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Pair Production in a Variable and Homogeneous Electric Field as an Oscillator Problem

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The calculation of the probability w of pair production in a strong electric field E(t) which is homogeneous in space reduces to the problem of the parametric excitation of a quantum oscillator (or, what is equivalent, to the calculation of the reflection coefficient for a plane wave incident on a one-dimensional potential barrier). Because of this relation, the calculation of the pair-production probability does not require the determination of the exact solutions of the relativistic wave equations and the Green's function. In the particular case when $E(t)=E(\cosh\omega t)^{-2}$ the probability w coincides with the coefficient of reflection from the Eckart potential. The relation between the exact formulas for w and the quasiclassical approximation (the "imaginary time" method) is investigated in detail, and the limits of applicability of the quasiclassical approximation are indicated.

1. INTRODUCTION

THE question of pair production in strong fields has attracted an ever increasing amount of attention in recent years. In electrodynamics such processes were theoretically predicted many years ago;[1-2] however, until recent times they were regarded as inaccessible to experimental observation. One can separate the articles on pair production into three groups. In the first place the problem of pair production has turned out to be very important in cosmology in connection with investigation of the initial stage of evolution of the universe.[4,5] In the second place, the production of an e^+e^- pair with the subsequent escape of the positron should occur in the Coulomb field of superheavy nuclei with charge $Z > Z_c \approx 170$ or during the collision of two bare (i.e., stripped of electrons) nuclei with total charge $Z_1 + Z_2 > Z_c$ (see[6-8]). Finally, there are articles[9-14] in which Schwinger's well-known result[3] about pair production in a constant and homogeneous electric field is generalized to a field of more complicated form.

The probability w of pair production is calculated in[9,10] in the presence of a constant magnetic field in addition to the electric field (the corresponding formula for w in implicit form is contained in[3]). An estimate of the possibility of experimental observation of the production of e^+e^- pairs at the focus of an intense laser beam is carried out in[9]. In articles[10-12] pair production is treated with the aid of Feynman's formalism and, in particular, the case of a pulsed field of the special form $E(t) = E(\cosh\omega t)^{-2}$ is analyzed in detail; for this case the exact solution of the Dirac and Klein-Gordon equations is found, the causal Green's function is constructed, and the pair production probability is calculated. The quasiclassical method of "imaginary time" is developed in[14]; this method is applicable to a very broad class of fields provided that $\omega \ll m$ and $E \ll E_0 = m^2c^3/e\hbar$ (here E and ω denote the characteristic amplitude and frequency of the electric field, and E_0 denotes the field strength at which the nonlinearities in quantum electrodynamics become essential).

In the present article we show that the example treated by Narozhnyĭ and Nikishov[11] can be generalized. It is found that the problem of the production of a pair of charged scalar particles in a homogeneous electric field $\mathbf{E}(t)$, depending on the time in an arbitrary manner, is exactly equivalent to the solution of the problem of parametric excitation of a quantum oscillator. This enables us to apply well-developed methods of quantum mechanics[15-19] to the calculation of the probability w.

The problem of the pair production of spinor particles can also be reduced to the oscillator problem, but the corresponding frequency $\omega(t)$ becomes complex. Therefore, in the fermion case the problem of pair production leads more naturally to a different problem, namely, the precession of the spin in a variable magnetic field.

2. ONE-PARTICLE SOLUTIONS

It is convenient to start with the classical Hamilton-Jacobi equation for the action $S = S(\mathbf{r}, t)$:

$$(\nabla S - e\mathbf{A})^2 - (\partial S/\partial t + e\varphi)^2 + m^2 = 0. \quad (1)$$

We take the potential of the homogeneous field $\mathbf{E}(t)$ in the form[1)]

$$\mathbf{A} = 0, \quad \varphi = -(\mathbf{E}\mathbf{r}). \quad (2)$$

We seek the action in the form $S = s(t) + \mathbf{p}(t)\cdot\mathbf{r}$; in order to satisfy Eq. (1) it is necessary to set

$$d\mathbf{p}/dt = e\mathbf{E}(t), \quad s(t) = \pm\int_{-\infty}^{t} dt[m^2 + p^2(t)]^{1/2}.$$

Everywhere below by $\mathbf{p}(t)$ we mean the momentum of a classical particle moving in a homogeneous field:

$$\mathbf{p}(t) = \mathbf{p}_- + e\int_{-\infty}^{t} \mathbf{E}(t')\,dt', \quad (3)$$

and we denote its initial and final momenta by $\mathbf{p}_-$ and $\mathbf{p}_+$:

$$\mathbf{p}_\pm = \mathbf{p}(t \to \pm\infty)$$

1) Another gauge is chosen in [11,12]:

$$\varphi' = 0, \mathbf{A}'(t) = -\int_{-\infty}^{t} \mathbf{E}(t')\,dt'.$$

The transition from one gauge to the other is accomplished with the aid of the gauge transformation

$$A_\mu' = A_\mu + \partial_\mu\Lambda, \qquad S' = S + e\Lambda, \qquad \Lambda = -\mathbf{r}\int^{t} \mathbf{E}(t')\,dt'.$$

(it is assumed that the field vanishes as $t \to \pm\infty$ or is adiabatically switched off[2]).

Passing on to the quantum case, let us consider the Klein-Gordon equation ($\hbar = c = 1$):

$$[(i\partial/\partial t - e\varphi)^2 + \Delta - m^2]\psi = 0. \qquad (4)$$

Since $\psi \propto e^{iS}$ for motion in a homogeneous field (see[20], the footnote on p. 376), we assume in analogy with the classical case

$$\varphi(\mathbf{r}, t) = \xi(t) e^{i\mathbf{p}(t)\mathbf{r}}. \qquad (5)$$

One can easily see that for arbitrary $\xi(t)$

$$(i\partial/\partial t - e\varphi) e^{i\mathbf{p}(t)\mathbf{r}} \xi(t) = ie^{i\mathbf{p}(t)\mathbf{r}} d\xi/dt,$$

if the potential $\varphi(t)$ is of the form (2). Therefore (4) reduces to the equation of motion of a one-dimensional oscillator having a variable frequency:[3]

$$d^2\xi/dt^2 + \omega^2(t)\xi = 0, \quad \omega^2(t) = m^2 + \mathbf{p}^2(t). \qquad (6)$$

