Functional Approach to the Turbulent Dynamo

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A new method in the theory of the turbulent dynamo is proposed. A functional approach is formulated and a variational differential equation is derived for the characteristic functional, and is an analog of the Schwinger equation in quantum mechanics. The equation is solved by expansion into a functional series. It is shown that already the first approximation is equivalent to summation of an infinite set of perturbation-theory diagrams. New results are obtained in the second approximation in the functional series. The turbulence diffusion coefficient for a regular magnetic field is obtained with higher accuracy. It is shown that in this approximation the nongyrotropic (reflection-invariant) turbulence does not induce field generation. The new approximation is equivalent to summation of a broader diagram set and hence to a definite method of diagram summations. On the other hand the second approximation can be obtained by equating to zero the semi-invariants, a procedure which is extensively employed in other branches of theoretical physics.

BY now there are many known theories of the turbulent dynamo. Thus, for example, it has been demonstrated that it is possible to generate regular magnetic fields[1,2]. The question of the excitation of random fields has also been considered[3-5]. In essence, the problem reduces in the cited papers to a search of the statistical properties of H knowing the statistical properties of v, using the equation

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{vH}] + \nu_m \Delta \mathbf{H}, \qquad (1)^*$$

where ν_m is the magnetic viscosity. In the general case it is impossible to express H analytically in terms of v, and in this lies the main difficulty. It is therefore customary to use a perturbation-theory series in the velocity. In the applications one is usually interested in a situation in which the magnetic Reynolds number $R_m \gg 1$. But in this case it is no longer possible to terminate the perturbation-theory series in v (except in the case of acoustic turbulence[5]), and a complete summation of the series is, naturally, impossible. Kazantsev[4], nevertheless, was able to sum successfully by using a turbulence model with a δ-like correlation in time. There is also another method, wherein the chain for the moments is terminated with the aid of Millionshchikov's hypothesis[6]. It would be of interest, of course to carry out a selective summation of the series, using various diagram methods.

The functional approach proposed in this article is frequently equivalent to the aforementioned methods, and is at the same time more compact. It will be shown that this approach can be regarded as the next step in the theory of the turbulent dynamo, since the earlier problems follow from it as particular cases. In addition to the proposed method, certain concrete results of importance to applications are also obtained (the coefficient of turbulent diffusion is determined). In view of the large computational difficulties, the functional approach could be used so far to obtain new results only for regular magnetic field. The method described below is close to the theory of scattering of electromagnetic waves by inhomogeneities[7].

*[vH] ≡ v × H.

1. DERIVATION OF VARIATIONAL DIFFERENTIAL EQUATION

We shall be operating in Fourier space. We therefore change over to H(k, t) and u(k, t), the Fourier transforms of the magnetic field and of the velocity. We introduce the vector-functional and the tensor-functional

$$G_i(\mathbf{k}, t, \theta) = \left\langle H_i(\mathbf{k}, t) \exp\left[i \int u_j(\mathbf{q}, \tau)\theta_j(\mathbf{q}, \tau)\, d\mathbf{q}\, d\tau \right] \right\rangle \qquad (2)$$

$$G_{ij}(\mathbf{k}, \mathbf{k}', t, \theta) = \left\langle H_i(\mathbf{k}, t) H_j(\mathbf{k}', t) \exp\left[i \int u_f(\mathbf{q}, \tau)\theta_f(\mathbf{q}, \tau)\, d\mathbf{q}\, d\tau \right] \right\rangle;$$
$$h_i = \frac{G_i}{\Phi}, \quad g_{ij} = \frac{G_{ij}}{\Phi}, \quad \Phi = \left\langle \exp\left[i \int u_j(\mathbf{q}, \tau)\theta_j(\mathbf{q}, \tau)\, d\mathbf{q}\, d\tau \right] \right\rangle \qquad (3)$$

(θ is the argument of the functional). Introducing G_i, we assume naturally that the field has a regular component whose dynamics is indeed the object of the study. In the problem in which G_{ij} is introduced there is no regular component, and the spectral function is investigated. It is clear that if the velocity field is assumed to be a homogeneous and isotropic random field, then

$$G_i(0) = g_i(0) = \langle H_i \rangle = B_i, \quad G_{ij}(0) = g_{ij}(0)$$
$$= \langle H_i(\mathbf{k}, t) H_j(\mathbf{k}', t) \rangle = F(k, t)\,\delta(\mathbf{k} + \mathbf{k}')\,(\delta_{ij} - k_i k_j / k^2).$$

