reaches the value of the velocity of second sound at very high frequencies.

From (8) and (9) we can also determine the sound absorption coefficient

$$\text{Im } k_{||1} = \begin{cases} \frac{\omega}{v_{||1}} g^{-1} (\sqrt{q^2 + g^2} - q) + \frac{\omega^2}{6v_{||1}} \frac{\rho_n^2 d^2}{\rho_s \eta}, & k_1^2 \leq k_2^2 \\ \frac{\omega}{v_{||1}} \left(1 + \frac{a \rho_s}{\rho - \rho_s b}\right), & k_1^2 > k_2^2 \end{cases} \quad (12)$$

The last term in (12) for the case $k_1^2 \leq k_2^2$ is connected with the slipping of the normal component of the liquid. It is interesting to note that it is inversely proportional to the viscosity coefficient η . This is connected with the fact that v_n decreases when η increases, and consequently the sound absorption connected with the oscillation of v_n decreases. An analogous situation obtains also in the case of fourth sound^[10].

As seen from Fig. 2a, for thin films ($d < 10^{-5}$) the absorption is small; with increasing film thickness, temperature oscillations come into play, the absorption increases sharply, reaches a maximum, and then decreases with a change over to a new oscillation regime. The sound absorption increases with increasing frequency (Fig. 2b).

For the second mode we obtain from (8) and (9) the following expression for the velocity

$$\text{Im } k_{||2} = \begin{cases} \frac{\omega}{v_{||2}} g^{-1} (\sqrt{g^2 + 1} + 1)^{1/2}, & k_1^2 \leq k_2^2 \\ \frac{\omega}{v_{||2}} \frac{\rho}{\rho_n m_1} \left\{ \left[\left(1 - \frac{\rho_n}{\rho} m_2\right)^2 + \frac{\rho_n^2}{\rho^2} m_1^2 \right]^{1/2} + \frac{\rho_n}{\rho} m_2 - 1 \right\}, & k_1^2 > k_2^2 \end{cases} \quad (13)$$

Figure 3 shows the dependence of the velocity of sound on the film thickness and on the sound frequency. At low thicknesses, the principal role is played by

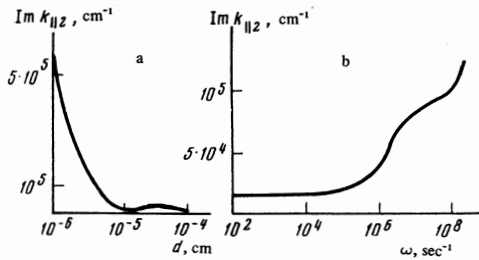


FIG. 4. Dependence of the absorption coefficient of the second mode on the film thickness (a) at $\omega = 2\pi \times 10^3$ sec^{-1} and on the sound frequency (b) at $d = 5 \times 10^{-6}$ cm for $T = 1.8^\circ K$.

waves connected with evaporation of helium from the surface of the film, and with decreasing film thickness contributions are made also by waves connected with the van der Waals interaction. The transition from one oscillation regime to another is clearly seen on the plot of the sound velocity against the frequency (Fig. 3b). At low frequencies, where evaporation effects are significant, there is a horizontal section, followed by a region of strong dispersion, and at high frequencies the evaporation effects cease to play a role and a horizontal section connected with the van der Waals waves begins.

From (8) and (9) we can also determine the sound absorption coefficient

$$v_{in} = \begin{cases} \delta \left(\frac{3\rho gfd}{\rho_n} \right)^{1/2} [\gamma g^2 + 1 - 1]^{-1/2}, & k_1^2 \leq k_2^2 \\ \sqrt{2fd} \left\{ \left[\left(1 - \frac{\rho_n m_2}{\rho} \right)^2 + \frac{\rho_n^2 m_1^2}{\rho^2} \right]^{1/2} + 1 - \frac{\rho_n m_2}{\rho} \right\}^{1/2}, & k_1^2 > k_2^2 \end{cases} \quad (14)$$

Figure 4 shows plots of the dependences of the absorption coefficient of the second mode. With increasing film thickness, the absorption decreases, but there is also a small maximum, which is due to the viscous forces in the region of partial deceleration of the normal component. With increasing frequency, the absorption coefficient increases. It should be noted that this mode is strongly damped in the entire considered range of film thicknesses and sound frequencies, unlike the first mode, for which weak damping (third sound) is observed in the region of low frequencies and thin films.

We have considered sound absorption mechanisms connected with the slipping of the normal component of the liquid and with evaporation of the helium from the surface of the film. Sound absorption can be due also to the bulk viscosity coefficients η , ζ_1 , ζ_2 , and ζ_3 and to heat transfer through the wall. These absorption mech-

anisms can be easily taken into account^[12]. Their contribution to the sound absorption is small and can be noticeable only in the limit when the normal component is completely stopped. Under conditions when third sound can propagate, the only second-viscosity coefficient playing any role is ζ_3 .

We note that the foregoing examination of oscillatory processes pertains only to saturated films, and also to helium layers that are not too thick, when effects connected with the partial deceleration of the normal component of the liquid are significant. In the case of thin unsaturated films, when $v_n = 0$ and the density ρ_s of the superfluid component depends on the film thickness, an important role is assumed by the connection between the film oscillations and the oscillations of the gas over the film, due to the periodic evaporation and condensation of the helium^[13]. On the other hand, in the second limiting case of thick helium films, $d = 0.1-1$ cm , ordinary capillary-gravitational waves are observed^[14].

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