For $t \to \pm\infty$ the frequency of the oscillator tends to the following constant limit:

$$\omega(t) \to \omega_\pm = [m^2 + \mathbf{p}_\pm^2]^{1/2}. \qquad (6')$$

Here, generally speaking, $\omega_+ \neq \omega_-$ because

$$\mathbf{p}_+ - \mathbf{p}_- = e\int_{-\infty}^{\infty} \mathbf{E}(t)\,dt = \mathbf{q},$$

that is, it is equal to the momentum transferred to the particle by the field. Below (see Sec. 3) we introduce the solution $\xi(t)$ of Eq. (6), which is determined by the initial condition

$$\xi(t) = \exp(-i\omega_- t), \quad t \to -\infty. \qquad (7')$$

For $t \to +\infty$ the function $\xi(t)$ has the following asymptotic behavior:

$$\xi(t) = C_1 \exp(-i\omega_+ t) + C_2 \exp(i\omega_+ t). \qquad (7'')$$

The appearance here of the term $\exp(i\omega_+ t)$ characterizes the excitation of the oscillator associated with a change of its frequency (parametric excitation). Following[16], we shall characterize the degree of excitation of the oscillator by the parameter ρ:

$$\rho = |C_2/C_1|^2 \quad (0 \leqslant \rho < 1). \qquad (8)$$

Since $\omega^2(t)$ is real, the Wronskian

$$W(\xi, \xi^*) = \xi \frac{d\xi^*}{dt} - \frac{d\xi}{dt}\xi^*$$

does not depend on the time (and is equal to $2i\omega_-$). This gives the relation

$$|C_1|^2 - |C_2|^2 = \omega_-/\omega_+,$$

whence

$$|C_1| = \left(\frac{\omega_-}{\omega_+(1-\rho)}\right)^{1/2}, \quad |C_2| = \left(\frac{\omega_-\rho}{\omega_+(1-\rho)}\right)^{1/2}.$$

The parameter ρ immediately determines the probability of pair production in an alternating electric field (see Eq. (18)). For its determination it is necessary to solve Eq. (6) in the interval from $t = -\infty$ to $t = +\infty$. For the calculations it is convenient to change to a nonlinear equation of the Riccati type, in analogy to what is done in the method of phase functions.[21] Assuming

$$\xi(t) = A(t)[e^{-i\varphi} + R(t)e^{i\varphi}], \quad d\xi(t)/dt = i\omega A[-e^{-i\varphi} + Re^{i\varphi}],$$

where

$$\varphi = \varphi(t) = \int_0^t \omega(t')\,dt',$$

it is not difficult to obtain a system of equations for the functions $A(t)$ and $R(t)$. Eliminating A and dA/dt from this system, we find

$$\frac{dR}{dt} = \frac{1}{2\omega}\frac{d\omega}{dt}(e^{-2i\varphi} - R^2 e^{2i\varphi}), \qquad (9)$$

where $R(-\infty) = 0$ and $\rho = |R(+\infty)|^2$. This equation is convenient for numerical calculation, and also for the derivation of the formulas of perturbation-theory for ρ, of the adiabatic approximation, and so forth. The interpretation of the quantity ρ as the coefficient of reflection from a potential barrier is also useful on occasions. This interpretation (due to L. P. Pitaevskiĭ) is based on the fact that Eq. (6), upon replacing t by x and $\omega(t)$ by $k(x)$, is equivalent to the one-dimensional Schrödinger equation $\xi'' + k^2(x)\xi = 0$.

To obtain a number of specific formulas, it is convenient to specialize somewhat the form of the field $\mathbf{E}(t)$. Let $\mathbf{E}(t)$ preserve its direction in space (for example, along the z axis):

$$E_z(t) = Ef(\omega t), \qquad (10)$$

where ($\tau = \omega t$)

$$|f(\tau)| \leqslant f(0) = 1, \quad f(-\tau) = f(\tau).$$

Here E denotes the amplitude of the electric field, ω is the characteristic frequency; the moment $t = 0$ corresponds to the maximum of the field (the condition that $f(\tau)$ be an even function does not have any fundamental significance and is assumed only for the sake of simplicity). In this case

$$\omega^2(t) = \varepsilon^2 + \frac{2mp_\parallel}{\gamma}F(\tau) + \frac{m^2}{\gamma^2}F^2(\tau), \qquad (10')$$

where

$$F(\tau) = -F(-\tau) = \int_0^\tau f(\tau')\,d\tau',$$

$\gamma = m\omega/eE$ is the adiabaticity parameter,[13] $\epsilon = \sqrt{m^2 + \mathbf{p}^2}$ and $\mathbf{p} = (p_\parallel, \mathbf{p}_\perp)$ denote the energy and momentum of the particle at the moment $t = 0$, where $p_\parallel$ denotes the component of the momentum along the direction of the field. As the parameter specifying the trajectory, for fields of the type (10) it is convenient to take not $\mathbf{p}_\pm$, but $\mathbf{p}(0) = \mathbf{p}$, which is clear from the quasiclassical case: the moment $t = 0$ is naturally interpreted as the moment of escape of the particle from under the barrier.[13,14] The constituents of the pair have momenta $\pm\mathbf{p}$ (the total momentum of the pair is equal to zero as a consequence of the homogeneity of the field), and the limiting frequencies of the oscillator (6′) are given by

$$\omega_\pm = [m^2 + p_\perp^2 + (p_\parallel \pm q/2)^2]^{1/2},$$
$$q = (2m/\gamma)F(\infty).$$

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[2]For example, in the case $E(t) = E\cos\omega t$ or any other periodic dependence.

[3]Equation (6) was derived by A. M. Perelomov by a somewhat different method (private communication). For the case when the field $\mathbf{E}(t)$ has a fixed direction, the solution in the form $f(t)\exp(i\mathbf{p}\cdot\mathbf{r})$ was considered by Nikishov and Narozhnyĭ.[11,12]

3. THE HEISENBERG REPRESENTATION

In order to calculate the probability for pair production, we find it necessary to change to the second-quantized theory, which is related to the transition to the Heisenberg representation for the creation and annihilation operators a^+, a and so forth (for an oscillator with a variable frequency $\omega(t)$ this representation is investigated in detail in[16,18]). We also note a certain analogy between the following calculations and the methods developed in the problem of pair production in an expanding universe.[4,5]