It will be shown below that the functionals (2) and (3) are fully sufficient for the turbulent-dynamo problem. We shall use the Fourier transform of (1)

$$\frac{\partial \mathbf{H}(\mathbf{k}, t)}{\partial t} = i \int [\mathbf{k}[\mathbf{u}(\mathbf{p}, t)\mathbf{H}(\mathbf{q}, t)]]\, d\mathbf{q} - \nu_m k^2 \mathbf{H}, \quad \mathbf{p} = \mathbf{k} - \mathbf{q}. \qquad (4)$$

We note that

$$\langle H_i(\mathbf{k}, t) u_j(\mathbf{k}', t) \rangle = \frac{1}{i} \left. \frac{\delta G_i(\theta)}{\delta \theta_j(\mathbf{k}', t')} \right|_{\theta=0}, \qquad (5)$$

$$\langle H_i(\mathbf{k}, t) H_j(\mathbf{k}', t) u_f(\mathbf{k}'', t'') \rangle = \frac{1}{i} \left. \frac{\delta G_{ij}(\theta)}{\delta \theta_f(\mathbf{k}'', t'')} \right|_{\theta=0}. \qquad (6)$$

We multiply first the i-th component of (4) by $\exp[i \int u_j(\mathbf{q}, \tau)\theta_j(\mathbf{q}, \tau) d\mathbf{q} d\tau]$ and average, and then by $H_j(\mathbf{k}', t) \exp[i \int u_f(\mathbf{q}, \tau)\theta_f(\mathbf{q}, \tau) d\mathbf{q} d\tau]$ and add to the j-th component of (4) multiplied by $H_i(\mathbf{k}, t)\exp[i \int u_f(\mathbf{q}, \tau)\theta_f(\mathbf{q}, \tau) d\mathbf{q} d\tau]$ and averaged; this yields

$$\frac{\partial h_i(\mathbf{k}, t, \theta)}{\partial t} = \varepsilon_{imn}\varepsilon_{naf} k_m \int d\mathbf{q} \left(\frac{\delta h_f(\mathbf{q}, t, \theta)}{\delta \theta_a(\mathbf{p}, t)} + h_f(\mathbf{q}, t, \theta) \frac{\delta \ln \Phi(\theta)}{\delta \theta_a(\mathbf{p}, t)} \right) - \nu_m k^2 h_i(\mathbf{k}, t, \theta), \qquad (7)$$

$$\frac{1}{2}\frac{\partial g_{ij}(\mathbf{k},\mathbf{k}',t,\theta)}{\partial t}=\varepsilon_{(imn}\varepsilon_{naf}k_m\int d\mathbf{q}\Big(\frac{\delta g_{fj)}(\mathbf{q},\mathbf{k}',t,\theta)}{\delta\theta_a(\mathbf{p},t)}+g_{fj)}(\mathbf{q},\mathbf{k}',t,\theta)\frac{\delta\ln\Phi(\theta)}{\delta\theta_a(\mathbf{p},t)}\Big)+\nu_m k^2 g_{(ij)}(\mathbf{k},\mathbf{k}',t,\theta). \quad (8)$$

The employed symbolism is that universally used (for example, in relativity theory): the parentheses contain that part of the tensor which is symmetrical with respect to the indices. It should be borne in mind that according to (9) if the substitution i → j is made, it is necessary to make simultaneously the substitution $\mathbf{k}\to\mathbf{k}'$ (and of course also in the right-hand side of (8)!):

$$T_{ij}(\mathbf{k},\mathbf{k}')={}^1/_2(T_{ij}(\mathbf{k},\mathbf{k}')+T_{ji}(\mathbf{k}',\mathbf{k})).$$

It is clear that (7) and (8) are analogous to the Schwinger equation in quantum field theory. We call attention to the obvious property of g_{ij}:

$$g_{ij}(\mathbf{k},\mathbf{k}',t,\theta)=g_{ji}(\mathbf{k}',\mathbf{k},t,\theta). \quad (9)$$

