Turbulence of Ion Sound in a Plasma Located in a Magnetic Field

E. A. KUZNETSOV

Institute of Nuclear Physics, Siberian Section, USSR Academy of Sciences

Submitted March 19, 1971

Zh. Eksp. Teor. Fiz. **62**, 584–592 (February, 1972)

The turbulence of ion sound in a plamsa in a magnetic field is considered. In a sufficiently dense plasma ($\omega_{pi} \gg \omega_{Hi}$, where ω_{pi}, ω_{Hi} are the plasma and cyclotron ion frequencies) and for frequencies $\omega < \omega_{Hi}$, the major nonlinear mechanism is three-plasmon interaction. Within the framework of weak turbulence, we have found the kinetic equation for the waves; in the region $k_{\perp} r_s < 1$ ($r_s = c_s/\omega_{Hi}$, c_s the sound velocity) the solution is in the form of a power law. The solution may be interpreted as representing the spectrum in the region of universal equilibrium. It is shown that the turbulence in this region is of a local nature.

1. INTRODUCTION

IN the theory of turbulence, a central position is occupied by the concept of the turbulence spectrum, i.e., the energy distribution over the size scales. The solution of Kolmogorov and Obukhov[1,2] is well known for hydrodynamic turbulence, the so-called Kolmogorov spectrum $\epsilon_k \sim k^{-5/3}$. This solution is based on the hypothesis of the self-similar character of the spectrum and the local nature of the turbulence. The latter supposition means that modes of the same order of scale interact strongly with one another. In media with weak dispersion, such solutions are obtained within the framework of the theory of weak turbulence.[3-6] The theory of weak turbulence assumes smallness of amplitude, which allows us to change over to a statistical description of the waves and to a kinetic equation for the waves by using the hypothesis of random phase.

In plasma, the turbulence of the waves is due to two types of interaction: scattering of waves by particles and processes of decay, coalescence and scattering of the waves with one another. The scattering of waves by particles leads to energy exchange between the waves and the particles. Nevertheless, situations are possible in which a rather broad region of transparency exists in k space. The character of the turbulence in this region depends a great deal on the transfer of energy from the pumping region to the damping region. Spectra of such form have been found by Zakharov for Langmuir oscillations in plasma in the absence of a magnetic field[7] and in a number of other systems.[8-10] In the researches mentioned, only isotropic turbulence has been considered. In the present research, a similar problem is solved for a nonisotropic medium. We consider weak turbulence of the acoustic type in a magnetized plasma ($\beta = 8\pi p/H^2 \ll 1$) with hot electrons ($T_e \gg T_i$). For ion-sound oscillations in an isotropic plasma, it is characteristic that the principal contribution to the interaction between the waves is scattering from particles in most cases.[6,11,12] Turning on a magnetic field materially changes the dispersion law and leads to the result that three-wave interactions become fundamental. Therefore, we have used the hydrodynamic approach in the research.

It is known[13-15] that conservative hydrodynamical systems are Hamiltonian. For the latter, analysis of the equations of hydrodynamics is conveniently transformed to canonical variables, with the help of which the kinetic equation for the waves can easily be obtained. The kinetic equation has an exact solution $\epsilon_k = \text{const}\, k_{\perp}^{-1} k_z^{-3/2}$. This same solution is obtained from considerations of dimensionality, while the constant is proportional to $I^{1/2}$, where I is the energy flux in the damping region. Furthermore, it is possible to prove the local nature of the turbulence.

2. THE KINETIC EQUATION

We now consider the low frequency potentials of the oscillation in a nonisothermal plasma ($T_e \gg T_i$) in a strong magnetic field ($\beta \ll 1$). We shall describe the slow motions of the ions by the equations of hydrodynamics. We close this set of equations by the equation of quasineutrality. It is assumed that the electrons have a Boltzmann distribution:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{e}{M}\nabla\varphi + [\mathbf{v}\omega_{Hi}], \tag{2*}$$

$$\rho = \rho_0 \exp(e\varphi / T_e). \tag{3}$$

Here ρ and $\mathbf{v}$ are respectively the density and the velocity of the ions, ρ_0 the unperturbed density, φ the electrostatic potential, and $\omega_H \equiv \omega_{Hi} = eH/Mc$. Equations (1)–(3) are suitable for the description of the oscillations with phase velocities along the magnetic field ω/k_z that are less than the Alfvên velocity v_A.