For $t \to -\infty$ the boson field operator $\psi(\mathbf{r}, t)$ can be expanded in terms of the solutions of the free equation:

$$\Psi(\mathbf{r},t)=\sum_{\mathbf{p}}\frac{e^{i\mathbf{pr}}}{\sqrt{2\omega_-}}[a_{\mathbf{p}}\exp(-i\omega_-t)+b_{-\mathbf{p}}{}^+\exp(i\omega_-t)], \tag{11}$$

where $\omega_- = \sqrt{m^2 + \mathbf{p}^2}$, $\mathbf{p} \equiv \mathbf{p}_-$, and $a_{\mathbf{p}}$ and $b_{\mathbf{p}}^+$ are operators which do not depend on the time (the Schrödinger representation). Taking the form (5) of the one-particle solutions in a homogeneous field into account, we change over to the Heisenberg operators $\hat{a}_{\mathbf{p}}(t)$, $\hat{b}_{\mathbf{p}}(t)$, and so forth:

$$\Psi(\mathbf{r},t)=\sum_{\mathbf{p}}\hat{Q}_{\mathbf{p}}(t)e^{i\mathbf{p}(t)\mathbf{r}},\quad \hat{Q}_{\mathbf{p}}=\frac{1}{\sqrt{2\omega_-}}\{\hat{a}_{\mathbf{p}}(t)+\hat{b}_{-\mathbf{p}}(t)\}. \tag{12}$$

Since $\psi(\mathbf{r}, t)$ satisfies Eq. (4), $\hat{Q}_{\mathbf{p}}(t)$ satisfies the same equation as the coordinate of the one-dimensional oscillator:

$$d^2\hat{Q}_{\mathbf{p}}/dt^2+\omega^2(t)\hat{Q}_{\mathbf{p}}=0, \tag{13}$$

whose frequency $\omega(t)$ coincides with (6). The solution of Eq. (13) has the form of the Bogolyubov canonical transformation:[4)]

$$\hat{a}_{\mathbf{p}}(t)=u(t)a_{\mathbf{p}}+v(t)b_{-\mathbf{p}}{}^+,\quad \hat{b}_{-\mathbf{p}}{}^+(t)=v^*(t)a_{\mathbf{p}}+u^*(t)b_{-\mathbf{p}}{}^+, \tag{14}$$

where the coefficients u and v can be expressed[5)] in terms of the function $\xi(t)$ introduced above:

$$u(t)=\frac{1}{2\omega_-}\left(i\frac{d\xi}{dt}+\omega_-\xi\right),\quad v(t)=\frac{1}{2\omega_-}\left(i\frac{d\xi^*}{dt}+\omega_-\xi^*\right).$$

Equation (14) determines the time evolution of the Heisneberg operators $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{-\mathbf{p}}$ in explicit form, and by the same token it demonstrates the absence of any kind of difficulties in the problem of pair production by an external field in the second-quantized theory (such difficulties appeared in connection with the single-particle description[1]). For $t \to -\infty$ we have: $\omega(t) \to \omega_-$, $u(t) = e^{-i\omega_- t}$, $v(t) = 0$, $\hat{a}_{\mathbf{p}}(t) = a_{\mathbf{p}}\exp(-i\omega_- t)$. Conservation in time of the commutators

$$[\hat{a}_{\mathbf{p}}(t),\hat{a}_{\mathbf{p}'}{}^+(t)]=[\hat{b}_{\mathbf{p}}(t),\hat{b}_{\mathbf{p}'}{}^+(t)]=\delta(\mathbf{p}-\mathbf{p}')$$

imposes the condition

$$|u(t)|^2-|v(t)|^2=1, \tag{15}$$

as a consequence of which the transformation (14) pertains to the group SU(1, 1). One can also verify the validity of Eq. (15) by a direct check:

$$|u(t)|^2+|v(t)|^2=\frac{1}{2i\omega_-}W(\xi(t),\xi^*(t))=1$$

(the Wronskian $W(\xi, \xi^*)$ does not depend on t).

The operators $\hat{a}_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}^+$ or $\hat{b}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}^+$ satisfy the commutation relations which are characteristic for an oscillator. Since there are two pairs of such operators, the problem reduces to a two-dimensional isotropic oscillator with a variable frequency which is a given function of the time t. The properties of this quantum system (equivalent to an investigation of a particle moving in a homogeneous magnetic field $\mathbf{H} = (0, 0, H(t))$ are investigated in[19], which contains a general formula for the probability of a transition between arbitrary levels. Since there are no pairs for $t \to -\infty$ (the oscillator is found in its ground state), the formula which we need for the transition probability can be derived more simply.

For this purpose let us consider an f-dimensional isotropic oscillator with frequency $\omega(t)$. Its states are specified by the collection of non-negative integers $(n_1, n_2, \ldots, n_f)$, and its energy levels are given by

$$E_n=(n+f/2)\omega,\quad n=\sum_{i=1}^{f}n_i.$$

Let us denote by $w(n_1, \ldots, n_f)$ the probability of a transition from the vacuum state $(n_1 = \ldots = n_f = 0)$ to the state $(n_1, \ldots, n_f)$ under the influence of the variable frequency $\omega(t)$, and let us denote the total probability for excitation of the n-th level by c_n:

$$c_n=\sum_{n_1+\cdots+n_f=n}w(n_1,\ldots,n_f) \tag{16}$$

(for $f > 1$ all of the levels, except $n = 0$, are degenerate). Let us consider the generating function for c_n:

$$G_f(z)=\sum_{n=0}^{\infty}c_nz^n=\sum_{n_1=0}^{\infty}\ldots\sum_{n_f=0}^{\infty}w(n_1,\ldots,n_f)z^n=[G_1(z)]^f,$$

where z is an auxiliary variable, and $G_1(z) = [(1-\rho)/(1-\rho z^2)]^{1/2}$ is the generating function of the transition probability w_{0n} for the one-dimensional oscillator. From this we obtain

$$G_f(z)=\left(\frac{1-\rho}{1-\rho z^2}\right)^{f/2},\quad c_{2n}=\frac{\Gamma(n+f/2)}{n!\,\Gamma(f/2)}(1-\rho)^{f/2}\rho^n \tag{17}$$

(only levels with even values of n are excited). Since $c_{2n+1} = 0$, we shall write c_n instead of c_{2n} in what follows (n then refers to the number of pairs).

