Theory of Synchrotron Radiation in Matter

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The radiation emitted by an ultrarelativistic particle moving in matter along or across a magnetic field is investigated. The spectral distribution of the radiation intensity is determined for frequencies that can be treated classically. It is shown that in the case of motion along a strong magnetic field the spectral distribution of the radiation intensity depends nonlinearly on the time of motion of the particle in the matter. For particles moving perpendicular to a weak magnetic field the spectrum has a plateau that depends on the properties of the medium.

RADIATION PRODUCED WHEN AN ELECTRON MOVES ALONG A MAGNETIC FIELD

1. The question of the radiation of relativistic electrons in a magnetic field in vacuum has been considered by a number of workers[1-3]. It is of interest to determine the character of the radiation when the charged particle moving in the magnetic field is simultaneously scattered by the atoms of the medium. In this case one should assume, in general, that the particle enters the medium at an arbitrary angle to the magnetic-field direction. However, such an analysis and the resultant formulas are too complicated in the case of an arbitrary entrance angle, so that it is advisable to consider two limiting cases: (a) the particle enters the medium parallel the magnetic field, and (b) the particle moves perpendicular to the magnetic field. We start the analysis with the case when the particle moves along the magnetic field prior to entering the medium.

Since the particle velocity was parallel to the magnetic field prior to entering the medium, the synchrontron radiation connected with the rotation of the magnetic field can appear only as a result of scattering of the particle by the atoms of the medium, and consequently will be due to multiple scattering in the substance. The solution of the problem, for frequencies that are low compared with the particle energy ($\omega \ll E$), can be obtained in pure classical manner. In the limiting case of high energies of the emitted quanta and not too long times of particle motion in the medium, the spectral distribution of the irradiation intensity obeys the usual formula for the bremsstrahlung energy spectrum[4].

Multiple scattering in a medium is considered under the usual assumption that the mean-square angle of multiple scattering during the entire time of flight T of the particle through the medium is much smaller than unity[5]:

$$\langle\theta_T^2\rangle = E_s^2 T / E^2 L \ll 1, \tag{1.1}$$

where $E_S^2 = 1700 m^2$, $L = [4Z^2 n_0 e^6 \ln(191 Z^{-1/3})/m^2]^{-1}$ is the cascade length unit[6], n_0 is the density of the atoms of the medium, and Z is the atomic number of the medium. The magnetic field is assumed sufficiently strong:

$$\omega_H \tau \gg 1. \tag{1.2}$$

Here $\tau = (n_0\sigma)^{-1}$ is the time traversed by the particle in the medium between two collisions, and $\omega_H = eH/E$ is the Larmor frequency. The physical meaning of the latter condition is obvious: during the time between two successive collisions the particles should execute many revolutions in the medium. Using the cross section given in[7] for the scattering of relativistic particles in an external field, and choosing the screened Coulomb potential for the scattering center, we can rewrite the estimate (1.2) in the form

$$H \gg e^{-1} n_0 \sigma E = 4\pi Z^2 e^3 n_0 E (a_0 Z^{-1/3})^2, \tag{1.3}$$

where a_0 is the Bohr radius. For electrons with energy $E = 10m = 5$ MeV and for a medium density $n_0 = 10^{18}$ cm^{-3}, the magnetic field should satisfy the condition $H \gg 300\ Z^{4/3}$ Oe.

Let us determine now the physical picture of the motion of the particle in a medium in a longitudinal magnetic field. Prior to the scattering, since $\mathbf{H} \parallel \mathbf{v}_0$, the magnetic field does not affect the motion of the particle. As a result of a collision with the atoms of the medium, the particle acquires velocity components perpendicular to H, and moves in a magnetic field along a helix between collisions. Were there no magnetic field, the particle, after colliding with the scattering center and acquiring a velocity component $\mathbf{v}_\perp \perp \mathbf{v}_0$, would move freely between the collisions and would traverse after a time t a distance $\mathbf{r}_\perp \sim \mathbf{v}_\perp t$ relative to the trajectory of the initial motion. If we confine ourselves to the customary assumption in the theory of multiple scattering, that the successive collisions are statistically independent, then $\langle v_\perp^2(t)\rangle \sim t$ (the mean value is taken over the positions of the scattering atoms), and the mean-squared perpendicular displacement is