We eliminate φ from Eq. (2) and as a result obtain

$$\partial \mathbf{v} / \partial t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla w + [\mathbf{v}\omega_H], \tag{4}$$

where $w = c_s^2 \ln(\rho/\rho_0)$ is the enthalpy, and c_s the sound velocity. The system of equations (1) and (4) conserves the energy

$$\mathcal{H} = \int \{{}^1/{}_2 \rho v^2 + \varepsilon(\rho)\} d\mathbf{r},$$

where the first term is the kinetic energy, and the second the internal energy of the electron gas, which is connected with the enthalpy by the relation $w = \partial\epsilon/\partial\rho$.

Following Zakharov,[15] we make the transition to canonical variables by the formula

*$[\mathbf{v}W_{Hi}] \equiv \mathbf{v} \times W_{Hi}$.

$$\mathbf{v} = {}^1/{}_2\rho^{-1}(\lambda\nabla\mu - \mu\nabla\lambda) + \nabla\Phi - (e/Mc)\mathbf{A}, \qquad (5)$$

where $\mathbf{A} = {}^1/{}_2\mathbf{H}\times\mathbf{r}$ is the vector potential of the constant magnetic field. Here (λ, μ) and (ρ, Φ) are pairs of canonically conjugate quantities. The equations for the new variables have the form of Hamilton's equations:

$$\frac{\partial\lambda}{\partial t} = \frac{\delta\mathscr{H}}{\delta\mu}, \quad \frac{\partial\mu}{\partial t} = -\frac{\delta\mathscr{H}}{\delta\lambda}, \quad \frac{\partial\rho}{\partial t} = \frac{\delta\mathscr{H}}{\delta\Phi}, \quad \frac{\partial\Phi}{\partial t} = -\frac{\delta\mathscr{H}}{\delta\rho}. \qquad (6)$$

In the next stage, we eliminate $\mathbf{r}$ from Eq. (5):

$$\lambda \to \lambda + \omega_H^{1/2}\rho^{1/2}x,$$

$$\mu \to \mu - \omega_H^{1/2}\rho^{1/2}y,$$

$$\Phi \to \Phi - \omega_H^{1/2}\rho^{1/2}(\lambda y + \mu x)$$

This gives

$$\mathbf{v} = {}^1/{}_2\rho^{-1}(\lambda\nabla\mu - \mu\nabla\lambda) + \nabla\Phi - (\omega_H/\rho)^{1/2}(\mathbf{i}\lambda + \mathbf{j}\mu).$$

We note that the transformation is canonical, i.e., the equations (6) keep their form.

We next expand $\mathbf{v}$ in terms of the canonical variables

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \ldots,$$

$$\mathbf{v}_1 = \nabla\Phi - (\omega_H/\rho_0)^{1/2}(\mathbf{i}\lambda + \mathbf{j}\mu),$$

$$\mathbf{v}_2 = {}^1/{}_2\rho_0^{-1}(\lambda\nabla\mu - \mu\nabla\lambda) + {}^1/{}_2(\omega_H/\rho_0)^{1/2}(\mathbf{i}\lambda + \mathbf{j}\mu).$$

We perform a similar operation with the Hamiltonian:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_{int};$$

$$\mathscr{H}_0 = \int\left\{\frac{\rho_0 v_1^2}{2} + \frac{c_s^2\delta\rho^2}{2\rho_0}\right\}d\mathbf{r},$$

$$\mathscr{H}_{int} = \int\left\{\frac{\delta\rho v_1^2}{2} + \rho_0\mathbf{v}_1\mathbf{v}_2 - \frac{c_s^2\delta\rho^2}{2\rho_0^2}\right\}d\mathbf{r}.$$