In particular, for $f = 2$ we have $c_n = (1-\rho)\rho^n$. Here $c_0 = c_0(\mathbf{p})$ is the probability that in the state with momentum $\mathbf{p}$ no pairs are created, and $c_n = c_n(\mathbf{p})$ is the probability for the creation of n pairs in this state. The usual method of calculation of the matrix elements[11] does not give the absolute probabilities c_n, but the relative probability $w_{\mathbf{p}}$:

$$w_{\mathbf{p}}=c_1(\mathbf{p})/c_0(\mathbf{p})=\rho, \tag{18}$$

which coincides thus with the coefficient ρ for reflection from the "barrier." As is evident from Eq. (6), the frequency $\omega(t)$ (and consequently also the value of ρ) depends on the particle momentum $\mathbf{p}$. In this connection $\omega^2(t) > 0$, that is, we are dealing with above-barrier re-

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4)Since the electromagnetic interaction conserves the charge, only the operators a and b^+ (and also a^+ and b) can be intermingled during the process of evolution. The coefficients u(t) and v(t) of the canonical transformation depend on the momentum **p** (through the frequency $\omega(t)$, see formula (6)). In what follows this is not specifically stated, and we shall omit the subscript **p**.

5)This is shown in [16]. We note that our definition of the function $\xi(t)$ differs from that of [16] by complex conjugation.

flection which, as a rule (differentiable functions $\omega(t)$ and so forth), is exponentially small: $\rho \ll 1$. Formula (18) reduces the calculation of the probability of pair production to the one-dimensional equation (6), which substantially simplifies all the calculations.

4. CERTAIN EXACT SOLUTIONS AND ASYMPTOTIC FORMS. THE LIMITS OF APPLICABILITY OF THE QUASICLASSICAL METHOD.

The probability of pair production becomes of the order of unity for $E \sim E_0 = m^2/e$. The electric field strengths experimentally accessible in the foreseeable future are much weaker[6)] than E_0. Under these conditions the probability for the production of a single pair $c_1(p) \ll 1$, and the probabilities c_n for the production of n = 2, 3, etc. pairs of bosons in the same quantum state is negligible, and therefore it is convenient to introduce the probability of pair production per unit volume:

$$w = \frac{P_1}{V} = \int \frac{d^3p}{(2\pi)^3} w_p \tag{19}$$

where it is assumed that the field is contained in a finite volume V, where the total probability for pair production $P_1 = wV \ll 1$).

Let us start with an example[11] which admits an exact solution:

$$E(t) = E/\mathrm{ch}^2\,\omega t, \quad f(\tau) = (\mathrm{ch}\,\tau)^{-2}. \tag{20}$$

Here $F(\tau) = \tanh\tau$; formula (10′) gives an expression for $\omega^2(t)$ which agrees with the well-known Eckart potential ($\tau = \omega t$):

$$\omega^2(t) = \frac{\omega_+^2 + \omega_-^2}{2} + \frac{\omega_+^2 - \omega_-^2}{2}\,\mathrm{th}\,\tau + \frac{\Omega^2 - \omega^2}{4\,\mathrm{ch}^2\tau}. \tag{21}$$

This function depends on three free parameters: ω_-, ω_+, and Ω. In the case (21) Eq. (6) can be solved exactly (in terms of hypergeometric functions) and the coefficient of reflection is given by[22,23]

$$\rho = [\mathrm{ch}(\alpha - \beta) + \cos\lambda]/[\mathrm{ch}(\alpha+\beta) + \cos\lambda], \tag{22}$$

where

$$\alpha = \pi\omega_-/\omega, \quad \beta = \pi\omega_+/\omega, \quad \lambda = \pi\Omega/\omega,$$
$$\omega_\pm = (m^2 + p_\pm^2)^{1/2}, \quad \Omega = [\omega^2 - (2m/\gamma)^2]^{1/2}.$$

Expression (22) agrees with the relative probability w_p for the production of pairs calculated in[11] (see formula (13a) of[11]; in order to compare it with Eq. (22) it is necessary to take the identity $\cosh 2x + \cos 2y = 2(\sinh^2 x + \cos^2 y)$ into account). Thus, the reduction of the problem to the nonrelativistic Schrödinger equation enables us to obtain the probability w_p almost without any calculations and, in addition, it explains why in the case of the field (20) it was found possible to determine the exact solution of the Klein-Gordon equation.

Let us consider limiting cases. If $\omega \ll m$ (more precisely $\omega\gamma \ll 2\pi m$), then $\lambda = i\lambda'$, $\lambda' \approx 2\pi m/\omega\gamma$ and the inequalities $\alpha + \beta \gg \lambda' \gg |\alpha - \beta|$, $\alpha + \beta - \lambda' \gg 1$ are satisfied. Therefore

$$w_p = \exp[-(\alpha + \beta - \lambda')]$$
$$= \exp\left\{-\frac{\pi E_0}{E}\left[\frac{2}{1 + (1+\gamma^2)^{1/2}} + \frac{1}{(1+\gamma^2)^{1/2}}\left(\frac{p_\perp}{m}\right)^2 + \frac{\gamma^2}{(1+\gamma^2)^{3/2}}\left(\frac{p_\parallel}{m}\right)^2\right]\right\}, \tag{23}$$

which agrees with the result of the quasiclassical approximation.[13] Expression (23) describes a continuous transition from a constant electric field ($\omega = \gamma = 0$) to a field which is rapidly changing in the characteristic tunneling time ($\omega \gg \omega_t$, $\gamma \gg 1$). Here the distributions in $p_\parallel$ and $p_\perp$ are narrow, that is, the constituents of the pair are primarily created at the instant t = 0 (when the field is maximal) with momenta $p \sim \sqrt{m\omega}$ close to zero.

Now let $\omega\gamma \gg 2\pi m$. Then the momentum transferred to each of the pair's particles is $q = m/\gamma \ll \omega$ and in Eq. (22) one can expand in powers of the parameter $q/\omega \ll 1$. This gives

$$w_p = \frac{(\alpha - \beta^2)/2}{\mathrm{ch}(\alpha+\beta) - 1} = \frac{\pi^2 p_\parallel^2}{(\omega\gamma)^2\varepsilon^2\,\mathrm{sh}^2(\pi\varepsilon/\omega)}, \tag{24}$$

where $\epsilon = (m^2 + p_\perp^2 + p_\parallel^2)^{1/2}$ denotes the particle energy at the instant t = 0.