$$\langle \mathbf{r}_\perp^2(t)\rangle \sim t^3.$$

In a magnetic field the particle does not move freely between collisions, its trajectory is a helix, and if the condition (1.2) is satisfied the particle has no displacement perpendicular to v_0 between collisions. Only the collisions lead to a displacement in a direction perpendicular to the unperturbed trajectory. Therefore, for large travel times, $t > \omega_H^{-1}$, the means-squared perpendicular displacement should increase linearly:

$$\langle \mathbf{r}_\perp^2(t)\rangle \sim t.$$

In the case or radiation connected with the scattering in the medium, motion of the particle along a helix in the magnetic field between collisions alters the spectrum radically, as will be shown below.

2. To obtain the spectral distribution of the energy radiated by a particle moving in a medium in a longitudinal magnetic field, we can use expression (2) of the paper of Landau and Pomeranchuk (see, for example,[4]):

$$dI = \frac{e^2\omega\, d\omega}{\pi}\int_0^T dt_1 \int_0^T dt_2 \frac{\exp[i\omega(t_1-t_2)]}{|\mathbf{r}_1-\mathbf{r}_2|}\left[\mathbf{v}_1\mathbf{v}_2 - \frac{(\mathbf{v}_1\mathbf{g})(\mathbf{v}_2\mathbf{g})}{g^2}\right]\sin g. \tag{2.1}$$

Here

$$\mathbf{v}(\tau) = \mathbf{v}_0(1-{}^1/_2\theta_\tau^2) + v_0\theta_\tau, \quad \mathbf{v}_0\theta_\tau = 0, \tag{2.2}$$

$$\mathbf{g} = \omega(\mathbf{r}_1-\mathbf{r}_2), \tag{2.3}$$

$$\mathbf{r}_1 = \mathbf{r}(t_1), \quad \mathbf{r}_2 = \mathbf{r}(t_2), \quad \mathbf{r}_1-\mathbf{r}_2 = \int_{t_2}^{t_1}\mathbf{v}(\tau)\,d\tau. \tag{2.4}$$

θ_T is the angle of multiple scattering in magnetic field after a time τ. Formula (2.1) was derived under the assumption that the effective values of g are large compared with unity. In this problem, the essential time interval of the emission is determined both by the characteristic time of the bremsstrahlung[4]

$$t_e \sim E^2/\omega m^2, \tag{2.5}$$

and by the characteristic time of the synchrotron radiation[7]

$$t_H \sim m/E\vartheta\omega_H. \tag{2.6}$$

Consequently, if we consider frequencies such that

$$\omega t_H \sim \omega m/E\vartheta\omega_H \gg 1, \quad \text{т. е.} \quad \omega \gg E\vartheta\omega_H/m \tag{2.7}$$

(ϑ is the angle between the direction of the magnetic field and the velocity), then the use of formula (2.1) is legitimate. We shall henceforth assume that the last inequality is satisfied. Thus, the region of frequencies under consideration is bounded by the inequalities

$$E\vartheta\omega_H/m \ll \omega \ll E. \tag{2.8}$$

Using the smallness of θ_T and taking (2.2)–(2.4) into account we can easily reduce the expression for the spectral intensity of the radiation to the form

$$dI = \frac{e^2\omega\, d\omega}{\pi}\int_0^T dt_1\int_0^T dt_2 \Big\langle \Big[\theta_{t_1}\theta_{t_2} + \frac{1}{(t_1-t_2)^2}\Big(\int_{t_2}^{t_1}\theta_\tau\, d\tau\Big)^2 - \frac{\theta_{t_1}+\theta_{t_2}}{t_1-t_2}\int_{t_2}^{t_1}\theta_\tau\, d\tau\Big]\Big\rangle \frac{\exp[i\omega(t_1-t_2)]\sin\langle g\rangle}{|t_1-t_2|}. \tag{2.9}$$

Just as in[4], when averaging all the possible trajectories of the particle in the medium, the mean products are replaced by products of the means, and the mean of the sine function is replaced by the sine of the mean.