The equations for small oscillations can be obtained by variation of the Hamiltonian $\mathscr{H}_0$ in the corresponding variables. Analysis of these equations gives the following dispersion equation:

$$\omega^2 = c_s^2\left(k_z^2 + \frac{\omega^2}{\omega^2 - \omega_H^2}k_\perp^2\right).$$

This equation has two branches of oscillations. We shall be interested only in the low frequency branch ($\omega \ll \omega_H$, $k_\perp r_s \ll 1$). For these oscillations, $\omega = k_z c_s(1 - {}^1/{}_2 k_\perp^2 r_s^2)$, where $r_s = e_s/\omega_H$. Departure from quasineutrality would lead to a dispersion contribution $-{}^1/{}_2 k^2 r_d^2$. However, under the condition $\omega_{pi} \gg \omega_H$ (ω_{pi} is the plasma frequency of the ions) this contribution can be neglected; it is significant only in the region of small angles of propagation relative to the magnetic field. The resultant spectrum is a decay spectrum, i.e., it satisfies the condition:

$$\omega_k = \omega_{k_1} + \omega_{k_2}, \quad k = k_1 + k_2.$$

Let us simplify Eqs. (6). For this purpose, we limit ourselves to a quadratic nonlinearity and assume that the dispersion is weak. It is convenient to carry out this procedure in the Hamiltonian, which gives

$$\mathscr{H}_0 = \int\left\{\frac{\rho_0}{2}\left[\left(\frac{\partial\Phi}{\partial z}\right)^2 - r_s^2\left(\nabla_\perp\frac{\partial\Phi}{\partial z}\right)^2\right] + \frac{c_s^2\delta\rho^2}{2\rho_0}\right\}d\mathbf{r},$$

$$\mathscr{H}_{int} = \int\left\{\frac{\delta\rho}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2 - \frac{c_s^2\delta\rho^3}{6\rho_0^2}\right\}d\mathbf{r}.$$

By varying the Hamiltonian in $\delta\rho$ and Φ, we obtain

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\frac{\partial^2}{\partial z^2}(1 + r_s^2\Delta_\perp)\Phi + \frac{\partial}{\partial z}\left(\delta\rho\frac{\partial\Phi}{\partial z}\right) = 0,$$

$$\frac{\partial\Phi}{\partial t} + c_s^2\frac{\delta\rho}{\rho_0} = \frac{c_s^2}{2}\frac{\delta\rho^2}{\rho_0^2} - \frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2.$$

The resultant equations are close to the equations for potential motion of one-dimensional gasdynamics. Their only difference lies in the dispersion term.

For waves traveling in one direction along the magnetic field, we can obtain an equation for $u = \partial\Phi/\partial z$ within the framework of the assumptions specified:

$$\frac{\partial u}{\partial t} - c_s\frac{\partial}{\partial z}\left(u + \frac{1}{2}r_s^2\Delta_\perp u - \frac{1}{2}\frac{u^2}{c_s}\right) = 0.$$

This equation is the analog of the well known Korteweg-de Vries equation.

We carry out a Fourier transformation over the coordinates and transform to a representation in which the Hamiltonian $\mathscr{H}_0$ is diagonal:

$$\delta\rho_k = (\rho_0\omega_k/2c_s^2)^{1/2}(a_k + a_{-k}^*),$$

$$\Phi_k = -i(c_s^2/2\rho_0\omega_k)^{1/2}(a_k - a_{-k}^*).$$

The equations of motion can be written in variational n form:

$$\partial a_k/\partial t = -i\delta\mathscr{H}/\delta a_k^*,$$

where the Hamiltonian $\mathscr{H}_0$ has the form

$$\mathscr{H} = \int\omega_k a_k a_k^*\,d\mathbf{k} + \int V_{kk_1k_2}\left\{\frac{1}{3}(a_k a_{k_1} a_{k_2} + \text{c. c.})\,\delta(k + k_1 + k_2) + (a_k^* a_{k_1} a_{k_2} + \text{c. c.})\,\delta(k - k_1 - k_2)\right\}d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2.$$