The equation $\omega\gamma = m$ corresponds to the frequency $\omega = \omega_1 = m\sqrt{E/E_0}$. A transition from the quasiclassical approximation to perturbation theory occurs in the range of frequencies $\omega \sim \omega_1$. In fact, one can represent the exact formula (22) in the form

$$w_p \approx D\exp(-2\pi\varepsilon/\omega) \tag{25}$$

(provided that $\gamma \gg 1$, that is, in the anti-adiabatic region). The factor D appearing in front of the exponential is given by

$$D = \left[\frac{2(1+\cos\lambda)}{u^2} + \left(\frac{p_\parallel}{\varepsilon}\right)^2\right]u^2e^{-u} = \begin{cases}(p_\parallel/\varepsilon)^2u^2, & u \ll 1\\ 1 - \pi^2/2u, & u \gg 1\end{cases} \tag{26}$$

where $u = 2\pi m/\omega\gamma = 2\pi(\omega_1/\omega)^2$ and $\lambda = \sqrt{\pi^2 - u^2}$. So long as $\omega \ll \omega_1$, the quantity $u \gg 1$ and expression (25) agrees with the quasiclassical asymptotic behavior described by (23). It should be emphasized that here the quasiclassical treatment not only gives the exact form of the exponential factor in the probability w_p, but also gives the correct pre-exponential factor ($D \approx 1$). However, when ω becomes of the order of ω_1, the pre-exponential factor D begins to change substantially and it no longer agrees with the result (23) of the quasiclassical calculation. For $\omega_1 \ll \omega$ expression (25) goes over into (24), that is, it agrees with the perturbation theory formula in an external field E(t).

We note that similar results are also valid for particles with spin 1/2. In this case the probability for pair production in the electric field (20) is given by formula (36), whose expansion provided $\omega \gg \omega_t$ has the form (25) as before, but now

$$D = \left[\left(\frac{\mathrm{sh}\,u/2}{u/2}\right)^2 - \left(\frac{p_\parallel}{\varepsilon}\right)^2\right]u^2e^{-u} = \begin{cases}(\varepsilon^2 - p_\parallel^2)u^2/\varepsilon^2, & u \ll 1\\ 1 + O(u^2e^{-u}), & u \gg 1\end{cases}. \tag{27}$$

Let us explain the physical meaning of the frequency ω_1 which appears in the pre-exponential factor D. In the pulsed field (10) the particle acquires energy $\Delta\epsilon \sim eET \sim m/\gamma$; single quantum absorption occurs provided $\Delta\epsilon < \omega$, that is, $\omega \gtrsim \omega_1$. Therefore, in this range of

6)Even for the lightest particles, namely electrons, $E_0 = 1.3 \times 10^{16}$ V/cm, and for π mesons $E_0 \sim 10^{21}$ V/cm. For comparison we indicate that in the Coulomb field of a point charge e the field strength E_0 is reached at a distance $r_0 = \sqrt{\alpha}\,\hbar/mc$ from the center ($r_0 = 33$ F for electrons and 0.12 F for π mesons).

frequencies the expression for w_p reduces to the result of first-order perturbation theory.

There is another characteristic frequency, namely, the tunneling frequency $\omega_t = mE/E_0$. Under the condition $E \ll E_0$, which at the present time is satisfied experimentally with a large margin, we have $\omega_t \ll \omega_1 \ll m$. The quasiclassical treatment assumes the multiple quantum nature of the absorption; therefore it is valid in the frequency range $\omega \ll \omega_1$ and here it gives the correct expression for the probability w_p, including the pre-exponential factor. For larger frequencies ω the pre-exponential factor D cannot be calculated quasiclassically; however, the major (exponential) factor $\exp\{-\pi E_0 g(\gamma)/E\}$ in the probability w_p retains the same form as in the quasiclassical treatment for all frequencies $\omega \ll m$.

In the region $\omega \gg \omega_1$ it is not difficult to obtain a closed expression for the probability w_p in the presence of an arbitrary field E(t). In this case $\gamma \gg 1$, and the variable part of the frequency (10′) is small:

$$\omega(t) = \varepsilon\left[1 + \frac{mp_\parallel}{\gamma\varepsilon^2}F(\tau) + \ldots\right], \quad \frac{mp_\parallel}{\gamma\varepsilon^2} \sim \frac{1}{\gamma}\sqrt{\frac{\omega}{m}} \tag{28}$$

and therefore the reflection coefficient ρ can be determined according to perturbation theory. Assuming $|R(t)| \ll 1$, from Eq. (9) we obtain

$$\rho = |R(\infty)|^2,$$

$$R(\infty) = \frac{1}{2\varepsilon}\int_{-\infty}^{\infty} dt\, e^{-2i\varepsilon t}\frac{d\omega(t)}{dt} = \frac{1}{4\varepsilon^2}\int e^{-2i\varepsilon t}\, d\omega^2(t).$$

For the electric field (10) we find

$$w_p = \rho = \left|\frac{\pi p_\parallel}{\gamma\varepsilon^2}\psi\left(\frac{2\varepsilon}{\omega}\right)\right|^2, \tag{29}$$

$$w = \frac{m^3}{6\gamma^2}\int_1^\infty d\varepsilon\left(1 - \frac{1}{\varepsilon^2}\right)^{3/2}\left|\psi\left(\frac{2\varepsilon}{\omega}\right)\right|^2.$$

Here w_p determines the pulsed spectrum,[7] and $w = P_1/V$ denotes the total probability of pair production per unit volume. The Fourier transform of the field $f(\tau)$ is denoted by $\psi(s)$:

$$\psi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, f(\tau)\, e^{-is\tau}.$$

Let us analyze the following limiting cases: $\omega \ll m$ and $\omega \gg m$. In the first case $2\epsilon/\omega \gg 1$. If the singularity of the function $f(\tau)$ nearest to the real axis is located at the point $\tau = \pm i\tau_0$, then $|\psi(s)| \sim e^{-s\tau_0}$ for $s \to \infty$. Therefore, to within the pre-exponential factor, from Eq. (29) we obtain

$$w_p \sim |\psi(2\varepsilon/\omega)|^2 = D\exp(-4\tau_0\varepsilon/\omega). \tag{30}$$

Expression (30) also follows from the quasiclassical treatment. According to formula (21) of[13] we have:

$$w_p = C\exp\left\{-\frac{\pi E_0}{E}\left[g(\gamma) + c_1\left(\frac{p_\perp}{m}\right)^2 + c_2\left(\frac{p_\parallel}{m}\right)^2\right]\right\}.$$

For $\gamma \gg 1$ we have $g(\gamma) = l(\gamma)/\gamma$, where $l(\gamma)$ is a function of logarithmic type: $l(\gamma) = c_0(\ln\gamma)^\alpha$, see the table.[8] Hence $c_1 \approx c_2 \approx (1/2)g(\gamma)$ for $\gamma \gg 1$, that is

$$w_p \approx C\exp\left\{-\frac{\pi m}{\omega}l(\gamma)\left(1 + \frac{p^2}{2m}\right)\right\} = C\exp\left\{-l(\gamma)\frac{\pi\varepsilon}{\omega}\right\}. \tag{31}$$

The distribution with respect to **p** is narrow and almost isotropic. If $f(\tau)$ has the nearest singularity for $\tau = i\tau_0$, then $l(\gamma) = 4\tau_0/\pi$ and (31) agrees with (30). Thus, in the range of frequencies $\omega \gg \omega_t$ formula (29) matches the results of the quasiclassical calculation (to within the exponential factor; in order to obtain agreement of the pre-exponential factors it is still required that $\omega \ll m\sqrt{E/E_0}$).