3. For further calculation of the spectral density of the radiation intensity, it is necessary to substitute in (2.9) the classical trajectory of the motion. To find it, we start from the Lagrange equation for the motion of the particle in a magnetic field in a medium:

$$d\mathbf{p}/dt = -\nabla U[\mathbf{r}(t)] + e[\mathbf{vH}]. \tag{3.1}$$

Here $U(\mathbf{r})$ is the potential of the atoms of the medium, and can be chosen in the form of a sum over the individual atoms:

$$U(\mathbf{r}) = \sum_a U_0(|\mathbf{r}-\mathbf{R}_a|); \tag{3.2}$$

$\mathbf{R}_a$ is the coordinate of the a-th atom. Assuming the particle energy to be sufficiently high, we can replace $\mathbf{r}(t)$ in $U[\mathbf{r}(t)]$ in the right-hand side of (3.1) by $\mathbf{r}_0(t) = \mathbf{v}_0 t$ and make in the left-hand side the substitution

$$d\mathbf{p}/dt = E\,d\mathbf{v}/dt. \tag{3.3}$$

Further, expanding $U(\mathbf{r})$ in a Fourier integral, taking into account the smallness of the angle of deviation of the particle in the field of the a-th center, i.e., $q_\parallel \ll q_\perp$, we obtain

$$U(\mathbf{r}) \equiv U[\mathbf{r}_0(t)] = \sum_a 2\pi\int d^2q_\perp U_0(\mathbf{q}_\perp) \times \exp(-i\mathbf{q}_\perp\mathbf{R}_{a\perp})\,\delta(v_0t - z_a), \tag{3.4}$$

where $U_0(\mathbf{q})$ is the Fourier component of the potential $U_0(\mathbf{r})$. The direction of motion of the particle (and of the magnetic field) is chosen as the z axis. Substituting (3.2), (3.3), and (3.4) in (3.1), writing out (3.1) in terms of its components, and solving this equation with initial conditions $v_x(0) = v_y(0) = 0$, we can easily obtain

$$v_x(t) = -\int_0^t \frac{dt'}{E}\cos[\omega_H(t-t')]\sum_a\int d^2q_\perp U_0(\mathbf{q}_\perp)\,iq_x \times 2\pi\delta(v_0t'-z_a)\exp(-i\mathbf{q}_\perp\mathbf{R}_{a\perp}) + \int_0^t\frac{dt'}{E}\sin[\omega_H(t-t')] \times \sum_a\int d^2q_\perp U_0(\mathbf{q}_\perp)\,iq_y 2\pi\delta(v_0t'-z_a)\exp(-i\mathbf{q}_\perp\mathbf{R}_{a\perp}),$$

$$v_y(t) = -\int_0^t\frac{dt'}{E}\cos[\omega_H(t-t')]\sum_a\int d^2q_\perp U_0(\mathbf{q}_\perp)\,iq_y \times 2\pi\delta(v_0t'-z_a)\exp(-i\mathbf{q}_\perp\mathbf{R}_a) - \int_0^t\frac{dt'}{E}\sin[\omega_H(t-t')]\sum_a\int d^2q_\perp U_0(\mathbf{q}_\perp)\,iq_x \times 2\pi\delta(v_0t'-z_a)\exp(-i\mathbf{q}_\perp\mathbf{R}_{a\perp}). \tag{3.5}$$

With the aid of expressions (3.5) it is easy to calculate the mean values over the positions of the atoms. We assume in the averaging that the medium is homogeneous in the mean and occupies the half-space from z = 0 to z = ∞, and is infinite along x and y. In the averaging they use the formula

$$\Big\langle\sum_a \exp(-i\mathbf{q}\mathbf{R}_a)\Big\rangle = (2\pi)^2\delta(\mathbf{q}_\perp)\int_0^\infty d\xi \cdot \exp(-iq_\parallel\xi)\,n_0, \tag{3.6}$$

and also the assumption that the positions of the atoms of the medium are not correlated with one another. Using (3.5) and (3.6), we easily obtain

$$\langle\theta_{t_1}\theta_{t_2}\rangle = (E_s^2/E^2L)\cos[\omega_H(t_1-t_2)]\min(t_1, t_2),$$

$$\langle\theta_\tau^2\rangle = E_s^2\tau/E^2L,$$

$$\Big\langle\Big(\int_0^t\theta_\tau d\tau\Big)^2\Big\rangle = \frac{2E_s^2\omega_H^{-2}}{E^2L}\Big(t - \frac{\sin\omega_H t}{\omega_H}\Big). \tag{3.7}$$