For $k_\perp r_s^2 \ll 1$, the matrix element takes the form

$$V_{kk_1k_2} = \frac{1}{4}\left(\frac{c_s}{\rho_0}\right)^{1/2}\left\{\left(\frac{|k_z|}{|k_{z1}||k_{z2}|}\right)^{1/2}k_{z1}k_{z2} + \left(\frac{|k_{z1}|}{|k_z||k_{z2}|}\right)^{1/2}k_z k_{z2} + \left(\frac{|k_{z2}|}{|k_{z1}||k_z|}\right)^{1/2}k_z k_{z1} - (|k_z||k_{z1}||k_{z2}|)^{1/2}\right\}. \qquad (7)$$

We now proceed to a description of a system in a weakly turbulent regime. As was shown above, weak turbulence assumes smallness of amplitude. However, in media with weak dispersion, this is insufficient.[8] A limitation should be placed on the amplitude: $\delta\rho/\rho_0 \ll k_\perp^2 r_s^2$, but for waves with $k_\perp = 0$ this criterion is violated. Here it is necessary to take into account dispersion at the Debye radius. Then the criteria would be written in the form

$$\delta\rho/\rho_0 \ll k^2 r_d^2 \quad \text{or} \quad W/nT_e \ll (kr_d)^4,$$

where W is the energy density of the noise.

We now proceed to the statistical description of the wave system. We make a hypothesis on the chaotic character of the oscillation phases with different k and transform to the new variables

$$\langle a_k a_{k_1}^*\rangle = n_k\delta(k - k_1),$$

where the angular brackets denote averaging over the random phases. We obtain the kinetic equation for the waves:

$$\partial n_k/\partial t = \text{St}(n_k) + \gamma_k n_k;$$

$$\text{St}(n_k) = 2\pi\int|V_{kk_1k_2}|^2\{\delta(k - k_1 - k_2)\,\delta(\omega_k - \omega_{k_1} - \omega_{k_2}) \times (n_{k_1}n_{k_2} - n_k n_{k_1} - n_k n_{k_2}) + 2\delta(k + k_1 - k_2)\,\delta(\omega_k + \omega_{k_1} - \omega_{k_2}) \times (n_{k_1}n_{k_2} - n_k n_{k_1} + n_k n_{k_2})\}\,d\mathbf{k}_1\,d\mathbf{k}_2 \qquad (8)$$

In the kinetic equation, the term $\gamma_k n_n$ has been introduced; it describes the scattering of the waves by the particles of the plasma. This term preserves the number of quasiparticles but does not conserve their energy.

3. DIMENSIONAL ESTIMATES

We shall first ascertain what sort of contribution is made by the term $\gamma_k n_k$. The expression for the increment can generally be written in terms of the probability R of scattering of a wave with momentum $\mathbf{k}$ on a particle of type α with momentum $\mathbf{p}$ and the creation of a wave with momentum $\mathbf{k}'$ ($\hbar = 1$):[11,12]

$$\gamma_k = \sum_\alpha \int R^\alpha(kk'p)\, n_{k'}(k_z - k_z')\, v_z \frac{\partial f}{\partial \varepsilon} d\mathbf{v}\, d\mathbf{k}', \tag{9}$$

$$R^\alpha(kk'p) \sim \delta(\omega_k - \omega_{k'} - (k_z - k_z')v_z).$$

In Eq. (9), we have discarded all the higher gyroresonances ($\omega \ll \omega_H$). We first note that the basic contribution to scattering is made by the ions inasmuch as in the scattering of an ion-sound quantum by an electron, the phase velocity of the quantum along $\mathbf{H}$ is of the order of or smaller than the velocity of the electrons participating in the processes of Cerenkov radiation. In an equilibrium plasma ($\partial f/\partial\epsilon < 0$) the scattering leads to a transfer of the energy into the region of smaller ω_k, i.e., in the given case, into a region of small k_z and large $k_\perp$. For scattering by ions, the following criteria must be satisfied:

$$\omega_k - \omega_{k'} < v_{Ti} |k_z - k_z'|. \tag{10}$$

It is then seen that on the sound part of the spectrum ($k_\perp^2 r_s^2 \ll 1$), interaction leads to the "isotropization" of the oscillations: waves from the region $k_z > 0$ and the region $k_z > 0$ with $\Delta\,|k_z| \approx 0$ strongly interact with one another.