2. CONSTRUCTION OF THE SOLUTIONS OF (7) AND (8)

Of course, it is possible to set up a perturbation-theory series for (7) and (8), but this would be equivalent to the usual iteration theory. It is more useful to expand the solution in the functional series

$$h_i(\mathbf{k},t,\theta)=h_i^0(\mathbf{k},t)+\int h_{ij}^{(1)}(\mathbf{k},t,\mathbf{q},\tau)\theta_j(\mathbf{q},\tau)d\mathbf{q}d\tau+{}^1/_2\int h_{ijf}^{(2)}(\mathbf{k},t,\mathbf{q},\tau,\mathbf{q}_1,\tau_1)\theta_j(\mathbf{q},\tau)\theta_f(\mathbf{q}_1,\tau_1)d\mathbf{q}\,d\tau\,d\mathbf{q}_1\,d\tau_1 \quad (10)$$

$$g_{ij}(\mathbf{k},\mathbf{k}',t,\theta)=g_{ij}^0(\mathbf{k},\mathbf{k}',t)+\int g_{ijf}^{(1)}(\mathbf{k},\mathbf{k}',t,\mathbf{q},\tau)\theta_f(\mathbf{q},\tau)d\mathbf{q}\,d\tau+{}^1/_2\int g_{ijfm}^{(2)}(\mathbf{k},\mathbf{k}',t,\mathbf{q},\tau,\mathbf{q}_1,\tau_1)\theta_f(\mathbf{q},\tau)\theta_m(\mathbf{q}_1,\tau_1)d\mathbf{q}\,d\tau\,d\mathbf{q}_1\,d\tau_1 \quad (11)$$

We shall terminate the series (10) at (11) at certain terms. The physical meaning of such a procedure will be explained later. Naturally, the expression $\delta\ln\Phi/\delta\theta$ will also be expanded in a functional series

$$\frac{\delta\ln\Phi}{\delta\theta_i(\mathbf{k},t)}=-\int T_{ij}(\mathbf{k},t,\mathbf{q},\tau)\theta_j(\mathbf{q},\tau)d\mathbf{q}\,d\tau+i\int F_{ijf}(\mathbf{k},t,\mathbf{q},\tau,\mathbf{q}_1,\tau_1)\theta_j(\mathbf{q},\tau)\theta_f(\mathbf{q}_1,\tau_1)d\mathbf{q}\,d\tau\,d\mathbf{q}_1\,d\tau_1 \quad (12)$$

Terminating the functional series and equating the coefficients of like powers, we obtain in lieu of (7) and (8) a closed system of integro-differential equations. The first trivial result is obtained by putting $g^{(1)}=0$ and $h^{(1)}=0$ (zeroth approximation)

$$\frac{\partial\mathbf{B}}{\partial t}+\nu_m k^2\mathbf{B}=0,\qquad \frac{\partial F}{\partial t}+2\nu_m k^2F=0. \quad (13)$$

In this approximation the turbulence is not taken into account at all. Assume now that $g^{(2)}=0$ and $h^{(2)}=0$. We then have the following systems:

$$\frac{1}{2}\frac{\partial g_{(ij)}^{(0)}(\mathbf{k},\mathbf{k}',t)}{\partial t}=\varepsilon_{(imn}\varepsilon_{naf}k_m\int d\mathbf{q}\,g_{fj)a}^{(1)}(\mathbf{q},t,\mathbf{k}',\mathbf{p},t)-\nu_m k^2 g_{(ij)}^{(0)}(\mathbf{k},\mathbf{k}',t)$$

$$\frac{1}{2}\frac{\partial g_{(ij)c}^{(1)}(\mathbf{k},t,\mathbf{b},t')}{\partial t}=-\varepsilon_{(imn}\varepsilon_{naf}k_m\int d\mathbf{q}g_{fj)}^{(0)}(\mathbf{q},\mathbf{k}',t)T_{ac}(\mathbf{p},t,\mathbf{b},t')-\nu_m k^2 g_{(ij)c}^{(1)}(\mathbf{k},t,\mathbf{b},t'). \quad (14)$$

The system for h is similar to (14), but has neither the subscript j nor symmetrization. The velocity field is assumed homogeneous and isotropic

$$T_{ij}(\mathbf{k},t,\mathbf{k}',t')=\delta(\mathbf{k}+\mathbf{k}')\sigma_{ij}(\mathbf{k},t-t'),\qquad \sigma_{ij}(\mathbf{k},t-t')=u(k,t-t')(\delta_{ij}-k_ik_j/k^2). \quad (15)$$

When solving the system (14) and similar systems, we shall assume that at t = 0 we have

$$g_{ijc}^{(1)}(\mathbf{k},t,\mathbf{b},t')=0,\quad g^{(2)}=0\ \text{etc.},\qquad h^{(1)}=0,\quad h^{(2)}=0\ \text{etc.} \quad (16)$$

The initial conditions (16) correspond to an initial statistical independence of the magnetic field and the velocity field. To obtain real equations it is necessary to consider all quantities at $t\gg\tau$, where τ is the correlation time, such that the system has "forgotten" the initial data.