The opposite case $\omega \gg m$ falls outside the limits of applicability of the quasiclassical method. The region of integration in (29) extends right up to $\epsilon \sim \omega \gg 1$, as a consequence of which one can replace $(1 - 1/\epsilon^2)^{3/2}$ by unity and integrate from zero:

$$w = bm^3\left(\frac{E}{E_0}\right)^2\frac{m}{\omega}, \quad b = \frac{1}{48\pi}\int_{-\infty}^{\infty} f^2(\tau)\, d\tau. \tag{32}$$

The relation $w \propto E^2$ corresponds to perturbation theory (a homogeneous field transfers only energy to the particle, and therefore pair production at $\omega > 2m$ is now possible in first-order perturbation theory). In the range of frequencies $\omega \gg m$ the pulsed distribution of the created particles is broad ($p \sim \epsilon \gg m$), and only the numerical constant b in (32) depends on the specific form of the field $f(\tau)$.

The parameters $l(\gamma)$ for $\gamma \gg 1$ and for several characteristic fields are cited in the Table.

5. THE FERMION CASE

Let us consider briefly the generalization of the method to the case of particles with spin $s = 1/2$. For this we start from the squared equation

$$\{\pi_\mu\pi_\mu + m^2 - {}^1\!/_2 e(1+\varkappa)\sigma_{\mu\nu}F_{\mu\nu}\}\psi = 0, \tag{33}$$

where $\pi_\mu = -(i\partial_\mu + eA_\mu)$, $\kappa = (g/2) - 1$, and g is the gyromagnetic ratio. Equation (33) describes a particle with magnetic moment $\mu = (1+\kappa)e\hbar/2mc$ in its rest frame. It includes as special cases the usual Dirac equation (for $\kappa = 0$) and the usual Klein-Gordon equation (for $\kappa = -1$). Substituting the solution in the form

$$\psi(\mathbf{r}, t) = \xi(t)\exp\{i\mathbf{p}(t)\mathbf{r}\},$$

into (33) with the gauge (2), where $\xi(t)$ is a bispinor independent of **r**, we obtain:

$$\{d^2/dt^2 + m^2 + \mathbf{p}^2(t) + ie(1+\varkappa)\sigma\mathbf{E}(t)\}\xi = 0. \tag{34}$$

$f(\tau)$	$l(\gamma)$	$2\psi(s)$	$f(\tau)$	$l(\gamma)$	$2\psi(s)$
$(\mathrm{ch}\,\tau)^{-2}$	2	$s/\mathrm{sh}(\pi s/2)$	$(1+\tau^2)^{-3/2}$	$4/\pi$	$(2s/\pi)K_1(s)$
$(\mathrm{ch}\,\tau)^{-1}$	2	$1/\mathrm{ch}(\pi s/2)$	$e^{-\tau^2/2}$	$1{,}8\sqrt{\ln\gamma}$	$(2/\pi)^{1/2}\exp(-s^2/2)$
$(1+\tau^2)^{-1}$	$4/\pi$	e^{-s}	$\cos\tau$	$1{,}27\ln\gamma$	$2\delta(s^2-1)$

[7]We note that for $p_\parallel = 0$ the probability w_p strictly speaking does not vanish, but becomes a small quantity of the order of γ^{-4}.

[8]The exponent α may be equal to zero; then the function $l(\gamma)$ reduces to a constant.

For electric fields of the form (10), which preserve their direction in space, the operator (34) is diagonal, and therefore the components of the bispinor $\xi(t)$ satisfy the oscillator equations (6). The frequency $\omega(t)$ is now complex:

$$\begin{aligned}\omega^2(t) &= m^2 + p^2(t) \pm ie(1+\varkappa)E(t)\\ &= \varepsilon^2 + (m/\gamma)[2p_{\|}F(\tau) \pm i\omega(1+\varkappa)f(\tau)]\\ &\quad + (m^2/\gamma^2)F^2(\tau).\end{aligned} \tag{35}$$

For the pulsed field (20) the dependence $\omega = \omega(t)$ has as before the form (21) corresponding to the Eckart potential, where

$$\omega_\pm^2 = \varepsilon^2 + \frac{m^2}{\gamma^2} \pm \frac{2mp_{\|}}{\gamma},$$

$$\Omega^2 = \omega^2 - \frac{4m^2}{\gamma^2} \pm 4i(1+\varkappa)\frac{m\omega}{\gamma}.$$

At $\kappa = -1$ we return to Eq. (22), and in the case $\kappa = 0$ (a Dirac particle without an anomalous magnetic moment) $\Omega = \omega \pm 2im/\gamma$ and $\cos\lambda = -\cosh(2\pi m/\omega\gamma)$ independently of the $\pm$ sign in Ω. Therefore the coefficient ρ, determined according to formula (22), remains real:[9]

$$\begin{aligned}w_p = -\rho &= \frac{\operatorname{ch}\lambda' - \operatorname{ch}(\alpha-\beta)}{\operatorname{ch}(\alpha+\beta) - \operatorname{ch}\lambda'}\\ &= \frac{\operatorname{sh}[(\lambda'+\alpha-\beta)/2]\operatorname{sh}[(\lambda'-\alpha+\beta)/2]}{\operatorname{sh}[(\alpha+\beta+\lambda')/2]\operatorname{sh}[(\alpha+\beta-\lambda')/2]},\end{aligned} \tag{36}$$

where $\lambda' = 2\pi m/\omega\gamma$, and α and β have the same values as in (22) (the equation $w_p = -\rho$ for calculations with a complex frequency $\omega(t)$ follows from a comparison of formulas (A.6) and (A.8) of the Appendix). Formula (36) for the relative probability of pair production was derived in[11] by directly solving the Dirac equation.