At $\omega_H = 0$, formulas (3.7) go over, as expected, into the usual formulas of the theory of multiple scattering[4]. It follows from the last formula of (3.7) that for times $t > \omega_H^{-1}$ the mean squared perpendicular displacement of the particle increases linearly:

$$\Big\langle\Big(\int_0^t\theta_\tau d\tau\Big)^2\Big\rangle = 2E_s^2t/E^2L\omega_H^2,$$

which agrees with the qualitative arguments presented above.

4. Substituting (3.7) in the expression (2.9) for the spectral density of the radiation intensity, and introducing new integration variables $\tau = t_2$ and $t_1 - t_2 = t$, we readily obtain

$$dI = dI_1 + dI_2;$$

$$dI_1 = -\frac{e^2E_s^2\omega d\omega}{\pi E^2L}\int_0^T d\tau \int_0^{T-\tau}\frac{dt}{t}\Big[\frac{2}{t^2\omega_H^2}\Big(t - \frac{\sin\omega_H t}{\omega_H}\Big) - \frac{1-\cos\omega_H t}{t\omega_H^2}\Big]\sin\Big[\frac{\omega m^2 t}{2E^2} + \frac{E_s^2\omega t}{E^2L}\Big(\frac{t}{4} - \frac{\omega_H t - \sin\omega_H t}{t^2\omega_H^3}\Big) + \tau\frac{E_s^2\omega t}{E^2L}\Big(\frac{1}{2} - \frac{1-\cos\omega_H t}{t^2\omega_H^2}\Big)\Big]; \quad (4.1)$$

$$dI_2 = -\frac{e^2E_s^2\omega d\omega}{\pi E^2L}\int_0^T \tau d\tau\int_0^{T-\tau}\frac{dt}{t}\Big[\cos\omega_H t + \frac{2(1-\cos\omega_H t)}{\omega_H^2t^2} - \frac{2\sin\omega_H t}{\omega_H t}\Big]\sin\Big[\frac{\omega m^2 t}{2E^2} + \frac{E_s^2\omega t}{E^2L}\Big(\frac{t}{4} - \frac{\omega_H t - \sin\omega_H t}{\omega_H^3t^2}\Big) + \tau\frac{E_s^2\omega t}{E^2L}\Big(\frac{1}{2} - \frac{1-\cos\omega_H t}{\omega_H^2t^2}\Big)\Big] \quad (4.2)$$

As is easily seen from (4.1) and (4.2), dI_1 and dI_2 depend differently on the time of flight of the particle through the medium. In addition, putting $\omega_H = 0$ (no magnetic field) we find that dI_2 vanishes and dI_1 goes over into the formula describing bremsstrahlung in a medium, with allowance for multiple scattering[4].

Expressions (4.1) and (4.2) cannot be integrated in general form, but if the time of motion in the medium is such that $\langle\theta_T^2\rangle \gg m^2/E^2$, then the synchrotron-radiation times

$$t_H \infty m/E(\langle\theta_T^2\rangle)^{1/2}\omega_H \ll \omega_H^{-1} \quad (4.3)$$

are significant, and the integrands of (4.1) and (4.2) should be expanded in powers of the small quantity $\omega_H t_H \ll 1$, while the upper limit of the integral with respect to t should be replaced by infinity. We then arrive at the following expressions for the spectral density of the radiation energy:

$$dI_1 = \frac{e^2E_s^2\omega d\omega}{6\pi E^2L}\int_0^T d\tau\int_0^\infty dt\sin\Big[\frac{\omega m^2 t}{2E^2} + \frac{E_s^2\omega t^2}{12E^2L} + \frac{\langle\theta_\tau^2\rangle\omega\omega_H^2t^3}{24}\Big],$$

$$dI_2 = -\frac{e^2E_s^2\omega_H^2\omega d\omega}{12\pi E^2L}\int_0^T\tau d\tau\int_0^{T-\tau}t dt\sin\Big[\frac{\omega m^2 t}{2E^2} \quad (4.4)$$

$$+\frac{E_s^2\omega t^2}{12E^2L} + \frac{\langle\theta_\tau^2\rangle\omega\omega_H^2t^3}{24}\Big]. \quad (4.5)$$

If we neglect the last term in the square brackets of (4.4), then we arrive immediately at the formula obtained by Landau and Pomeranchuk[4] for the bremsstrahlung intensity in a medium, with allowance for multiple scattering.