The second equation of (14) can be easily solved with respect to $g^{(1)}$ by expressing $g^{(1)}$ in terms of $g^{(0)}$; we substitute $g^{(1)}$ in the first equation of (14). The time dependence of the right-hand side can be determined by assuming $t\gg\tau$, i.e., by considering the asymptotic expression. This yields the following equations:

$$\partial\mathbf{B}/\partial t+(\chi+\nu_m)k^2\mathbf{B}=0, \quad (17)$$

$$\frac{\partial F}{\partial t}+2(\chi+\nu_m)k^2F=\int F(p)\nu(q)\left(k^2-\frac{(\mathbf{kq})(\mathbf{kp})(\mathbf{pq})}{p^2q^2}\right)d\mathbf{q}; \quad (18)$$

$$\nu(k)=\int_{-\infty}^{+\infty}u(k,s)ds,\qquad \chi={}^1/_3\int\nu(k)d\mathbf{k}.$$

It is clear that (18) is the equation obtained in[3,4]. Thus, the first approximation yields immediately the sum of the infinite series. A similar circumstance arises in the theory of the propagation of electromagnetic waves in the presence of inhomogeneities[7].

3. PHYSICAL MEANING OF THE FIRST APPROXIMATION AND OF EQ. (18)

It is easily seen that $g^{(2)}=0$ ($h^{(2)}=0$) is equivalent to the Millionshchikov-Chandrasekhar hypothesis[6]. Indeed, the functional coefficients $g^{(n)}$ are something like the semi-invariants in magnetohydrodynamics (cf., e.g.,[8]); in addition, it follows from $g^{(2)}=0$ that

$$\langle H_iH_ju_fu_m\rangle=\langle H_iH_j\rangle\langle u_fu_m\rangle. \quad (19)$$

The Millionshchikov hypothesis should yield also correlations of the type $\langle H_iu_j\rangle\langle H_ju_m\rangle$, but according to Chandrasekhar $\langle u_jH_i\rangle=0$. We note that Chandrasekhar did not solve the dynamo problem, but considered the stationary state and obtained an equation for a stationary space-time correlation function. Nonetheless, by using the hypothesis (19) it is also possible to obtain (18). In fact,

$$\partial\langle H_i(\mathbf{x}_1,t)H_j(\mathbf{x}_2,t)\rangle/\partial t=\langle H_i(\mathbf{x}_1,t)\mathrm{rot}_j[\mathbf{v}(\mathbf{x}_2,t)\mathbf{H}(\mathbf{x}_2,t)]\rangle+\langle H_j(\mathbf{x}_2,t)\mathrm{rot}_i[\mathbf{v}(\mathbf{x}_1,t)\mathbf{H}(\mathbf{x}_1,t)]\rangle, \quad (20)$$

$$\partial\langle H_i(\mathbf{x}_1,t)H_j(\mathbf{x}_2,t)v_f(\mathbf{x}_3,t')\rangle/\partial t=\langle\mathrm{rot}_i[\mathbf{v}(\mathbf{x}_1,t)\mathbf{H}(\mathbf{x}_1,t)]H_j(\mathbf{x}_2,t)v_f(\mathbf{x}_3,t')\rangle+\langle H_i(\mathbf{x}_1,t)\mathrm{rot}_j[\mathbf{v}(\mathbf{x}_2,t)\mathbf{H}(\mathbf{x}_2,t)]v_f(\mathbf{x}_3,t')\rangle.$$

Equations (2) demonstrate the usual situation in turbulence theory, where the second moments are expressed in terms of the third, the third in terms of the fourth, etc. Using the hypothesis (19) and taking the Fourier transform of (20), we obtain a system equivalent to (14), and consequently also Eq. (18). Finally, if we write down the system for the moments not in the form (20), but as follows:

$$\langle\mathbf{H}'\mathbf{H}\rangle=\langle\mathbf{H}'\mathbf{H}_0\rangle+\left\langle\mathbf{H}'\int_0^t\mathrm{rot}[\mathbf{vH}]dt_1\right\rangle,$$

$$\langle \mathbf{H}' \operatorname{rot}[\mathbf{vH}] \rangle = \langle \mathbf{H}' \operatorname{rot}[\mathbf{vH}_0] \rangle + \left\langle \mathbf{H}' \operatorname{rot}\left[\mathbf{v} \int_0 \operatorname{rot}[\mathbf{vH}]\, dt_1 \right] \right\rangle$$

etc., then the obtained chain can be terminated with, say, the fourth-order moment $\langle v_i v_j v_f H_m H_b \rangle = 0$[1,3]. This approximation is usually called "allowance for the quadratic correction" (since the highest-order term contains the square of the velocity). In this case we again obtain (18). All the foregoing pertains equally well to (17), and therefore the hypothesis (19) is replaced by

$$\langle H_i u_j u_f \rangle = B_i \langle u_i u_f \rangle. \tag{21}$$

The hypotheses (19) and (21) can also be treated as statistical independence of the magnetic field and the velocity fields at the corresponding moments.