It should be noted that in the case of the field (20) expression (35) reduces as before to the Eckart potential for a reason that is, to some extent, accidental. In general three independent functions enter in (35), namely $F(\tau)$, $F^2(\tau)$, and $f(\tau)$; whereas in (21) only two variable terms are present: $\tanh\tau$ and $(\cosh\tau)^{-2}$. However, the identity $F^2(\tau) = 1 - f(\tau)$ holds for the field (20) and reduces $\omega^2(t)$ to the form (21). In the quasiclassical limit $\omega \ll m\sqrt{E/E_0}$, by expanding (36) in powers of $p_{\|}/m$ and $p_\perp/m$, we again arrive at formula (23). This indicates that in the quasiclassical region the probability for pair production in an alternating electric field does not depend on the spin (also see[14]).

Thus, in the fermion case ($s = 1/2$) the calculation of the probability w_p reduces to the oscillator problem (6) but with a complex frequency $\omega(t)$. Furthermore, the coefficients of the canonical transformation (14) no longer satisfy condition (15). Since the anticommutators $\hat{a}_p(t)$, $\hat{a}_p^+(t)$ and so forth preserve their value in time, then

$$|u(t)|^2 + |v(t)|^2 = 1, \tag{37}$$

that is, the transformation (14) no longer is contained in the group SU(1, 1) but in the unitary group SU(2). Therefore, in a mathematical sense one can anticipate that the production of a pair of spinor particles is equivalent to the problem of the precession of the spin in the presence of a variable magnetic field $\mathbf{H}(t)$. From this point of view, the appearance of complex quantities in (35) is not surprising, since one can reduce the equation for the rotation of the spin in a field $\mathbf{H}(t)$ to the equation of motion of an "oscillator" with a complex frequency.[17] We hope to investigate the case $s = 1/2$ in more detail later.

Although the transition from (34) to an equation of the type $d\mathbf{s}/dt = [\Omega(t), \mathbf{s}]$ has a number of advantages, especially from a group theoretic point of view,[10] nevertheless, for calculational purposes, Eq. (35) may turn out to be preferable in view of its simplicity. In particular, it is not difficult to derive with its aid the analog of formula (29) for particles with spin 1/2:

$$w_p = \frac{\pi^2(\varepsilon^2 - p_{\|}^2)}{\gamma^2\varepsilon^4}\left|\psi\left(\frac{2\varepsilon}{\omega}\right)\right|^2,$$

$$w = \frac{m^3}{3\gamma^2}\int_1^\infty \frac{d\varepsilon}{\varepsilon^3}(2\varepsilon^2+1)(\varepsilon^2-1)^{1/2}\left|\psi\left(\frac{2\varepsilon}{\omega}\right)\right|^2. \tag{38}$$

This formula determines the probability for pair production of spinor particles in the anti-adiabatic region $\gamma \gtrsim \sqrt{E_0/E} \gg 1$, where perturbation theory is valid.

6. CONCLUSION

We wish to make several concluding remarks.

1) As is well known,[3] pairs cannot be created in the field of a plane wave of arbitrary polarization and spectral composition, since both of the invariants $\mathcal{F} = \mathbf{E}^2 - \mathbf{H}^2$ and $\mathcal{G} = (\mathbf{E}\cdot\mathbf{H})$ vanish. For other fields $\mathcal{F}$ and $\mathcal{G}$ do not vanish,[11] and pair production becomes possible (for example, for a spherical wave converging to a focus).

2) Thus, a field of the electric type is required for pair production. For example, the field considered above (a homogeneous electric field $\mathbf{E}(t)$ under the assumption that $\mathbf{H}(t) = 0$) bears a model character. Strictly speaking such a field cannot be created in vacuum—its creation calls for a current $\mathbf{j} = -(1/4\pi)d\mathbf{E}(t)/dt$. However, the field which arises upon focusing can be approximately regarded as uniform in a region of size $R \sim \lambda$, where λ denotes the wavelength. As long as $\omega \ll m$ we have $R \gg \hbar/mc$, which justifies the applicability of the present method.

3) We have investigated the problem of pair production in terms of field oscillators. It is possible to obtain the exact solution of the problem (not within the framework of perturbation theory) because the oscillators of a charged boson field remain independent in a

[9]Upon making the substitution $t \to x$ we obtain the problem concerning the propagation of a wave through a barrier with absorption. However, as is evident from (35), there is no absorption at $x \to \pm\infty$ since $E(t) \to 0$. This enables us to determine the coefficient of reflection from the barrier by the usual method.

[10]The connection between the problem of pair production in an external field and group theory is helpful in a number of problems (for example, in connection with deriving the equation $w_p = -\rho$ for spinor particles). In this connection, see the Appendix. The group-theoretical aspects of this problem for particles of arbitrary spin have recently been investigated in detail by A. M. Perelomov.

[11]One can easily see this even in the simplest example of the interference of two plane waves with wave vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Here $F = E_1E_2(1-\cos\theta)\cos(\psi-\phi)$ and $G = E_1E_2(1-\cos\theta)\sin(\psi-\phi)$, where ψ, θ, and ϕ are the Euler angles defining the rotation of the triplet of vectors $\mathbf{E}_1$, $\mathbf{H}_1$, $\mathbf{n}_1$ to $\mathbf{E}_2$, $\mathbf{H}_2$, $\mathbf{n}_2$.

homogeneous electric field **E**(t), even though they have a variable frequency $\omega_p(t)$.

4) The excitation of a quantum oscillator is not only possible by means of a change of its frequency, but also under the influence of external forces (see[16,18]). This case can be realized in the problem of pair production, provided that external sources are added to the right hand sides of Eqs. (4) and (33).

5) Let us discuss finally the connection between the present work and the "imaginary time" method. This method, which is a generalization of the quasiclassical method to the case of fields **E** and **H** that vary in time, is valid in the region where the pair production probability $w \ll 1$. In principle it is applicable to a broad class of fields: it does not require the electric field to be uniform, it is possible to take account of **H**(t), etc.; the only requirement is the analyticity of the fields in t (which, however, follows from Maxwell's equations). On the other hand, the reduction of the problem to the parametric excitation of a set of independent oscillators is only possible for a uniform field **E**(t), but in return E and ω can be arbitrary. For $E \ll E_0$ a region of matching exists, $\omega \ll m\sqrt{E/E_0}$, in which both methods give identical results. This agreement can be regarded as substantiation of the "imaginary time" method.

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APPENDIX

One can express[17] the matrix elements of a transition between n-quantum states of an oscillator with variable frequency in terms of the matrix elements of the group SU(1, 1). For our purposes it is sufficient to know the probability of the vacuum—vacuum transition; let us present a simple derivation of this formula.