For the calculations that follow, it is convenient to change over in (4.4) and (4.5) from t to a new variable

$$x = (\langle\theta_\tau^2\rangle\omega\omega_H^2)^{1/3}t/2^{1/3}. \quad (4.6)$$

It is easy to verify directly that the right-hand side of the inequality (2.8), which imposes the lower limit on the frequency of the quanta in question, implies satisfaction of the condition

$$\langle\theta_H^2\rangle \lesssim \frac{m^3}{E^3}\sqrt{\langle\theta_T^2\rangle} \quad (4.7)$$

$2\pi\langle\theta_H^2\rangle$ is the square of the multiple-scattering angle for one revolution in the magnetic field. The condition (4.7) enables us to discard the second terms in the argument of the sine function in (4.4) and (4.5). Now the expression (4.5) reduces to the derivative of the Airy function, and the energy can be written in the form

$$dI_1 = \frac{e^2E_s^2\gamma^2d\omega}{3\pi E^2L}\int_0^T\frac{d\tau}{(\langle\theta_\tau^2\rangle)^{1/3}}\int_0^\infty dx\sin\Big[\frac{\gamma^2m^2x}{E^2(\langle\theta_\tau^2\rangle)^{1/3}} + \frac{x^3}{3}\Big]; \quad (4.8)$$

$$dI_2 = -\frac{e^2E_s^2\gamma\omega_H d\omega}{3\pi^{1/2}E^2L}\int_0^T\frac{\tau d\tau}{(\langle\theta_\tau^2\rangle)^{2/3}} \times \Phi'\Big(\gamma^2\frac{m^2}{E^2(\langle\theta_\tau^2\rangle)^{1/3}}\Big), \quad (4.9)$$

$\gamma^3 \equiv \omega/\omega_H$. Let us investigate the obtained expressions in the regions of high and low frequencies. For high frequencies

$$\omega \gg \omega_H(E/m)^3(\langle\theta_T^2\rangle)^{1/2}. \quad (4.10)$$

Using the asymptotic behavior of the Airy function at large values of the argument, we have

$$dI_1 = \frac{e^2E_s^2Td\omega}{3\pi m^2L}; \quad (4.11)$$

$$dI_2 = \frac{e^2E_s^2\gamma^{-3}\omega_H d\omega}{3\pi m^2L}\int_0^T\tau d\tau \times \exp\Big[-\frac{2}{3}\Big(\gamma^2\frac{m^2}{E^2\langle\theta_\tau^2\rangle^{1/3}}\Big)^{3/2}\Big]. \quad (4.12)$$

Expression (4.11) represents the spectral density of the bremsstrahlung energy[4]. It depends linearly on the time of flight of the particle in the medium, as it should.

It is convenient to rewrite (4.12) in the form

$$dI_2 = \frac{e^4H^2\langle\theta_T^2\rangle E^2T}{3m^4\omega}\frac{1}{T^2}\int_0^T\tau d\tau \cdot \times\exp\Big[-\frac{2}{3}\Big(\gamma^2\frac{m^2}{E^2\langle\theta_\tau^2\rangle^{1/3}}\Big)^{3/2}\Big] \sim \frac{e^4H^2\langle\theta_T^2\rangle E^2T}{6m^4\omega}\times\exp\Big[-\frac{2}{3}\Big(\gamma^2\frac{m^2}{E^2\langle\theta_T^2\rangle^{1/3}}\Big)^{3/2}\Big]. \quad (4.13)$$

In this form, it coincides in structure with the spectral density of the intensity of synchrotron radiation of an ultrarelativistic particle at high frequencies, when the particle moves at an angle $\langle\theta_T^2\rangle^{1/2}$ to the magnetic-field direction.