Let us summariz the physical meaning and the justification of Eqs. (1' and (18). They are obtained under the following conditions:

1) The chain is terminated and the quadratic correction taken into account (see[1,3]). The method is valid for small perturbations, and in our case if $R_m \ll 1$ or if $v\tau/l \ll 1$, where $v = (\langle v^2 \rangle)^{1/2}$ and l is the correlation length.

2) The following turbulence model is used: a Gaussian distribution of the velocity field and δ-like correlation in time[4]. It is natural to use the case when τ is much smaller than the period of the process. The fact that (1) and (2) yield an equivalent result is a situation usually encountered in problems of this type.

3) The Chandrasekhar hypothesis (19) is used; Chandrasekhar himself did not derive the equation. The hypothesis has a purely statistical character.

4) The functional series is terminated at $g^{(2)} = 0$, as was done in Sec. 2. The physical meaning of this action is the same hypothesis (19).

4. THE NEXT STEP IN THE THEORY OF THE TURBULENT DYNAMO

It is of interest to determine the result of the quadratic correction of the functional series: $g^{(3)} = 0$, $h^{(3)} = 0$. We shall need in what follows those terms of the series (12) which are quadratic in θ, i.e., essentially the third moments of the velocity field. We shall assume for simplicity that the series (12) is limited to only the first term. This is the situation for a Gaussian distribution (of the velocity field only!), when the entire random process is described by the correlation tensor. We note that introduction of the third moment and, in general, deviations from a Gaussian distribution, cause no fundamental difficulty. This is one of the advantages of the functional approach; in perturbation theory it is quite difficult to use a non-Gaussian process. It seems that the use of a Gaussian velocity field in the problem of the turbulent dynamo imposes hardly any fundamental limitations on the general character of the solution.

We write out a system for the second-approximation h_i (the system for g_{ij} is analogous):

$$\frac{\partial h_i^0(\mathbf{k}, t)}{\partial t} = \varepsilon_{imn}\, \varepsilon_{naf}\, k_m \int d\mathbf{q}\, h_{fa}^{(1)}(\mathbf{q}, t, \mathbf{p}, t) - \nu_m k^2 h_i^0(k, t)$$

$$\partial h_{ic}^{(1)}(\mathbf{k}, t, \mathbf{b}, t')/\partial t = \varepsilon_{imn}\, \varepsilon_{naf}\, k_m \int d\mathbf{q}\, \{h_{fa}^{(2)}(\mathbf{q}, t, \mathbf{p}, \mathbf{b}, t') - h_f^0(\mathbf{q}, t)\, T_{ac}(\mathbf{p}, t, \mathbf{b}, t')\} - \nu_m k^2 h_{ic}^{(1)}(\mathbf{k}, t, \mathbf{b}, t');$$

$$^1/_2\, \partial h_{i(cl)}^{(2)}(\mathbf{k}, t, \mathbf{b}, t', \mathbf{d}, t'')/\partial t = -\varepsilon_{imn}\, \varepsilon_{naf}\, k_m \int d\mathbf{q}\, h_{f(c}^{(1)}(\mathbf{q}, t, \mathbf{b}, t')\, T_{al)}(\mathbf{p}, t, \mathbf{d}, t'') - {}^1/_2\, \nu_m k^2 h_{i(cl)}^{(2)}(\mathbf{k}, t, \mathbf{b}, t', \mathbf{d}, t'');$$

$$k_i h_i^0 = 0, \quad k_i h_{ic}^{(1)} = 0, \quad k_i h_{icl}^{(2)} = 0. \tag{22}$$

Notice should be taken of the following mnemonic rule, which may be useful for the higher approximations. It is easily noted that in all the equations like indices correspond to like arguments of the tensors. This may make it easier to write down the system, and also simplify its derivation. We write the correspondence relations:

$$i \to \mathbf{k}, t, \quad a \to \mathbf{p}, t, \quad l \to \mathbf{d}, t'', \quad f \to \mathbf{q}, t, \quad c \to \mathbf{b}, t'.$$

The symmetrization symbols (...) denote simultaneous permutation of the indices in the corresponding arguments. The system of second-approximation equations for g, while having the same form as (22), has additional symmetrization with respect to the indices i and j, so that the number of terms in the right-hand sides of the equations is doubled. We shall therefore solve only the system (22), and determine by the same token the behavior of the regular magnetic fields, and solve only half-way, as it were, the problem of the pulsating fields. We shall assume henceforth for simplicity that $\nu_m = 0$, i.e., the plasma is highly conducting. It is clear that ν_m should combine in this case with the turbulent diffusion both in (17) and in (18). We shall not calculate the ohmic diffusion more precisely, since this is not the major part of our problem.