The expression for the probability $R(kk'p)$ can be obtained by the method of test particles.[12] Using the drift approximation for ions, we find

$$R = 4(2\pi)^3 (e^2/M)^2 \delta(\omega_k - \omega_{k'} - (k_z - k_z')v_z)\,(k_z v_z/\omega_{pi})^2.$$

The term $\gamma_k n_k$ has the order

$$\gamma_k n_k \sim \frac{1}{v_{Ti}^3}\omega_k \left(\frac{\omega_-}{k_{z-}}\right)^3 \frac{W}{nT_e} n_k,$$

where ω_-/k_{z-} is the phase velocity of the pulsation along the magnetic field.

The estimate for the collision term gives

$$\mathrm{St}(n_k) \sim \omega_k (W/nT_e) n_k,$$

i.e., the contribution to scattering in the interaction between waves is small (see (10)).

In addition to the processes considered, processes are also possible in which low-frequency non-potential oscillations of the Alfvén type (A waves) are excited. These are processes of decay and scattering of the waves by particles. In the first case in a plasma with $\beta \ll 1$, only decay of A waves into A waves and ion sound are possible.[16] Estimates show that the characteristic time of this process τ is of the order

$$\tau^{-1} \sim \omega_k (W/nT_e)\beta^{1/2}.$$

In the second case, only scattering by ions is important. In this case, estimates give

$$\tau^{-1} \sim \omega_k (W/nT_e)\beta(T_e/T_i)\frac{\omega_-}{k_{z-} v_{Ti}},$$

i.e., the effect of nonpotential oscillations can be neglected, at least for $\beta \lesssim T_i/T_e$.

We now consider Eq. (8). In the collision term, we replace the product of δ functions of k and ω by the expression:

$$\frac{1}{c_s r_s^2}\delta(\mathbf{k}_\perp - \mathbf{k}_{\perp1} \pm \mathbf{k}_{\perp2})\,\delta(|k_z| - |k_{z1}| \pm |k_{z2}|)$$
$$\times\ \delta(|k_z|k_\perp^2 - |k_{z1}|k_{\perp1}^2 \pm |k_{z2}|k_{\perp2}^2).$$

Generally speaking, such a substitution is not correct; the substitution is valid only in the region $k_\perp^2 r_s^2 \ll 1$. This subdivision means that the waves interact strongly with a single value of k_z; therefore, it suffices to consider only the region $k_z > 0$. In what follows, we shall consider only such a reduced kinetic equation. This equation possesses the following integrals of motion:

$$P = \int |k_z| n_k\, dk, \qquad E = \int |k_z| k_\perp^2 n_k\, dk.$$

The first quantity is the momentum of a system of waves traveling in one direction along the magnetic field, the second can be interpreted as the "energy."[1)]

It is well known[1,2] that the Kolmogorov spectrum in the inertia region is determined by only a single quantity—the energy flux in the region of high k (in the inertia region, of the energy is an integral of motion). Similar results were obtained for sound turbulence.[17] For waves on the surface of a liquid,[10] two spectra are possible: one corresponds to a constant flux of quasiparticles, the other to a constant energy flux. Here, in the region of transparency, the number of quasiparticles and the energy are integrals of motion. We make the hypothesis that to each integral in the transparent region there corresponds its own Kolmogorov spectrum. In our case, the flux of motion and of "energy" can easily be expressed in terms of the characteristics of the turbulence:

$$I_1 = \partial P/\partial t \sim k_z^5 k_\perp^2 n_k,$$
$$I_2 = \partial E/\partial t \sim k_z^5 k_\perp^4 n_k.$$

We then have for the energy spectral densities:

$$\varepsilon_k \sim I_1^{1/2} k_z^{-3/2} k_\perp^{-1}, \qquad \varepsilon_k \sim I_2^{1/2} k_z^{-3/2} k_\perp^{-2}. \tag{11}$$

We note that these estimates are explicitly based on the hypothesis of the local nature of the turbulence.