Thus, if the condition (4.10) is satisfied, the spectral distribution of the radiation intensity breaks up into a sum of two terms describing the bremsstrahlung and the synchrotron radiation of a particle having a velocity that fluctuates randomly in the medium. The physical explanation of this fact is that these two mechanisms are not coherent when the condition (4.10) is satisfied, because the times of quantum emission by the bremsstrahlung and synchrotron-radiation mechanisms satisfy the inequality $t_e \ll t_H$, and therefore the intensities add up. The nonlinear dependence of the spectral density of the synchrotron-radiation intensity on the time is connected with the fact that the latter is proportional to the square of the velocity component perpendicular to the magnetic field, which in the case of multiple scattering is proportional to the time of flight of the particle in the medium.

We proceed to the low-frequency region. Let the frequency ω be such that the following relation is satisfied

$$\omega \ll \omega_H (E/m)^3 \langle \theta_T^2 \rangle^{1/2}. \qquad (4.14)$$

In this case we can omit the first terms of the square brackets in (4.8) and (4.9). The integration is then elementary and the result is

$$dI_1 = \frac{\Gamma(1/3) e^2 E_s^2 \gamma^2 d\omega}{4\pi 3^{2/3} E^2 L} \frac{T}{(\langle \theta_T^2 \rangle)^{1/3}} \sim \omega^{2/3} T^{2/3} H^{-2/3}, \qquad (4.15)$$

$$dI_2 = \frac{3^{1/6}\Gamma(2/3) e^2 E_s^2 \gamma \omega_H d\omega}{8\pi E^2 L} \frac{T^2}{(\langle \theta_T^2 \rangle)^{2/3}} \sim \omega^{1/3} T^{4/3} H^{2/3}. \qquad (4.16)$$

It follows from (4.15) and (4.16) that the term connected with the bremsstrahlung decreases with increasing magnetic field intensity. The term describing the synchrotron radiation increases with increasing magnetic field. In addition, the last term increases much more rapidly with time, in proportion to $T^{4/3}$. Finally, both (4.15) and (4.16) increase with increasing frequency. We notice also that the dependence of (4.16) on the frequency and on the magnetic field intensity at low frequencies agrees with the general theory.

From a comparison of (4.12) and (4.16) we see that the maximum of the spectrum of the synchrotron radiation in a longitudinal magnetic field occurs at frequencies

$$\omega \sim \omega_H (E/m)^3 \langle \theta_T^2 \rangle^{1/2}. \qquad (4.17)$$

In addition, it follows from (4.8) and (4.9) that with continuing motion of the particle in the medium the quanta emitted by it become harder.

RADIATION OF AN ELECTRON MOVING PERPENDICULAR TO THE MAGNETIC FIELD IN A MEDIUM

5. We proceed to consider the spectral density of the radiation intensity of an ultrarelativistic electron in a magnetic field in a medium, the initial velocity of the electron being perpendicular to the magnetic field. It is well known[1-3] that an ultrarelativistic electron moving in a constant magnetic field H perpendicular to the electron velocity radiates a quasicontinuous frequency spectrum with a maximum occurring at

$$\omega_0 \sim \omega_H (E/m)^3. \qquad (5.1)$$

A quantum-mechanical study of the radiation in a transverse magnetic field has shown[2-3] that if the condition

$$\omega_0 \ll E \qquad (5.2)$$

is satisfied we can neglect both the quantum recoil produced when the photon is emitted, and the quantization of the electron motion, so that the radiation problem can be considered classically. We note that for motion in a transverse magnetic field in a medium, the collisions of the electrons with the atoms of the medium have a twofold influence on the radiation spectrum. First, the collision causes a "jump" of the center of the electron orbit, and in addition the electron acquires a fluctuating velocity component parallel to the magnetic field, leading, by virtue of the elasticity of the scattering by each atom, which we shall postulate below, to a decrease of the velocity component in the plane of rotation, and consequently to a decrease of the radius of the circle along which the electron moves. Thus, motion in a magnetic field in the presence of scattering is along a fluctuating helix. Second, bremsstrahlung is produced when the electron collides with the atoms of the medium. To simplify the final results, we disregard this bremsstrahlung, but we note that it can be accounted for in elementary fashion. Here, just as in the first part of this paper, it is assumed that condition (1.1) is satisfied and that, in addition, the total time of motion satisfies the inequality

$$T > 2\pi / \omega_H, \qquad (5.3)$$

i.e., the particle has time to execute at least one revolution in the magnetic field in the medium.