We proceed to solve the system (22). Substituting $h^{(2)}$ from the third equation of (22) into the second we obtain an equation for $h^{(1)}$. The complicated time dependence in the equation can be simplified by specifying the following temporal correlation:

$$u(k, t - t') = u_1(k)\, e^{-a|t-t'|}\tau(k)\, a. \tag{23}$$

In this expression a does not depend on k. In the "Kolmogorov" turbulence the time depends on k, but if it is assumed that a depends on k, then the problem cannot be solved in final form. The situation is simplified by the fact that u(k, s) itself enters only under the integral sign with respect to s, so that we can introduce a dimensionless coefficient ("weight") $a\tau(k)$ which makes the integral with respect to the temporal part $\int e^{-as}\tau(k)a\, ds$ already dependent on k. We note that in such a model the first-order approximation in the functional series yields again Eqs. (17) and (18), and $\nu(k)$ must be calculated by using (23).

To use (23), let us determine the time dependence of $h_{ic}^{(1)}(\mathbf{k}, t, \mathbf{b}, t')$. It is clear that $h^{(1)}$ corresponds to

$$\langle H_i(x, t)\, v_c(x', t') \rangle = R_{ic}.$$

Once the system "forgets" the initial data, i.e., at $t \gg \tau$, R_{ic} becomes invariant to shifts with respect to t' and x' if x and t are fixed, i.e.,

$$R_{ic} = M_{ij}(\mathbf{B}(\mathbf{x}, t))\, f_{cj}(\mathbf{r}, s); \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \quad s = t - t';$$

$$h_{ic}^{(1)}(\mathbf{k}, t, \mathbf{b}, t') = m_{ij}(\mathbf{B}(\mathbf{k} + \mathbf{b}, t))\, f_{ij}(\mathbf{b}, s). \tag{24}$$

The regular component $\mathbf{B}(\mathbf{k}, t)$ varies slowly in comparison with the pulsations, and it is therefore natural to subdivide $h^{(1)}$ into a slow component (m_{ij}) and a rapidly varying (correlation) component $f_{ij}(b, s)$. Examination of the equation for $h^{(1)}$ at $t \gg \tau$ and at $t \ll t'$

(this is perfectly sufficient, since we shall need only $h^{(1)}$ at $t = t'$), we can see that the substitution

$$f_{ij}(b, s) = f_{ij}' e^{as}, \qquad s \leqslant 0$$

satisfies the correlation dependence of the equation, i.e., such a substitution causes the entire dependence on s to disappear. This is indeed the simplification resulting from the correlation dependence (23).

We write out the equation for the slow variation of $h^{(1)}$:

$$\left.\frac{\partial h_{ic}^{(1)}(\mathbf{k}, t, \mathbf{b}, t')}{\partial t}\right|_{t'=t} + (a + \chi k^2) h_{ic}^{(1)}(\mathbf{k}, t, \mathbf{b}, t)$$
$$+ 2a^{-1} k_n \int d\mathbf{q}\, q_p h_{[p[i}^{(1)}(\mathbf{q} + \mathbf{b}, t, \mathbf{p}, t)\, \sigma_{n]]c}(\mathbf{b}) = -2k_n h_{[n}^{(0)}(\mathbf{k} + \mathbf{b}, t)\, \sigma_{i]c}(\mathbf{b}). \tag{25}$$