As also in hydrodynamics, the integral $\int \epsilon_k dk$ is divergent. The divergence of the integral at large $k_\perp$ for both spectra can be explained by the assumption that was made on the smallness of $k_\perp r_s$. The divergence for small k_z appears as in the case of hydrodynamics; therefore we must divide the whole region of k into energy-conserving, inertial, and dissipative regions. The obtained spectra (11) are valid in the inertial region.

4. EXACT SOLUTION OF THE REDUCED KINETIC EQUATION

We shall seek cylindrically symmetric stationary solutions of Eq. (8) for the assumptions made above. Inasmuch as the kernal of the equation does not depend on the angles in the $k_\perp$ plane, we carry out averaging over these angles. For this purpose, we represent the δ function of $k_\perp$ in the form

[1)]The total energy is $c_s(P - 1/2 r_s^2 E)$.

$$\delta(\mathbf{k}_\perp - \mathbf{k}_{\perp 1} \pm \mathbf{k}_{\perp 2}) = \frac{1}{(2\pi)^2}\int \exp[i(\mathbf{k}_\perp - \mathbf{k}_{\perp 1} \pm \mathbf{k}_{\perp 2})\mathbf{r}]\,d\mathbf{r}.$$

After averaging of this expression, we obtain

$$\langle\delta(\mathbf{k}_\perp - \mathbf{k}_{\perp 1} \pm \mathbf{k}_{\perp 2})\rangle = 1/\Delta(k_\perp, k_{\perp 1}, k_{\perp 2}),$$
$$\Delta(k_\perp k_{\perp 1} k_{\perp 2}) = {}^1/_2[2(k_{\perp 1}^2 k_{\perp 2}^2 + k_\perp^2 k_{\perp 1}^2 + k_\perp^2 k_{\perp 2}^2) - k_\perp^4 - k_{\perp 1}^4 - k_{\perp 2}^4]^{1/2}, \quad (12)$$

where Δ is the area of the triangle formed by the vectors $\mathbf{k}_\perp$, $\mathbf{k}_{\perp 1}$, $\mathbf{k}_{\perp 2}$. Averaging yields an identically zero result if these vectors do not form a triangle, i.e., when the radicand in (12) is negative.

We now make the substitutions $k_\perp^2 = s_{1,2,3}$ and $k_z = k$. We integrate over s_2 and k_2 in the collision term. As a result, we obtain

$$\int_0^k dk_1 \int_0^{sk/k_1} \frac{ds_1}{k_-} T^{kk_1k_-}_{ss_1s_-}(n_{k_1s_1}n_{k_-s_-} - n_{ks}n_{k_-s_-} - n_{ks}n_{k_1s_1}) + 2\int_0^\infty dk_1 \int_0^\infty \frac{ds_1}{k_+} T^{kk_1k_+}_{ss_1s_+}(n_{k_1s_1}n_{k_+s_+} + n_{ks}n_{k_+s_+} - n_{ks}n_{k_1s_1}) = 0, \quad (13)$$

where

$$T^{kk_1k_2}_{ss_1s_2} = |V_{kk_1k_2}|^2/\Delta(ss_1s_2),$$
$$k_\pm = k \pm k_1, \qquad s_\pm = (ks \pm k_1s_1)/(k \pm k_1).$$

Here T is a positive definite function of degree of homogeneity +3 in k, −1 in s. Therefore, we shall seek solutions of Eq. (13) in the form $n_{k,s} = Ak_s^{-\alpha-\beta}$, where A is a constant, and α and β are unknown quantities.