6. To find the spectral density of the synchrotron-radiation intensity it is again convenient to use formula (2) of[4] (see formula (2.1)). Just as in the first part of the paper, it is easy to show that in order for the use of this formula to be valid, it is necessary to impose a lower bound on the considered radiation frequencies:

$$\omega \gg \omega_H E/m. \qquad (6.1)$$

The condition (6.1) excludes from consideration only frequencies that are small compared with ω_0, and therefore, in accordance with the statements made above, expression (2.1) allows us to consider the most significant part of the synchrotron-radiation spectrum.

It will be shown below that the influence of the multiple scattering on the frequency spectrum of the synchrotron radiation becomes manifest only in the effective range of emission angles[7]:

$$\varphi_e \propto m/E, \qquad (6.2)$$

while the aforementioned "helical" motion of the particle does not influence the frequency spectrum of the radiation. Let us estimate the influence of multiple scattering on the quantum radiation in the effective region. The characteristic angles (of particle rotation along the circle) from which the radiation takes place are given by (6.2). Consequently, if the square of the multiple-scattering angle accumulated during the time of the radiation $t_H \sim m/\omega_H E$ is comparable with or larger than φ_e^2, then the multiple scattering has an effect on each individual radiation act:

$$\langle \theta_{t_H}^2 \rangle \gtrsim m^2/E^2 \text{ or } H \lesssim E_s^2/meL. \qquad (6.3)$$

7. For the subsequent calculation of the spectral radiation intensity it is necessary to know the classical trajectory of particle motion. To find it, we start from the classical equations of motion of the particle in a magnetic field in a medium, (3.1). If we note that the relation

$$r_0 = v_0 / \omega_H \gg a_0, \qquad (7.1)$$

is satisfied for all reasonable values of the magnetic field intensity, then the presence of the magnetic field can be disregarded in the analysis of each individual scattering act, and consequently it can be assumed that the collision occurs instantaneously. Then, using (7.1), we can write the equations of motion describing the scattering of an electron in a magnetic field by an individual center in the form*

$$d\mathbf{p}/dt = e[\mathbf{vH}], \quad t < t_0,$$

*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$.

$$dp/dt = e[\mathbf{v},\mathbf{H}], \quad t > t_a:$$
$$v_x(t_a - 0) = v_0 \cos\varphi_a, \quad v_x(t_a + 0) = v_0(1 - {}^1/_2\theta_{za}{}^2) \times \cos(\varphi_a - \vartheta_a),$$
$$v_y(t_a - 0) = -v_0 \sin\varphi_a, \quad v_y(t_a + 0) = -v_0(1 - {}^1/_2\theta_{za}{}^2)\sin(\varphi_a - \vartheta_a),$$
$$\varphi_a = \omega_H t_a, \quad v_z = v_0\theta_{za}.$$

Here t_a is the instant of time at which the electron collides with a-th center, ϑ_a is the angle of scattering of the electron in the plane of the orbit, θ_{za} is the angle of scattering along the magnetic field direction $H \parallel z$, and the factor $1 - \theta_{za}^2/2$ is due to the fact that the energy conservation law must be satisfied in elastic scattering. It is easy to verify that $v^2 = v_0^2$.