The group SU(1, 1) is, in a certain sense, an analytic continuation of the unitary group SU(2), which is related to ordinary three-dimensional rotations. If ψ, θ, and φ denote the Euler angles for the spatial rotation $R = \exp(-i\psi J_3)\exp(-i\theta J_1)\exp(-i\varphi J_3)$, then in order to change to SU(1, 1) it is sufficient to replace θ by $i\beta$ and to assume that $0 \le \beta < \infty$ (the parameters ψ and φ describe rotations in the Euclidean plane x_1, x_2 and as usual they vary in a finite interval). The commutation relations between the generators of these groups are analogous to each other ($J_\pm = J_1 \pm iJ_2$):

$$[J_+, J_-] = 2\varepsilon J_3, \quad [J_3, J_\pm] = \pm J_\pm; \tag{A.1}$$

here $\epsilon = 1$ in the case of the group SU(2) and $\epsilon = -1$ in the case of SU(1, 1). Upon realization in three-dimensional space, $\mathbf{x}^2 = x_\mu x^\mu = \epsilon(x_1^2 + x_2^2) + x_3^2$ is an invariant quadratic form, that is, SU(1, 1) is isomorphic to the three-dimensional Lorentz group $\mathscr{L}_3$.

In the problem of pair production the raising operator J_+ is obviously given by a^+b^+. Assuming in the boson case that

$$J_+ = a^+b^+, \quad J_- = ab, \quad J_3 = \tfrac{1}{2}(a^+a + b^+b + 1), \tag{A.2}$$

and in the fermion case

$$J_+ = a^+b^+, \quad J_- = ba = -ab, \quad J_3 = \tfrac{1}{2}(a^+a + b^+b - 1), \tag{A.3}$$

it is not difficult to verify that the commutators have the form (A.1) with $\epsilon = -1$ for bosons and $\epsilon = 1$ for fermions, and the square of the angular momentum (the Casimir operator) is given by

$$\mathbf{J}^2 = \varepsilon(J_1^2 + J_2^2) + J_3^2 = \varkappa[1 - (a^+a - b^+b)^2], \tag{A.4}$$

where $\kappa = -1/4$ for bosons and $\kappa = 3/4$ for fermions.

The fermion case is somewhat simpler. Not more than one pair can be created in a state with a given value of **p** and σ, and therefore $J_3 = \pm 1/2$, $\mathbf{J}^2 = 3/4$, which corresponds to the spinor representation of the rotation group (this fact was already mentioned in[12]). The probability that no pair is created in the state **p**, σ is nothing other than the probability of the vacuum—vacuum transition for the corresponding field oscillator: $c_0 = |d^j_{-j,-j}(\theta)|^2$, where $d^j_{mm'}(\theta)$ denotes the matrix for a finite rotation (according to Eq. (A.3), the vacuum state corresponds to $m = -j = -1/2$; here m denotes the eigenvalue of the operator J_3). From this it follows that

$$c_0(\mathbf{p}) = \cos^2\frac{\theta}{2}, \quad c_1(\mathbf{p}) = \sin^2\frac{\theta}{2}, \quad w_\mathbf{p} = \frac{c_1}{c_0} = \mathrm{tg}^2\frac{\theta}{2}. \tag{A.5}$$

In the boson case n = 0, 1, 2, ... pairs can be found in a state with zero charge, and the spectrum of the operator J_3 has the form $m = n + 1/2$. In this connection $c_0 = 1 - \rho$, where ρ is the coefficient of reflection from the barrier (see formulas (17) and (8)). Let us determine the relation between ρ and the angle β of the corresponding hyperbolic rotation. For this purpose we consider a one-dimensional oscillator for which[24] $\mathbf{J}^2 = j(j + 1) = -3/16$, and moreover for the vacuum state $m = -j = 1/4$. The probability of a vacuum—vacuum transition under the influence of the variable frequency $\omega(t)$ is given by[15,16]

$$w_{00} = \sqrt{1-\rho} = 1|d_{-j,}{}^j{}_{-j}(\beta)|^2, \quad j = -\tfrac{1}{4}. \tag{A.6}$$

Noting that $d^j_{-j,-j}(\theta) = \cos^{2j}(\theta/2)$ in the case of the group SU(2), and analytically continuing this formula for the group SU(1, 1), we find: $d^{-1/4}_{1/4,1/4}(\beta) = \cosh^{-1/2}(\beta/2)$. For the two-dimensional oscillator $c_0 = 1 - \rho = |w_{00}|^2$, hence

$$c_0(\mathbf{p}) = \mathrm{ch}^{-2}\frac{\beta}{2}, \quad w_\mathbf{p} = \frac{c_1}{c_0} = \rho = \mathrm{th}^2\frac{\beta}{2}. \tag{A.7}$$

Formulas (A.5) and (A.7) establish a relation between the relative probabilities w_p for pair production and the group parameters θ and β. In this connection it is obvious that the angles θ and β depend on the momentum p, because it appears in expressions (10) and (35) for the frequency $\omega(t)$ (for brevity the subscript **p** on the operators a and b etc. is omitted). The probability that the vacuum remains the vacuum (that is, the probability that not even a single pair is created) is given by

$$P_v = \prod_\mathbf{p} c_0(\mathbf{p}) = \begin{cases} \prod_\mathbf{p} \mathrm{ch}^{-2}(\beta_\mathbf{p}/2) & \text{for bosons} \\ \prod_{\mathbf{p},\sigma} \cos^2(\theta_\mathbf{p}/2) & \text{for fermions} \end{cases}$$

The necessity of changing the sign upon analytic continuation of w_p from the scalar case to the spinor case ($\beta \to i\theta$) follows from a comparison of formulas (A.5) and (A.7). This fact was utilized in Sec. 5.

For fields of the form (10) the probability for the creation of fermion pairs in the state p, σ does not depend on the spin component σ and is given by

$$w_{p\sigma} = -R_1R_2^*, \tag{A.9}$$

where $R_1 = R[\omega]$ and $R_2 = R[\omega^*]$ denote the amplitudes of reflection for the frequencies $\omega(t)$ and $\omega^*(t)$ (in terms of the notation of formula (7′) we have: $R = C_2/C_1$). These quantities can be calculated from the oscillator equation (35). The structure of the formula (A.9) reflects the analyticity of $w_{p\sigma}$ with respect to the group parameter β.

The derivation of formula (A.7) given here is not, of course, mathematically rigorous. A more complete exposition of the theory of representations of the group SU(1, 1) is contained in[25], and its application to the oscillator is discussed in articles[17,19,24].

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