We carry out a bilinear transformation of the region of integration of the second integral in (13) to the region of integration of the first:

$$k_1 \to \frac{k-k_1}{k_1}k, \qquad dk_1 \to -\left(\frac{k}{k_1}\right)^2 dk_1;$$
$$s_1 \to \frac{(ks-k_1s_1)s}{(k-k_1)s_1}, \qquad ds_1 \to -\frac{k}{k-k_1}\left(\frac{s}{s_1}\right)^2 ds.$$

In this substitution, the integrals in (13) are joined together and the expression under the integral can be reduced to the form:

$$\int dk_1\,dk_2\,ds_1\,ds_2(kk_1k_2)^{-\alpha}(ss_1s_2)^{-\beta}T^{kk_1k_2}_{ss_1s_2} \times (k^\alpha s^\beta - k_1^\alpha s_1^\beta - k_2^\alpha s_2^\beta)(k^{2\alpha-4}s^{2\beta-1} - k_1^{2\alpha-4}s_1^{2\beta-1} - k_2^{2\alpha-4}s_2^{2\beta-1}) \times \delta(k-k_1-k_2)\delta(ks-k_1s_1-k_2s_2) = 0.$$

It is evident that the integrand vanishes for the following values of α and β:

1) $\alpha = 1$, $\beta = 0$; 2) $\alpha = 1$, $\beta = 1$;
3) $\alpha = {}^5/_2$, $\beta = {}^1/_2$; 4) $\alpha = {}^{5}/_2$, $\beta = 1$.

Inasmuch as the function T is positive definite, there are no other power solutions of Eq. (13). The first two solutions correspond to the Rayleigh-Jeans distribution for the reduced kinetic equation. The total energy diverges for large k_z (the ultraviolet catastrophe). The third and fourth solutions were obtained previously from considerations of dimensionality:

$$\varepsilon_k = A_3 k_\perp^{-1} k_z^{-3/2}, \qquad \varepsilon_k = A_4 k_\perp^{-2} k_z^{-3/2}.$$

5. LOCAL NATURE OF THE TURBULENCE

In order to prove the local character of the turbulence, it is necessary that the interaction between waves with scales of the same order be larger than the interaction of waves with scales of different orders. In other words, it is necessary that the integrals in (13) converge. We first consider the convergence at small k_1, s. For $k \gg k_1$ and $s \gg s_1$, we have

$$\Delta(k_1s_1) = {}^1/_2[4ss_1 - s^2(k/k_1)^2]^{1/2}$$

and the integration is carried out only over the region where the radicand of the expression is greater than zero. Taking this into account, we collect all terms that go to infinity for $k_1 \to 0$ and $s_1 \to 0$:

$$\int dk_1 \int ds_1 \frac{k^2k_1}{\Delta(k_1s_1)} n_{k_1s_1}\{n_{k-k_1,\,s(1+k_1/k)} - n_{ks} + 2(n_{k+k_1,\,s(1-k_1/k)} - n_{ks})\}.$$

These terms have the order

$$\left(\frac{\partial n}{\partial s}\frac{s}{k} - \frac{\partial n}{\partial k}\right)\frac{k^2}{s}\int dk_1\int ds_1 \frac{k_1^3}{\Delta(k_1s_1)} n_{k_1s_1}. \quad (14)$$

Integration in Eq. (14) is first carried out over k_1. After the substitution

$$\xi = 4s_1/s, \qquad \eta = k_1/k\xi^{1/2}$$

the integral is transformed to the form

$$sk^3\left(\frac{\partial n}{\partial s}\frac{s}{k} - \frac{\partial n}{\partial k}\right)\int_0^{\xi_0} d\xi\,\xi^{1-\beta-\alpha/2}\int_0^1 \frac{d\eta\,\eta^{2-\alpha}}{(1-\eta^2)^{1/2}}.$$

It is seen that these integrals converge for $\alpha < 3$, $\beta + \alpha/2 < 2$. The inequalities are satisfied for the distributions 1) – 3) but not for the fourth. This means that the contribution from the energy-conserving region in the inertial region is small for the distributions 1)–3). The fourth distribution is nonlocal.