The velocity of the electron at an arbitrary instant of time t in the presence of many scattering centers, can be written in the form

$$v_x(t) = v_0(1 - {}^1/_2\theta_z{}^2(t))\cos(\omega_H t - \vartheta_{xy}(t)),$$
$$v_y(t) = -v_0(1 - {}^1/_2\theta_z{}^2(t))\sin(\omega_H t - \vartheta_{xy}(t)), \tag{7.2}$$
$$v_z(t) = v_0\theta_z(t),$$

where $\vartheta_{xy}(t)$ is the component of the multiple-scattering angle in the x, y plane, acquired by the particle within a time t. By virtue of the independence of the positions of the centers we have

$$\vartheta_{xy}(t) = \sum_a \vartheta_{axy},$$

with each angle ϑ_{axy} taken in a direction perpendicular to the velocity at the instant of collision with the a-th center. Expressions (7.2) should be expanded accurate to terms of second order of smallness in ϑ_{xy}. It is more convenient, however, to carry out the expansion during a later stage. Integrating (7.5) with respect to the time, we obtain the formulas for the classical trajectory of the motion:

$$x(t) = x_0 + \int_0^t \cos[\omega_H t' - \vartheta_{xy}(t')]v_0(1 - {}^1/_2\theta_z{}^2(t'))\,dt',$$

$$y(t) = y_0 - \int_0^t \sin[\omega_H t' - \vartheta_{xy}(t')]v_0(1 - {}^1/_2\theta_z{}^2(t'))\,dt',$$

$$z(t) = v_0 \int_0^t \theta_z(t')\,dt'. \tag{7.3}$$

8. Substituting (7.3) in (2.1), carrying out the averaging just as in the first part of the paper, we can easily obtain for the spectral density of the radiation intensity per unit time

$$\frac{dI}{T} = \frac{e^2\gamma\omega_H d\omega}{\pi}\int_0^\infty x\,dx \sin\left[\frac{\gamma^2 m^2}{2E^2}x + \frac{x^3}{3} + \gamma\frac{\langle\theta_H{}^2\rangle}{3}\right]. \tag{8.1}$$

An interesting case is when

$$\langle\theta_H{}^2\rangle \gg m/E, \tag{8.2}$$

for when the inequality (8.2) is satisfied the multiple scattering has a strong effect on the radiation in the effective region (see also (6.3)). In the frequency region

$$(\gamma^2 m^2/2E^2)^2 \ll {}^1/_3\gamma\langle\theta_H{}^2\rangle \ll {}^1/_3 \tag{8.3}$$

the first and third terms in the square brackets can be emitted and as a result of integration we have

$$\frac{dI}{T} = -\pi^{-1/2}e^2\gamma\omega_H\,d\omega\,\Phi'(0) = 0{,}52\frac{e^4H^2}{m^2}\left(\frac{m}{E}\right)\left(\frac{\omega}{\omega_H}\right)^{1/3} d\omega \tag{8.4}$$

where $\Phi'(0)$ is the derivative of the Airy function. Formula (8.4) coincides with the results obtained for motion in vacuum[8].

In the frequency region

$$(\gamma^2 m^2/2E^2)^2 \ll {}^1/_3 \ll {}^1/_3\gamma\langle\theta_H{}^2\rangle \tag{8.5}$$

(8.1) reduces to the form

$$dI/T = 3e^2\omega_H d\omega / 2\pi\langle\theta_H{}^2\rangle, \tag{8.6}$$

i.e., the spectral density of the radiation intensity in the frequency region (8.5) flattens out into a plateau. Finally, in the frequency region

$${}^1/_3 \ll {}^1/_3\gamma\langle\theta_H{}^2\rangle \ll (\gamma^2 m^2/2E^2)^2 \tag{8.7}$$

the emission spectrum is given by the formula

$$\frac{dI}{T} = \frac{e^4H^2(\omega/\omega_H)^{1/2}}{4\pi^{1/2}m^2}\left(\frac{m}{E}\right)^{5/2}\exp\left[-\frac{2}{3}\gamma^3\left(\frac{m}{E}\right)^3\right]d\omega. \tag{8.8}$$

Thus, as seen from (8.3)–(8.8), when the condition (8.2) is satisfied, the maximum of the spectral density of the radiation intensity in the medium shifts from the frequency region $\omega_0 \sim \omega_H(E/m)^3$ into the frequency region $\omega_1 \sim 27\omega_H/(\langle\theta_H^2\rangle)^3 \ll \omega_0$. We note that the estimate (6.3) for elements with $Z \sim 10$ and for a medium density $n_0 \sim 10^{19}$ cm^{-3} leads to the inequality $H \lesssim 10$ Oe.

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