To abbreviate the notation, we have used in (25) the universal symbol [ ] in the subscript, designating the part of the tensor that is antisymmetrical in the indices in the brackets:

$$T_{[ij]} = {}^1/_2 (T_{ij} - T_{ji}).$$

The double brackets (as in the third term from the left in (25)) cannot be expanded unambiguously, and this constitutes an inconvenience. To determine the indices of the tensor we can stipulate that its form is antisymmetrical in the indices i and n, just as the tensor in the right-hand side of (25), i.e., the brackets are expanded in the following manner:

$$[p[in]] = {}^1/_2([pin] - [pni]).$$

It is clear that when $t \gg \tau = a^{-1}$ the quantity $h^{(1)}(t, t)$ assumes a quasistationary behavior determined by the inhomogeneous part of Eq. (25), i.e., by the right-hand part. The time dependence in the right-hand part is contained in $\mathbf{h}^0 = \mathbf{B}$, which varies slowly, and this causes the quasistationary behavior. To determine $h^{(1)}$ at these values of t we use that form of the equation which follows from (25) at $t \gg \tau$:

$$(a + \chi k^2) h_{ic}^{(1)}(\mathbf{k}, \mathbf{b}) + 2a^{-1} k_n \int q_p h_{[p[i}^{(1)}(\mathbf{q} + \mathbf{b}, \mathbf{p})\, \sigma_{n]]c}(\mathbf{b})\, a\tau(b)\, d\mathbf{q}$$
$$= -2k_n h_{[n}^{(0)}(\mathbf{k} + \mathbf{b})\, \sigma_{i]c}(\mathbf{b}) \tag{26}$$

(the dependence on t has been omitted). If we assume that the right-hand side is known, then Eq. (26) has the structure of an integral tensor equation with respect to $h^{(1)}$ or, equivalently, a system of nine integral equations with respect to $h_{ic}^{(1)}$. The solution of (26) can be substituted in the expression

$$2k_n \int d\mathbf{q}\, h_{[ni]}^{(1)}(\mathbf{q}, \mathbf{p})$$

(we have used antisymmetrization) in the first equation of the system (22). Bearing this in mind, we make the following changes of arguments in (26): $\mathbf{k} \to \mathbf{q}$, $\mathbf{b} \to \mathbf{p}$, $\mathbf{q} \to \mathbf{q}' - \mathbf{p}$ ($\mathbf{q}'$ is a new integration variable). Calculation of the second term of the left-hand side of (26) gives rise to the following integrals:

$$T_{ij}(\mathbf{k}) = \int h_{ij}^{(1)}(\mathbf{q}', \mathbf{k} - \mathbf{q}')\, d\mathbf{q}', \quad B_{c[ij]}(\mathbf{k}) = \int q_c' h_{[ij]}^{(1)}(\mathbf{q}', \mathbf{k} - \mathbf{q}')\, d\mathbf{q}'. \tag{27}$$

We can solve (26) by specifying $h_{ic}^{(1)}$ in a certain general form and stipulating that this form satisfy (26). It is simpler, however, to specify a general form of the tensors (27):

$$T_{ij}(\mathbf{k}) = A(k) h_i^0(\mathbf{k}) k_j + B(k) h_j^{(0)}(\mathbf{k}) k_i,$$
$$B_{c[ij]}(\mathbf{k}) = C(k) k_c h_{[i}^{(0)} k_{j]} + D(k) k^2 \delta_{c[i} h_{j]}^0. \tag{28}$$

Substituting (28) in (26), we obtain an expression for $h_{ic}^{(1)}$ in terms of the functions A, B, C, and D, and then form the tensors T_{ij} and B_{cij} in accordance with formulas (29) from this expression for $h_{ic}^{(1)}$. Comparing the obtained tensors with (28), we obtain a system of equations for A, B, C, and D. If a solution of the system exists, then the form (28) has been chosen correctly. We omit these rather cumbersome calculations, and write out only the tensors in terms of which A, B, C, and D are expressed:

$$\frac{1}{2a} \int \frac{\sigma_{ij}(\mathbf{q})\, a\tau(\mathbf{q})\, d\mathbf{q}}{a + \chi p^2} = \alpha_0 \delta_{ij} + \beta_0 \frac{k_i k_j}{k^2}; \tag{29}$$
$$\frac{1}{2a} \int \frac{q_c \sigma_{ij}(\mathbf{q})\, a\tau(\mathbf{q})\, d\mathbf{q}}{a + \chi p^2} = \gamma_1 \delta_{ij} k_c + \gamma_1'(\delta_{ic} k_j + \delta_{cj} k_i) + \delta_1 \frac{k_i k_j k_c}{k^2};$$
$$\frac{1}{2a} \int \frac{q_a q_b \sigma_{ij}(\mathbf{q})\, a\tau(\mathbf{q})\, d\mathbf{q}}{a + \chi p^2} = \alpha_2 \delta_{ab} \delta_{ij} + \alpha_2'(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})$$
$$+ \beta_2 \delta_{ij} k_a k_b + \beta_2'(\delta_{ab} k_i k_j + \delta_{ai} k_b k_j + \delta_{aj} k_b k_i + \delta_{bi} k_a k_j + \delta_{bj} k_a k_i)$$
$$+ \gamma_2 k_a k_b k_i k_j / k^2.$$

All the coefficients (α_0, β_0, γ_1, etc.) here are functions of k.