We now consider the convergence for $k_1 \to \infty$ and $s_1 \to \infty$. For $k_1 \gg k$, $s_1 \gg s$,

$$\Delta(k_1s_1) = {}^1/_2[4ss_1 - s_1^2(k/k_1)^2]^{1/2}.$$

In this case, the most dangerous terms will be

$$2\int dk_1\int ds_1 \frac{kk_1}{\Delta(k_1s_1)}(n_{k+k_1,\,s_1(1-k/k_1)} - n_{k,s})n_{k,s} \sim \sim k^2 n_{ks}\int dk_1\int ds_1 \frac{k_1}{\Delta(k_1s_1)}\left(\frac{\partial n}{\partial k_1} - \frac{\partial n}{\partial s_1}\frac{s_1}{k_1}\right).$$

Similarly, we carry out the integration by making the change of variables

$$\xi = 4s/s_1, \qquad \eta = k/k_1\xi^{1/2}.$$

As a result, we obtain

$$sk^3\left(\frac{\partial n}{\partial k} - \frac{\partial n}{\partial k}\frac{s}{k}\right)\int_0^{\xi_0} d\xi\,\xi^{\beta+\alpha/2-2}\int_0^1 \frac{d\eta\,\eta^{\alpha-2}}{(1-\eta^2)^{1/2}}.$$

The convergence of the integrals is guaranteed for $\alpha > 1$ and $\beta + \alpha/2 > 1$. It is then seen that these inequalities are satisfied only for the third and fourth distributions.

Thus, the contribution of the dissipative region to the inertial is small for the third and fourth distributions. The only local distribution is the spectrum corresponding to a constant momentum flux.

In conclusion, the author expresses his gratitude to V. E. Zakharov for valued advice and constant interest in the research.

[1]A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
[2]A. M. Obukhov, Izv. Akad. Nauk SSSR Ser. Geograf. i Geofiz. **5**, ,(1941).

[3]A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nucl. Fusion **1**, 82 (1961).
[4]W. Drummond and D. Pines, Nucl. Fusion Suppl. (Part III), 1049 (1962).
[5]A. A. Galeev, V. I. Karpman, and R. Z. Sagdeev, Nucl. Fusion **5**, 20 (1965).
[6]B. B. Kadomtsev, Voprosy teorii plazmy (Problems of Plasma Theory) Vol. 4, Atomizdat, 1964.
[7]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **51**, 688 (1966) [Sov. Phys.-JETP **24**, 455 (1967)].
[8]V. E. Zakharov Prikl Mater. Tekh. Fiz. **4**, 34 (1965).
[9]V. E. Zakharov and N. N. Filonenko, Prikl Mater. Tekh. Fiz. **5**, 62 (1967).
[10]V. E. Zakharov, Candidate's Dissertation, Novosibirsk, 1967.
[11]L. M. Kovrizhnykh, Zh. Eksp. Teor. Fiz. **48**, 1114 (1965) [Sov. Phys.-JETP **21**, 744 (1965)].
[12]V. N. Tsytovich, Nelineinye effekty v plazme (Nonlinear Effects in Plasma) Nauka, 1967.
[13]B. I. Davydov, Dokl. Akad. Nauk SSSR **69**, 165 (1949).
[14]V. E. Zakharov and E. A. Kuznetsov, Dokl. Akad. Nauk SSSR **194**, 1288 (1970) [Sov. Phys.-Dokl. **15**, 913 (1971)].
[15]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **60**, 1714 (1971) [Sov. Phys.-JETP **33**, 927 (1971)].
[16]V. N. Oraevskii, Nucl. Fusion **4**, 263 (1964).
[17]V. E. Zakharov and R. Z. Sagdeev, Dokl. Akad. Nauk SSSR **192**, 297 (1970) [Sov. Phys.-Dokl. **15**, 439 (1970)].

Translated by R. T. Beyer
73