A solution of the system for A, B, C, and D actually exists, but is very cumbersome; we therefore write out only the expression for A – B, which is the only one contained in the first equation of (22) and plays the role of turbulent viscosity. It is interesting that A – B, generally speaking, depends on k, so that strictly speaking, the equations for B are no longer of the diffusion type such as (17). To be sure this dependence is weak at small values of k. But the regular component constitutes precisely the Fourier components at small values of k, i.e., $k \ll 1/l$. At such values of k we have

$$A - B = 2a\alpha_0. \tag{30}$$

Substituting this expression in the first equation of (22), we obtain

$$\frac{\partial \mathbf{B}}{\partial t} + \chi_2 k^2 \mathbf{B} = 0, \quad \chi_2 = \frac{2}{3} \int \frac{u(q)\, a\tau(q)}{a + \chi q^2} d\mathbf{q}. \tag{31}$$

For comparison with (17), we express χ_2 in terms of $\nu(k)$:

$$\chi_2 = \frac{1}{3} \int \frac{a\nu(q)}{a + \chi q^2} d\mathbf{q}. \tag{32}$$

5. PHYSICAL MEANING OF THE SECOND APPROXIMATION AND POSSIBLE APPLICATIONS OF THE RESULTS

First, it is immediately seen from (32) that χ_2 differs strongly from the first-approximation χ (a has the physical meaning of the reciprocal correlation time: $a = 1/\tau$), although they have the same order of magnitude. This indicates that if this result were to be obtained by summing diagrams, then it would be necessary to add an infinite set of diagrams to those summed in[2,3].

Another important physical conclusion is that in the second approximation, when using the spectral tensor (15) (there is no gyrotropy), there is no turbulent dynamo, and all we have is turbulent damping of the field. We can therefore advance the hypothesis that turbulent generation of regular fields can occur only in the presence of gyrotropy.

The second approximation is obtained under the assumption that $h^{(3)} = 0$. If we take the third variational derivative of the vector-functional h_i at $\theta = 0$ and equate it to zero, then we can see that this assumption corresponds to the form

$$\langle H_i v_j v_f v_n \rangle = \langle H_i v_j \rangle \langle v_f v_n \rangle + \langle H_i v_f \rangle \langle v_j v_n \rangle + \langle H_i v_n \rangle \langle v_j v_f \rangle \qquad (33)$$

If there is no regular component and we are solving the pulsating-field problem, then the assumption $g^{(3)} = 0$ corresponds to the form

$$\langle H_i H_j v_n v_m \rangle = \langle H_i H_j v_f \rangle \langle v_n v_m \rangle + \langle H_i H_j v_n \rangle \langle v_f v_m \rangle + \langle H_i H_j v_m \rangle \langle v_f v_n \rangle. \qquad (34)$$

It is interesting to note that hypotheses (19), (21), (33), and (34) can be formulated as the splitting of the moments of the corresponding orders into a sum of all possible combination of the moments, and also as the vanishing of the semi-invariants of certain definite order. Such hypotheses are frequently used also in other branches of the theory (cf., e.g.,[9]). Expanding H_i in (33) and (34) in perturbation-theory series in the velocity in terms of the initial perturbation H_{0i} and recognizing that $\mathbf{H}_0$ is not correlated with the field $\mathbf{v}$, we obtain in lieu of (33) and (34) a number of relations that connect moments of only the velocity field. In particular, it follows from the zeroth, first, and second approximations that the third, fourth, fifth, and in general all odd moments of the velocity field behave like Gaussian moments. A deviation from the Gaussian law occurs already in the sixth moment. It is of great interest to choose a statistical distribution law (no longer Gaussian) for the velocity field, or, equivalently, a characteristic functional such that all the indicated relations are valid. This would mean that for such a distribution Eq. (31) is the exact solution of the problem (there is no small parameter in the problem!).

As to applications, it can be stated that in astrophysics one frequently encounters situations wherein regular (large-scale) magnetic fields coexist with turbulence, for example on the sun, on stars, and in the interstellar gas of the galaxy. Then the turbulent damping frequently competes with the generation mechanisms. It is therefore important to know more than just the order of magnitude of the diffusion coefficient. In particular, there are hopes at present to obtain experimentally the space-time spectrum of the velocity field on the surface of the sun, and formula (31) may be useful for this case.

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